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Interval Observers for Continuous-Time Systems with Parametric Uncertainties

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Abstract—In this paper, interval observers are designed for linear dynamic systems described by continuous-time models with exogenous disturbances, measurement noises, and parametric uncertainties. Jordan canonical form-based relations are presented for an interval observer that estimates the set of admissible values of a given linear function of the system state vector. The theoretical results are illustrated by a practical example.

Keywords: linear systems, interval observers, Jordan canonical form, estimation

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1. INTRODUCTION AND PROBLEM STATEMENT

The problem of constructing interval observers has received much attention in recent years. Solutions for different classes of systems—discrete- and continuous-time, delayed, and singular—can be found in [1–11], including practical applications of such observers in various fields. The corresponding results were thoroughly overviewed in [10, 11]. As a rule, the interval observers considered therein estimate the set of admissible values of the complete state vector. In practice, however, it may be of interest to get an interval estimate of admissible values only for some linear function of this vector. The corresponding interval observer may turn out to be much simpler than the observer for the full state vector, and the interval width may be noticeably smaller due to the possibility of minimizing the effect of exogenous disturbances. Moreover, when estimating a given linear function, the dynamics of the observer can be represented in canonical form, which simplifies the process of solving the problem and expands the class of systems with the possibility of constructing an interval observer for them.

In recent years, the Jordan canonical form (JCF) [4, 10] has been actively used to design interval observers. (Previously, it was employed to analyze the self-correction property of faults [12].) Under an appropriate choice of the eigenvalues, the system dynamics matrix implemented in the JCF ensures observer stability and is a Metzler matrix, i.e., its off-diagonal elements are nonnegative. Due to these properties, at each time instant, the interval observer produces an estimate of the set of admissible values of the system state vector with uncertainties. According to the analysis results, in addition to stabilization, the JCF simplifies the procedure of ensuring the observer's insensitivity to disturbances and, in some cases, reduces its dimension due to no need for stabilization by special means.

This paper is a logical continuation of the works [13, 14], devoted to the problem of constructing interval observers for systems described by linear continuous-time models with exogenous disturbances and measurement noises. As was demonstrated in [14], the JCF-based interval observer design allows reducing the dimension of observers and, in some cases, the interval width as well. In this paper, for systems described by continuous-time models with exogenous disturbances, measurement noises, and parametric uncertainties, we construct interval observers estimating the set of admissible values of a given linear function of the state vector.

Consider a class of systems with the linear model

$$\begin{aligned}\dot{x}(t) &= (F + \Delta F(\mu(t)))x(t) + Gu(t) + L\rho(t),\\ y(t) &= Hx(t) + v(t),\end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^l$ denote the state, control, and output vectors, respectively; F and G are constant matrices describing the linear dynamics; H and L are given constant matrices; $\rho(t) \in \mathbb{R}^p$ is an unknown bounded time-varying function describing exogenous disturbances of the system, $\|\rho(t)\| \leqslant \rho_*$; $v(t) \in \mathbb{R}^l$ is an unknown bounded time-varying function describing measurement noises, $\|v(t)\| \leqslant v_*$; finally, $\mu(t) \in \Pi \subset \mathbb{R}^s$ is a the bounded vector of variable parameters. By analogy with [10], the values of the vector $\mu(t)$ are unmeasurable, the set of admissible values Π is known, and the matrix function $\Delta F(\mu)$ is bounded for all $\mu(t) \in \Pi$, i.e., $\underline{\Delta F} \leqslant \Delta F \leqslant \overline{\Delta F}$, where $\underline{\Delta F}$ and $\overline{\Delta F}$ are given. As in the paper [2], for arbitrary vectors w_1, w_2 and matrices A_1, A_2, the relations $w_1 \leqslant w_2$ and $A_1 \leqslant A_2$ are understood elementwise.

It is required to construct a minimum-dimension interval observer that forms the lower $\underline{z}(t)$ and upper $\overline{z}(t)$ bounds of the variable $z(t) = Mx(t)$ with a known matrix M such that $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$ for all $t \geqslant 0$. The problems of considering measurement noises and exogenous disturbances for continuous-time systems were discussed in detail in [13]. Therefore, this paper focuses on parametric uncertainties, first assuming that $v(t) = 0$ and $\rho(t) = 0$.

2. MODEL BUILDING

This problem is solved using the minimum-dimension model of system (1.1) that estimates the variable $z(t)$ and some variable $y_*(t)$ determined during the solution process:

$$\begin{aligned}\dot{x}_*(t) &= (F_* + \Delta F_*(t) - PH_*)x_*(t) + (J_* + J')y(t) + Py_*(t) + G_*u(t),\\ y_*(t) &= H_*x_*(t),\\ z(t) &= H_z x_*(t) + Qy(t),\end{aligned} \tag{2.1}$$

where $x_*(t) \in \mathbb{R}^k$ and $k < n$ denotes the model dimension; F_*, G_*, J_*, H_*, H_z, and Q are the matrices to be found; the matrices ΔF_* and J' are defined below; finally, the choice of the matrix P is explained in Section 3. In [4], the stable observer designed previously was reduced to the JCF for the interval estimation of the vector $x(t)$. In contrast, in this paper, the matrix F_* will be immediately found in the JCF. Assuming that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ of this matrix are different and negative, we obtain the diagonal matrix

$$F_* = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0\\ 0 & \lambda_2 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & \lambda_k \end{pmatrix}. \tag{2.2}$$

Let us show the correctness of this assumption. According to [13], the matrix F_* can always be implemented in the identification canonical form

$$F_* = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

By introducing the residual signal feedback with the matrix K, it is reduced to the form $F_* - KH_*$; by choosing (different and negative) eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$ and by determining the corresponding elements of the matrix K, the matrix $F_* - KH_*$ can be made stable. Since the eigenvalues are assumed different, the corresponding eigenvectors $v_1, v_2, \dots, v_k$ form a linearly independent system, so the matrix $T^{-1} = (v_1\ \ v_2\ \ \dots\ \ v_k)$ will be nonsingular. The matrix T has the form

$$T = \begin{pmatrix} w_1 \\ w_2 \\ \dots \\ w_k \end{pmatrix},$$

where the vectors $w_1, w_2, \dots, w_k$ satisfy the condition $w_i v_i = 1$, $i = 1, \dots, k$, $w_i v_j = 0$, $i \neq j$. By the definition of eigenvalues and eigenvectors, $(F_* - KH_*)v_1 = \lambda_1 v_1$. Multiplying on the left by the matrix T, we transform this equation into $T(F_* - KH_*)T^{-1}Tv_1 = \lambda_1 T v_1$. Due to the form of T, $Tv_1 = (1\ \ 0\ \ \dots\ \ 0)^T$ and, consequently,

$$T(F_* - KH_*)T^{-1}(1\ \ 0\ \ \dots\ \ 0)^T = (\lambda_1\ \ 0\ \ \dots\ \ 0)^T.$$

Therefore, the first column of the matrix $T(F_* - KH_*)T^{-1}$ is $(\lambda_1\ \ 0\ \ \dots\ \ 0)^T$. Considering the equation $(F_* - KH_*)v_i = \lambda_i v_i$, $i = 2, 3, \dots, k$, by analogy, we finally arrive at the matrix (2.2).

In the absence of noises, disturbances, and uncertainties, let $x_*(t) = \Phi x(t)$ and $y_*(t) = R_* x(t)$, where the matrix Φ satisfies the well-known conditions [15]

$$\Phi F = F_* \Phi + J_* H, \quad H_* \Phi = R_* H, \quad \Phi G = G_*, \tag{2.3}$$

and an auxiliary condition due to the variable $z(t)$. We derive it below. From $z(t) = Mx(t)$ and (2.1) it follows that

$$M = H_z \Phi + QH = (H_z\ \ Q) \begin{pmatrix} \Phi \\ H \end{pmatrix}. \tag{2.4}$$

This equation has a solution if

$$\operatorname{rank} \begin{pmatrix} \Phi \\ H \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \Phi \\ H \\ M \end{pmatrix}. \tag{2.5}$$

Note that the condition $H_* \Phi = R_* H$ can be written as the equation

$$(H_*\ \ -R_*) \begin{pmatrix} \Phi \\ H \end{pmatrix} = 0, \tag{2.6}$$

which has a solution if

$$\operatorname{rank} \begin{pmatrix} \Phi \\ H \end{pmatrix} < \operatorname{rank}(\Phi) + \operatorname{rank}(H). \tag{2.7}$$

For the matrix (2.2), the first equation in (2.3) splits into k independent equations:

$$\Phi_i F = \lambda_i \Phi_i + J_{*i} H, \quad i = 1, \ldots, k.$$

They can be written as

$$(\Phi_i \quad -J_{*i}) \begin{pmatrix} F - \lambda_i I_n \\ H \end{pmatrix} = 0, \quad i = 1, \ldots, k, \tag{2.8}$$

where Φ_i and J_{*i} stand for the ith rows of the matrices Φ and J_*, respectively, and I_n denotes an identity matrix of dimensions $n \times n$.

When solving equation (2.8), the values $\lambda_i < 0$ must be set so that the resulting matrix Φ with the minimum number of rows satisfies conditions (2.5) and (2.7). After that, the matrices H_z and Q are determined from (2.4), the matrices H_* and R_* from (2.6), and the matrix G_* from (2.3). The uncertainty $\Delta F_*(t)$ figuring in (2.1) is determined from $\Delta F(\mu(t))$ as follows. By analogy with (2.3), formulas (1.1) and (2.1) imply

$$\Phi(F + \Delta F(\mu(t))) = (F_* + \Delta F_*(t))\Phi + J_* H + J' H.$$

Since the matrix F_* satisfies the condition $\Phi F = F_* \Phi + J_* H$, we obtain

$$\Phi \Delta F(\mu(t)) = \Delta F_*(t) \Phi + J' H,$$

or

$$\Phi \Delta F(\mu)) = (\Delta F_* \;\; J') \begin{pmatrix} \Phi \\ H \end{pmatrix}. \tag{2.9}$$

This equation has a solution if

$$\operatorname{rank} \begin{pmatrix} \Phi \\ H \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \Phi \\ H \\ \Phi \Delta F(\mu)) \end{pmatrix}. \tag{2.10}$$

After calculating the matrix Φ, this condition is checked and, if it holds, $\Delta F_*(t)$ and J' are determined from equation (2.9). The matrices $\underline{\Delta F}_*$ and $\overline{\Delta F}_*$ are also determined from (2.9). If condition (2.10) fails, another solution of equation (2.8) is sought. For simplicity, we consider the case where J' is independent of $\Delta F(\mu)$.

3. INTERVAL OBSERVER DESIGN

3.1. The First Variant

By analogy with [2, 10], we consider two variants of constructing interval observers, namely, a simple one with constraints on the original system and a more complicated one without them. Let us begin with the first variant, assuming that $\underline{\Delta F}_* \geqslant 0$ and $x_*(t) \geqslant 0$ for all $t \geqslant 0$. In this case, the observer based on model (2.1) is designed in the form

$$\begin{aligned}
\dot{\underline{x}}_*(t) &= (F_* + \underline{\Delta F}_* - \underline{P} H_*)\underline{x}_*(t) + \underline{P} y_*(t) + (J_* + J')y(t) + G_* u(t), \\
\dot{\overline{x}}_*(t) &= (F_* + \overline{\Delta F}_* - \overline{P} H_*)\overline{x}_*(t) + \overline{P} y_*(t) + (J_* + J')y(t) + G_* u(t), \\
\underline{z}(t) &= H_z \underline{x}_*(t) + Q y(t), \\
\overline{z}(t) &= H_z \overline{x}_*(t) + Q y(t), \\
\underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}.
\end{aligned} \tag{3.1}$$

We introduce the estimation errors

$$\begin{aligned} \underline{e}_*(t) &= x_*(t) - \underline{x}_*(t), \quad \overline{e}_*(t) = \overline{x}_*(t) - x_*(t), \\ \underline{e}_z(t) &= z(t) - \underline{z}(t), \qquad \overline{e}_z(t) = \overline{z}(t) - z(t). \end{aligned} \tag{3.2}$$

Due to (2.1) and (3.1), these errors satisfy the differential equations

$$\begin{aligned} \dot{\underline{e}}_*(t) &= (F_* - \underline{P}H_*)\underline{e}_*(t) + \Delta F_* x_*(t) - \underline{\Delta F}_*\underline{x}_*(t), \\ \dot{\overline{e}}_*(t) &= (F_* - \overline{P}H^*)\overline{e}_*(t) + \overline{\Delta F}_*\overline{x}_*(t) - \Delta F_* x_*(t). \end{aligned} \tag{3.3}$$

Equations (3.1) and (3.3) lead to the following requirements for the matrices $\underline{P}$ and $\overline{P}$: they are chosen so that $F_* + \underline{\Delta F}_* - \underline{P}H_*$ and $F_* + \overline{\Delta F}_* - \overline{P}H_*$ are stable matrices whereas $F_* - \underline{P}H_*$ and $F_* - \overline{P}H_*$ are Metzler matrices. Some recommendations on determining these matrices can be found in [2]. The matrix P in (2.1) is set equal to $\underline{P}$ for the first equation in (3.1) and to $\overline{P}$ for the second one.

Theorem 1. *Assume that* $0 \leqslant \underline{\Delta F}_* \leqslant \Delta F_* \leqslant \overline{\Delta F}_*$, $H_z \geqslant 0$, $x_*(t) \geqslant 0$ *for all* $t \geqslant 0$, $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$, *and there exist matrices* $\underline{P}$ *and* $\overline{P}$ *with the properties specified above. Then the interval observer* (3.3) *satisfies the relation* $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$.

Proof. If $x_*(t) \geqslant \underline{x}_*(t)$, we obtain $\underline{\Delta F}_*(x_*(t) - \underline{x}_*(t)) \geqslant 0$ and $\underline{\Delta F}_* x_*(t) \geqslant \underline{\Delta F}_*\underline{x}_*(t)$ because $\underline{\Delta F}_* \geqslant 0$. Since $\Delta F_* \geqslant \underline{\Delta F}_*$ and $x_*(t) \geqslant 0$, it follows that $(\Delta F_* - \underline{\Delta F}_*)x_*(t) \geqslant 0$ and $\Delta F_* x_*(t) \geqslant \underline{\Delta F}_* x_*(t)$. Due to the considerations above, we therefore have $\Delta F_* x_*(t) \geqslant \underline{\Delta F}_*\underline{x}_*(t)$, i.e., $\Delta F_* x_*(t) - \underline{\Delta F}_*\underline{x}_*(t) \geqslant 0$ in (3.3). By the assumptions made, for $t = 0$, $x_*(0) \geqslant \underline{x}_*(0)$, i.e., $\underline{e}_*(0) \geqslant 0$, and the matrix $F_* - \underline{P}H_*$ in (3.3) is Metzler; hence, $\underline{e}_*(t) \geqslant 0$ for all $t \geqslant 0$ by induction [2]. Since $z(t) = H_z x_*(t) + Qy(t)$, from (3.2) it follows that

$$\underline{e}_z(t) = H_z x_*(t) + Qy(t) - (H_z\underline{x}_*(t) + Qy(t)) = H_z\underline{e}_*(t),$$

which yields $\underline{e}_z(t) \geqslant 0$ due to $\underline{e}_*(t) \geqslant 0$ and the assumption $H_z \geqslant 0$. The inequality $\overline{e}_z(t) \geqslant 0$ is established by analogy. Obviously, the latter inequalities are equivalent to the desired result. The stable matrices $F_* + \underline{\Delta F}_* - \underline{P}H_*$ and $F_* + \overline{\Delta F}_* - \overline{P}H_*$ ensure that the bounds $\underline{x}_*(t)$ and $\overline{x}_*(t)$ (and hence, $\underline{z}(t)$ and $\overline{z}(t)$) are finite. The proof of Theorem 1 is complete.

Remark 1. In the case $H_z \leqslant 0$, the bounds $\underline{z}(t)$ and $\overline{z}(t)$ are given by

$$\underline{z}(t) = H_z\overline{x}_*(t) + Qy(t), \quad \overline{z}(t) = H_z\underline{x}_*(t) + Qy(t).$$

Indeed, in this case,

$$\underline{e}_z(t) = z(t) - \underline{z}(t) = H_z x_*(t) + Qy(t) - (H_z\overline{x}_*(t) + Qy(t)) = -H_z\overline{e}_*(t),$$

which implies $\underline{e}_z(t) \geqslant 0$ due to $\overline{e}_*(t) \geqslant 0$ and $H_z \leqslant 0$. The corresponding inequality for the bound $\overline{z}(t)$ is proved by analogy.

If the matrix H_z is oscillating, the final result will remain the same but with more complicated formulas for the upper and lower bounds.

Thus, an interval observer estimating the variable $z(t) = Mx(t)$ is constructed in the following steps: find solutions of equation (2.8) that give with minimum k a matrix Φ satisfying conditions (2.5) and (2.7); calculate the matrices J_*, G_*, R_*, and H_* and choose matrices $\underline{P}$ and $\overline{P}$ with the properties specified above. Such a choice is not always possible; in this case, the second observer design variant should be used.

3.2. The Second Variant

Consider a more complex interval observer without constraints on the original system. The presentation is preceded by an auxiliary result.

Lemma 1. *Assume that* $\underline{A} \leqslant A \leqslant \overline{A}$ *for some matrices* $\underline{A}$, A, *and* $\overline{A}$ *of dimensions* $k \times k$ *and* $\underline{x}_* \leqslant x_* \leqslant \overline{x}_*$ *for some* k*-dimensional vectors* $\underline{x}_*$, x_*, *and* $\overline{x}_*$. *Then*

$$\underline{A}^+\underline{x}_*^+ - \overline{A}^+\underline{x}_*^- - \underline{A}^-\overline{x}_*^+ + \overline{A}^-\overline{x}_*^- \leqslant Ax \leqslant \overline{A}^+\overline{x}_*^+ - \underline{A}^+\overline{x}_*^- - \overline{A}^-\underline{x}_*^+ + \underline{A}^-\underline{x}_*^-,$$

where $A^+ = \max(0, A)$ *and* $A^- = A^+ - A$, *and the analogous relations hold for the vector* x_*.

The proof was given in [1].

The interval observer based on model (2.1) with $P = 0$ is designed in the form

$$\begin{aligned}
\dot{\underline{x}}_*(t) &= F_*\underline{x}_*(t) + (\underline{\Delta F}_*^+\underline{x}_*^+ - \overline{\Delta F}_*^+\underline{x}_*^- - \underline{\Delta F}_*^-\overline{x}_*^+ + \overline{\Delta F}_*^-\overline{x}_*^-) + (J_* + J')y(t) + G_*u(t),\\
\dot{\overline{x}}_*(t) &= F_*\overline{x}_*(t) + (\overline{\Delta F}_*^+\overline{x}_*^+ - \underline{\Delta F}_*^+\overline{x}_*^- - \overline{\Delta F}_*^-\underline{x}_*^+ + \underline{\Delta F}_*^-\underline{x}_*^-) + (J_* + J')y(t) + G_*u(t),\\
\underline{z}(t) &= H_z\underline{x}_*(t) + Qy(t),\\
\overline{z}(t) &= H_z\overline{x}_*(t) + Qy(t),\\
\underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}.
\end{aligned} \tag{3.4}$$

Here, the equations for the estimation errors take the form

$$\begin{aligned}
\dot{\underline{e}}_*(t) &= F_*\underline{e}_*(t) + \Delta F_* x_*(t) - (\underline{\Delta F}_*^+\underline{x}_*^+ - \overline{\Delta F}_*^+\underline{x}_*^- - \underline{\Delta F}_*^-\overline{x}^+ + \overline{\Delta F}_*^-\overline{x}_*^-),\\
\dot{\overline{e}}_*(t) &= F_*\overline{e}_*(t) + (\overline{\Delta F}_*^+\overline{x}_*^+ - \underline{\Delta F}_*^+\overline{x}_*^- - \overline{\Delta F}_*^-\underline{x}_*^+ + \underline{\Delta F}_*^-\underline{x}_*^-) - \Delta F_* x_*(t).
\end{aligned} \tag{3.5}$$

Theorem 2. *Assume that* $\underline{\Delta F}_* \leqslant \Delta F_* \leqslant \overline{\Delta F}_*$ *and* $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$. *Then the interval observer* (3.4) *satisfies the relation* $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$.

Proof. Due to (3.2), from $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$ it follows that $\underline{e}_*(0) \geqslant 0$ and $\overline{e}_*(0) \geqslant 0$. Since F_* is a Metzler matrix, by (3.5) and the lemma we have $\underline{e}_*(t) \geqslant 0$ and $\overline{e}_*(t) \geqslant 0$ for all $t \geqslant 0$ [2]. The relation $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$ is established by analogy with the second part of Theorem 1 and Remark 1. The proof of Theorem 2 is complete.

Remark 2. Direct comparison shows that the first variant imposes noticeably more constraints on the system than the second one but has a simpler structure. In particular, the second variant does not need the matrices H_* and R_*; hence, condition (2.7) can be ignored when implementing this variant, and the corresponding interval observer can be constructed for a wider class of systems. When solving a particular problem, one should therefore begin with the first variant, passing to the second variant only if the former is impossible to implement.

Remark 3. An essential role in Theorems 1 and 2 is played by the condition $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$. Due to the stability of the matrices $F_* + \underline{\Delta F}_* - \underline{P}H_*$, $F_* + \overline{\Delta F}_* - \overline{P}H_*$, and F_*, the desired inequality $\underline{z}(t) \leqslant z(t) \leqslant \overline{z}(t)$ will hold for some $t > 0$ even without the condition $\underline{x}_*(0) \leqslant x_*(0) \leqslant \overline{x}_*(0)$: the effect of the initial conditions almost disappears after some time. Indeed, we demonstrate this fact for the second variant and the first equation in (3.5). Let us denote

$$v_0(t) = \Delta F_* x_*(t) - (\underline{\Delta F}_*^+\underline{x}_*^+ - \overline{\Delta F}_*^+\underline{x}_*^- - \underline{\Delta F}_*^-\overline{x}^+ + \overline{\Delta F}_*^-\overline{x}_*^-).$$

The lemma implies $v_0(t) \geqslant 0$. Clearly, the matrices figuring in the expression above, particularly $\overline{\Delta F}_*^+$, can be always chosen so that $v_0(t) \geqslant v_{0*}$ for some $v_{0*} > 0$.

Since the matrix F_* is diagonal, we consider equation (3.5) for the first component of the error $\underline{e}_*(t)$:

$$\dot{\underline{e}}_{*1}(t) = \lambda_1\underline{e}_{*1}(t) + v_{01}(t).$$

It has the solution

$$\underline{e}_{*1}(t) = exp(\lambda_1 t)\underline{e}_{*1}(0) + exp(\lambda_1 t)\int\limits_0^t v_{01}(\tau)exp(-\lambda_1\tau)d\tau.$$

Because $\lambda_1 < 0$ and $v_0(t) \geqslant v_{0*} > 0$, the first term will almost vanish and the error $\underline{e}_{*1}(t)$ will be positive after some time T_1, despite the possibly negative value $\underline{e}_{*1}(0)$. Similar results can be obtained for the other components of the vector $\underline{e}_*(t)$ and all components of the vector $\overline{e}_*(t)$.

Remark 4. The concepts of a confidence interval and a confidence probability from mathematical statistics can be adapted to interval observers: if $\Delta F_* \in (\underline{\Delta F}_*, \overline{\Delta F}_*)$ with some degree of belonging μ_*, then $z(t) \in (\underline{z}(t), \overline{z}(t))$ with the degree μ_* as well.

4. CONSIDERATION OF EXOGENOUS DISTURBANCES AND MEASUREMENT NOISES

The case $v(t) \neq 0$, $\rho(t) \neq 0$ was studied in detail in [13]. Note that the observer is designed based on model (2.1) with the term $J_*y(t)$ replaced by $J_*Hx(t)$. (It is necessary to consider the measurement noises according to the relation $y(t) = Hx(t) + v(t)$.) In this case, the first variant of the interval observer is described by

$$\begin{aligned}
\dot{\underline{x}}_*(t) &= (F_* + \underline{\Delta}F_* - \underline{P}H_*)\underline{x}_*(t) + \underline{P}y_*(t) + (J_* + J')y(t) + G_*u(t)\\
&\quad - (|J_*| + |J'|)E_k v_* - |L_*|E_k\rho_*,\\
\dot{\overline{x}}_*(t) &= (F_* + \overline{\Delta}F_* - \overline{P}H_*)\overline{x}_*(t) + \overline{P}y_*(t) + (J_* + J')y(t) + G_*u(t)\\
&\quad + (|J_*| + |J'|)E_k v_* + |L_*|E_k\rho_*,\\
\underline{z}(t) &= H_z\underline{x}_*(t) + Qy_0(t),\\
\overline{z}(t) &= H_z\overline{x}_*(t) + Qy_0(t),\\
\underline{x}_*(0) &= \underline{x}_{*0}, \quad \overline{x}_*(0) = \overline{x}_{*0}.
\end{aligned} \tag{4.1}$$

Here, the matrix $|A|$ consists of the absolute values of the corresponding elements of the matrix A, and E_k is a matrix of dimensions $k \times 1$ composed of unities. The variable $y_0(t)$ represents the components of the output vector $y(t)$ that are not affected by the exogenous disturbances, i.e., $y_0(t) = N_1y(t)$ for some matrix N_1. We introduce a maximum-rank matrix L_0 such that $L_0L = 0$. Since the vector $x'(t) = L_0x(t)$ is not disturbed, we obtain $y_0(t) = N_2x'(t)$ for some matrix N_2. Then the matrices N_1 and N_2 satisfy the equation $N_1H = N_2L_0$, which has a solution if

$$\operatorname{rank}\begin{pmatrix} H \\ L_0 \end{pmatrix} < \operatorname{rank}(H) + \operatorname{rank}(L_0).$$

Under this condition, the matrices N_1 and N_2 are determined from the equation

$$(N_1 \quad -N_2)\begin{pmatrix} H \\ L_0 \end{pmatrix} = 0.$$

If this condition fails, it is necessary to replace the variable $y_0(t)$ in (4.1) by $y(t)$, which will extend the interval $(\underline{z}(t), \overline{z}(t))$.

The equations for the estimation errors take the form

$$\begin{aligned}
\dot{\underline{e}}_*(t) &= (F_* - \underline{P}H_*)\underline{e}_*(t) + \Delta F_*x_*(t) - \underline{\Delta F}_*\underline{x}_*(t)\\
&\quad - (J_* + J')v(t) + (|J_*| + |J'|)E_k v_* - L_*\rho(t) + |L_*|E_k\rho_*,\\
\dot{\overline{e}}_*(t) &= (F_* - \overline{P}H_*)\overline{e}_*(t) + \overline{\Delta F}_*\overline{x}_*(t) - \Delta F_*x_*(t)\\
&\quad + (J_* + J')v(t) + (|J_*| + |J'|)E_k v_* + L_*\rho(t) + |L_*|E_k\rho_*.
\end{aligned}$$

The desired result directly follows from the proof of Theorem 1 and the obvious additional inequalities

$$\pm(J_* + J')v(t) + (|J_*| + |J'|)E_k v_* \geqslant 0, \quad \pm L_*\rho(t) + |L_*|E_k\rho_* \geqslant 0,$$

valid for all $t \geqslant 0$.

Remark 5. Due to (4.1), the width of the interval $(\underline{z}(t), \overline{z}(t))$ depends on the exogenous disturbances and measurement noises. To reduce this width, the matrices Φ and J_* should be determined from the equation

$$(S_i \quad -J_{*i})\begin{pmatrix} L_0(F - \lambda_i I_n) \\ H \end{pmatrix} = 0, \quad i = 1, \ldots, k, \tag{4.2}$$

which is derived from (2.8) for $\Phi = SL_0$ with some matrix S. (It ensures $L_* = 0$, i.e., the insensitivity of the model to the exogenous disturbances.) As before, here S_i denotes the ith row of the matrix S. If equation (4.2) has no solution, robust methods should be used; see below.

The sensitivity of model (2.1) to the exogenous disturbances $\rho(t)$ is often estimated by the norm $\|\Phi L\|_F$ of the matrix ΦL. To minimize this norm for $i = 1, \ldots, k$, we choose an eigenvalue $\lambda_i < 0$ for which equation (2.8) has a solution and then find all solutions in the form $\Phi_i^{(1)}, \ldots, \Phi_i^{(n_i)}$. These solutions can be written as

$$\Phi_{*i} = \begin{pmatrix} \Phi_i^{(1)} \\ \ldots \\ \Phi_i^{(n_i)} \end{pmatrix}, \quad J_i = \begin{pmatrix} J_{*i}^{(1)} \\ \ldots \\ J_{*i}^{(n_i)} \end{pmatrix}.$$

The singular-value decomposition of the matrix product $\Phi_{*i}L$ is given by

$$\Phi_{*i}L = U_L\Sigma_L V_L,$$

where U_L and V_L are orthogonal matrices,

$$\Sigma_L = (\mathrm{diag}(\sigma_1, \ldots, \sigma_{c_i})\ 0) \ \text{ or } \ \Sigma_L = \begin{pmatrix} \mathrm{diag}(\sigma_1, \ldots, \sigma_{c_i}) \\ 0 \end{pmatrix}$$

depending on the number of rows and columns in the matrix $\Phi_{*i}L$, $c_i = \min(n_i, kp)$, and $0 \leqslant \sigma_1 \leqslant \ldots \leqslant \sigma_{c_i}$ are the singular values of the matrix $\Phi_{*i}L$ arranged in ascending order [16]. The first transposed column of the matrix U_L is taken as the weight vector $w = (w_1 \ \ldots \ w_{n_i})$, and the rows $\Phi_i = w\Phi_{*i}$ and $J_{*i} = wJ_i$ are calculated consequently; if $n_i = 1$, then $\Phi_i := \Phi_i^{(1)}$ and $J_{*i} := J_i^{(1)}$. The resulting rows $\Phi_1, \ldots, \Phi_k$ and $J_{*1}, \ldots, J_{*k}$ form the matrices

$$\Phi = \begin{pmatrix} \Phi_1 \\ \ldots \\ \Phi_k \end{pmatrix}, \quad J_* = \begin{pmatrix} J_{*1} \\ \ldots \\ J_{*k} \end{pmatrix},$$

and the feasibility of estimating the variables $z(t)$ and $y_*(t)$ by criteria (2.5) and (2.7) is checked. If the outcome is positive, the problem is successfully solved; otherwise, for some $\lambda_i < 0$, it is necessary to find another vector $w = (w_1 \ \ldots \ w_{n_i})$ for a singular value $\sigma > \sigma_1$ and calculate a new matrix Φ_i.

Theorem 3. *This procedure for constructing the matrix Φ yields an optimal solution in terms of minimizing the norm $\|\Phi L\|_F$.*

Proof. The conclusion directly follows from the properties of the singular-value decomposition.

The determination of the matrices H_z, Q, ΔF_*, J', H_*, R_*, $G_* = \Phi G$, and $L_* = \Phi L$ completes the design of model (2.1) with the minimum sensitivity to the exogenous disturbances.

By analogy with [13, 14], the above results for estimating the variable $z(t)$ can be applied to estimate the full state vector $x(t)$. For this purpose, it is necessary to find the solutions of equation (4.2) with $\lambda_i < 0$ and the maximum possible value k, compile the matrix $\Phi^{(1)} =: M^{(1)}$, and construct an interval estimate for the vector $x^{(1)}(t) = M^{(1)}x(t)$ following one of the variants if criteria (2.5) and (2.7) hold. Note that this estimate will be independent of the exogenous disturbance $\rho(t)$. Next, it is necessary to determine a vector $x^{(2)}(t)$ that complements $x^{(1)}(t)$ to $x(t)$ and a matrix $M^{(2)}$ such that $x^{(2)}(t) = M^{(2)}x(t)$. The interval observer estimating the variable $x^{(2)}(t)$ is obtained based on equation (2.8). The composition of the two observers yields an interval $(\underline{x}(t), \overline{x}(t))$ that is narrower compared to the classical one [10].

5. A PRACTICAL EXAMPLE

Consider a three-tank system of the form

$$\begin{aligned}
\dot{x}_1(t) &= b_4 u_1(t)/\vartheta_1 - (b_1 - \delta_1(t))(x_1(t) - x_2(t)),\\
\dot{x}_2(t) &= b_5 u_2(t)/\vartheta_2 + (b_1 - \delta_1(t))(x_1(t) - x_2(t)) - (b_2 - \delta_2(t))(x_2(t) - x_3(t)),\\
\dot{x}_3(t) &= (b_2 - \delta_2(t))(x_2(t) - x_3(t)) - b_3(x_3(t) - b_6) + \rho(t),\\
y_1(t) &= x_2(t) + v_1(t), \quad y_2(t) = x_3(t) + v_2(t),
\end{aligned} \tag{5.1}$$

where the coefficients $b_1, \ldots, b_5$ are determined by the geometrical dimensions of the system; x_1, x_2, and x_3 are the fluid levels in the corresponding tanks (Fig. 1). Fluid flows into the first and second tanks and flows out of the third tank through a pipe located at a height b_6. The parametric uncertainties $\delta_1(t)$ and $\delta_2(t)$ are related to possible clogging of the pipes connecting the tanks and, consequently, a decrease in their capacity. The exogenous disturbance $\rho(t)$ reflects possible leaks in the third tank. For simplicity, we take $b_1 = \ldots = b_5 = 1$ and $b_6 = 0$; $0 \leqslant \delta_1(t) \leqslant 0.2$, $0 \leqslant \delta_2(t) \leqslant 0.1$; $|\rho(t)| \leqslant \rho_*$, $|v_1(t)| \leqslant v_{*1}$, $|v_2(t)| \leqslant v_{*2}$.

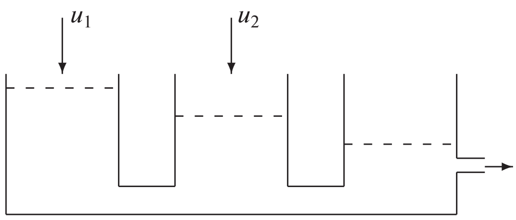

Fig. 1. Three-tank system.

This system is modeled by the matrices

$$F = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Direct comparison of the general model (1.1) and system (5.1) shows that the uncertainties are described by the matrix functions

$$\Delta_1 F(t) = \begin{pmatrix} \delta_1(t) & -\delta_1(t) & 0 \\ -\delta_1(t) & \delta_1(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Delta_2 F(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta_2(t) & -\delta_2(t) \\ -\delta_2(t) & \delta_2(t) & 0 \end{pmatrix}.$$

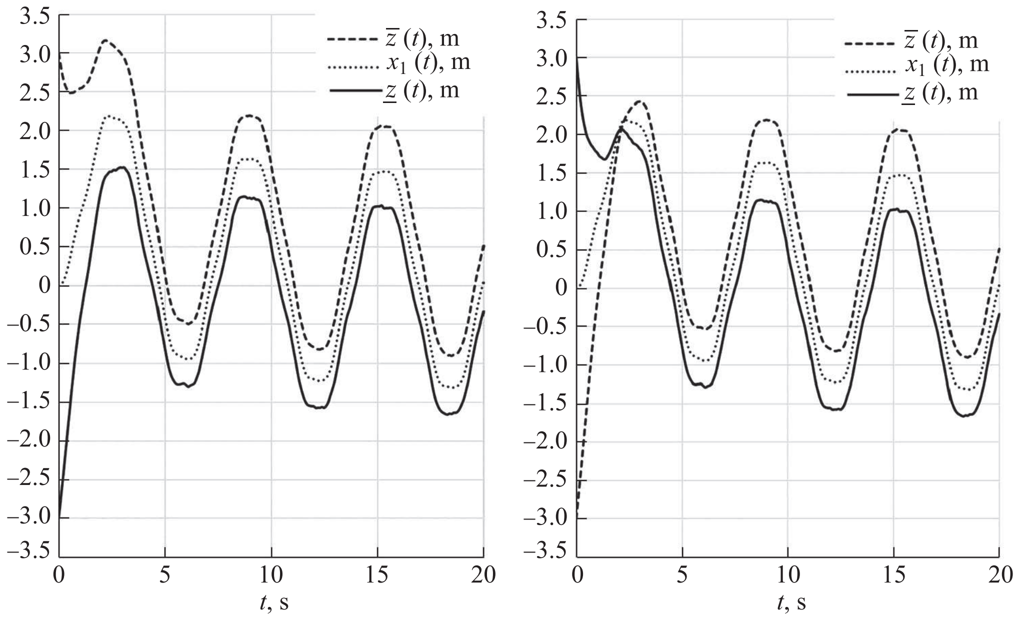

Fig. 2. The graphs of the functions $x_1(t)$, $\underline{z}(t)$, and $\overline{z}(t)$ under different initial conditions.

We construct an interval observer estimating the variable $z(t) = x_1(t)$. In this case, $M = (1\ \ 0\ \ 0)$ and $L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Equation (4.2) takes the form

$$(S_i \quad -J_{*i}) \begin{pmatrix} -1-\lambda_i & 1 & 0 \\ 1 & -2-\lambda_i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0.$$

Letting $\lambda = -1$ gives $S_i = J_{*i} = (1\ \ 0)$, $\Phi = (1\ \ 0\ \ 0)$, and $G_* = (1\ \ 0)$. The uncertainty $\Delta F_*(t)$ and the matrix J' are determined from equation (2.9); as a result, we obtain $\Delta_1 F_*(t) = \delta_1(t)$, $J_1' = (-\delta_1(t)\ \ 0)$, $\Delta_2 F_*(t) = 0$, and $J_2' = 0$. According to physical considerations, $x_*(t) \geqslant 0$ and $\delta_1(t) \geqslant 0$ for all $t \geqslant 0$; hence, the first variant can be used to design the observer. In view of $0 \leqslant \delta_1(t) \leqslant 0.2$, we assume that $P = 0$.

Since the variable $x_3(t)$ is affected by the disturbance $\rho(t)$ and, in addition, $y_2(t) = x_3(t) + v_2(t)$, we take $y_0(t) = y_1(t)$. Obviously, condition (2.5) holds and $H_z = 1$, $Q = 0$. The model becomes

$$\begin{aligned} \dot{x}_*(t) &= (\delta_1(t) - 1)x_*(t) + (1 - \delta_1(t))H_1 x(t) + u_1(t), \\ z(t) &= x_*(t), \end{aligned}$$

where $x_* = \Phi x(t) = x_1(t)$. The interval observer (4.1) take the form

$$\begin{aligned} \dot{\underline{x}}_*(t) &= (\underline{\delta_1} - 1)\underline{x}_*(t) + (1 - \underline{\delta_1})y_1(t) + u_1(t) - v_*, \\ \dot{\overline{x}}_*(t) &= (\overline{\delta_1} - 1)\overline{x}_*(t) + (1 - \overline{\delta_1})y_1(t) + u_1(t) + v_*, \\ \underline{z}(t) &= \underline{x}_*(t), \quad \overline{z}(t) = \overline{x}_*(t). \end{aligned}$$

Due to $\underline{\delta}_1 - 1 < 0$ and $\overline{\delta}_1 - 1 < 0$, the observer is stable.

Numerical simulation was performed for $u_1(t) = 2\sin(t)$, $u_2(t) = 2\sin(5t)$, the noises $v_1(t)$, $v_2(t)$ and disturbance $\rho(t)$ described by random independent variables with variances 0.1, and the uncertainty $\delta_1(t) = 0.1(1 + \sin(10t))$ with $\underline{\delta_1} = 0$ and $\overline{\delta_1} = 0.2$. The simulation results are presented in

Fig. 2: the graphs of the functions $x_1(t)$, $\underline{z}(t)$, and $\overline{z}(t)$ under the initial conditions $x(0) = (0\ \ 0\ \ 0)^T$, $\underline{x}_*(0) = -3$, and $\overline{x}_*(0) = 3$ (Fig. 2a) and $x(0) = (0\ \ 0\ \ 0)^T$, $\underline{x}_*(0) = 3$, and $\overline{x}_*(0) = -3$ (Fig. 2b). As has been demonstrated in Section 3, the "atypical" initial conditions are rather quickly "forgotten" and have no effect on the estimation process for $t \geqslant 2$.

6. CONCLUSIONS

In this paper, we have constructed interval observers for linear dynamic systems described by continuous-time models with exogenous disturbances, measurement noises, and parametric uncertainties. Jordan canonical form-based relations have been presented for a minimum-dimension interval observer estimating the set of admissible values of a given linear function of the system state vector. Two observer design variants have been considered and compared. The theoretical results have been illustrated by a practical example.

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REFERENCES

1. Efimov, D., Raissi, T., Perruquetti, W., and Zolghadri, A., Estimation and Control of Discrete-Time LPV Systems Using Interval Observers, *Proc. 52nd IEEE Conf. on Decision and Control*, Florence, 2013, pp. 5036–5041.

2. Chebotarev, S., Efimov, D., Raissi, T., and Zolghadri, A., Interval Observers for Continuous-Time LPV Systems with L1/L2 Performance, *Automatica*, 2015, vol. 51, pp. 82–89.

3. Efimov, D., Polyakov, A., and Richard, J., Interval Observer Design for Estimation and Control of Time-Delay Descriptor Systems, *Eur. J. Control*, 2015, vol. 23, pp. 26–35.

4. Kolesov, N., Gruzlikov, A., and Lukoyanov, E., Using Fuzzy Interacting Observers for Fault Diagnosis in Systems with Parametric Uncertainty, *Proc. 12th Inter. Symp. Intelligent Systems (INTELS'16)*, October 5–7, 2016, Moscow, pp. 499–504.

5. Kremlev, A.S. and Chebotarev, S.G., Synthesis of Interval Observer for Linear System with Variable Parameters, *Journal of Instrument Engineering*, 2013, vol. 56, no. 4, pp. 42–46.

6. Mazenc, F. and Bernard, O., Asymptotically Stable Interval Observers for Planar Systems with Complex Poles, *IEEE Trans. Automatic Control*, 2010, vol. 55, no. 2, pp. 523–527.

7. Blesa, J., Puig, V., and Bolea, Y., Fault Detection Using Interval LPV Models in an Open-Flow Canal, *Control Engineering Practice*, 2010, vol. 18, pp. 460–470.

8. Zheng, G., Efimov, D., and Perruquetti, W., Interval State Estimation for Uncertain Nonlinear Systems, *Proc. IFAC NOLCOS 2013*, Toulouse, 2013.

9. Zhang, K., Jiang, B., Yan, X., and Edwards, C., Interval Sliding Mode Based Fault Accommodation for Non-Minimal Phase LPV Systems with Online Control Application, *Int. J. Control*, 2019. https://doi.org/10.1080/00207179.2019.1687932

10. Efimov, D. and Raissi, T., Design of Interval State Observers for Uncertain Dynamical Systems, *Autom. Remote Control*, 2016, vol. 77, no. 2, pp. 191–225.

11. Khan, A., Xie, W., Zhang, L., and Liu, L., Design and Applications of Interval Observers for Uncertain Dynamical Systems, *IET Circuits Devices Syst.*, 2020, vol. 14, pp. 721–740.

12. Zhirabok, A., Self-correction of Errors in Discrete Dynamic Systems, *Autom. Remote Control*, 2006, vol. 67, no. 6, pp. 868–879.

13. Zhirabok, A.N., Zuev, V.V., and Ir, K.C., Method to Design Interval Observers for Linear Time-Invariant Systems, *J. Comput. Syst. Sci. Int.*, 2022, vol. 61, no. 4, pp. 485–495.

14. Zhirabok, A., Zuev, A., Filaretov, V., Shumsky, A., and Kim Chkhun Ir, Jordan Canonical Form in Diagnosis and Estimation Problems, *Autom. Remote Control*, 2022, vol. 83, no. 9, pp. 1355–1370.

15. Zhirabok, A., Shumsky, A., and Pavlov, S., Diagnosis of Linear Dynamic Systems by the Nonparametric Method, *Autom. Remote Control*, 2017, vol. 78, no. 7, pp. 1173–1188.

16. Low, X., Willsky, A., and Verghese, G., Optimally Robust Redundancy Relations for Failure Detection in Uncertain Systems, *Automatica*, 1996, vol. 22, pp. 333–344.

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LINEAR SYSTEMS

Robust Stability of Differential-Algebraic Equations under Parametric Uncertainty

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Abstract—This paper considers linear differential-algebraic equations (DAEs) representing a system of ordinary differential equations with an identically singular matrix at the derivative in the domain of its definition. The matrix coefficients of DAEs are assumed to depend on the uncertain parameters belonging to a given admissible set. For the parametric family under consideration, structural forms with separate differential and algebraic parts are built. As is demonstrated below, the robust stability of the DAE family is equivalent to the robust stability of its differential subsystem. For the structure of perturbations, sufficient conditions are established under which the separation of DAEs into the algebraic and differential components preserves the original type of functional dependence on the uncertain parameters. Sufficient conditions for robust stability are obtained by constructing a quadratic Lyapunov function.

Keywords: differential-algebraic equations, parametric uncertainty, arbitrarily high unsolvability index, robust stability

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1. INTRODUCTION

Consider a system of differential-algebraic equations (DAEs) of the form

$$A(\gamma)x'(t) + B(\gamma)x(t) = 0, \quad t \in T = [0, +\infty), \tag{1.1}$$

where $\gamma = \operatorname{col}(\gamma_1, \ldots, \gamma_l)$ denotes an uncertain vector parameter belonging to a given admissible set $\Gamma = \left\{\gamma \in \mathbf{R}^l : \ \|\gamma\|_{\mathbf{R}^l} \leqslant a\right\}$; $A(\gamma)$ and $B(\gamma)$ are known real matrices of dimensions $(n \times n)$; finally, $x(t)$ is the desired n-dimensional function. By assumption, $\det A(\gamma) \equiv 0$.

A crucial characteristic of DAEs is the unsolvability index, which reflects the complexity of the internal structure of the system. The higher value this index takes, the more difficult it will be to divide DAEs into differential and algebraic components. Without such separation, it is impossible to analyze the stability of nonstationary systems. See Subsection 2.1 for the exact definition of the index used in this paper. The closest concept to this definition is the differentiation index [1], introduced for unperturbed nonstationary systems of DAEs. If the unsolvability index exists, it is equal to the differentiation index.

This paper discusses the issue of asymptotic stability for the parametric family (1.1). Robust stability analysis is much more complicated for DAEs than for systems of ordinary differential equations resolvable with respect to the derivative; for example, see [2, pp. 186–225]. The explanation is that even in the simplest case of index 1, an arbitrarily small perturbation of the coefficients may violate the internal structure of the system and, consequently, change the properties and type of the general solution [3, p. 61].

There are relatively few publications on the robust stability of DAEs; in particular, we refer to [4–14]. The papers [4, 5] are considered to be the pioneering studies on this subject. Most authors investigated systems with the perturbed matrix coefficient at $x(t)$ [5–9]. Only a few works supposed perturbations of the matrix at the derivative of the desired vector function [4, 10–12]. Separate research was devoted to robust stability and estimation of the stability radius of nonstationary DAEs of index 1; see [6, 7, 13].

Robust stability is also analyzed when studying the admissibility of linear DAEs; for example, we mention [15–21]. In addition to stability, admissibility implies that the system possesses the properties of regularity and either causality (for discrete-time systems) or impulse freeness (in the continuous-time case).

Presently, it is still topical to investigate the robust stability of parametric families of the form (1.1) of an arbitrarily high unsolvability index with perturbations in all matrix coefficients.

For the family (1.1) with the vector and scalar parameters, we prove below the existence of structural forms with separated algebraic and differential subsystems as well as propose algorithms for building these forms. In our previous works (e.g., [22, 23]), the dimension of the solution space and the structure of the general solution of perturbed DAEs were ensured the same as those of the nominal system by imposing some finite relations on the perturbations. In this paper, the structural forms are defined simultaneously for the entire family (1.1), and their existence is established without additional constraints on the perturbations.

Unfortunately, the reduction of the DAEs (1.1) to a certain structural form generally complicates the original type of functional dependence on uncertain parameters. Under the sufficient conditions derived below, the system is separated into algebraic and differential parts while preserving the original type of this dependence in the differential subsystem.

Under the assumptions accepted here, the stability of system (1.1) is equivalent to the stability of its differential part, which represents a parametric family as well. Sufficient conditions for robust stability follow from the required existence of a general quadratic Lyapunov function for the differential subsystem.

2. SUFFICIENT CONDITIONS FOR ROBUST STABILITY

2.1. The Structural Form for DAEs with Parametric Uncertainty

For system (1.1) we define the following matrices:

$$\begin{aligned} \mathcal{B}_r\left[B(\gamma)\right] &= \operatorname{col}\left(B(\gamma), O, \ldots, O\right), \\ \mathcal{A}_r\left[A(\gamma), B(\gamma)\right] &= \operatorname{col}\left(A(\gamma), B(\gamma), O, \ldots, O\right) \end{aligned} \tag{2.1}$$

of dimensions $(n(r+1) \times n)$,

$$\Lambda_r\left[A(\gamma), B(\gamma)\right] = \begin{pmatrix} O & O & \ldots & O & O \\ A(\gamma) & O & \ldots & O & O \\ B(\gamma) & A(\gamma) & \ldots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \ldots & A(\gamma) & O \\ O & O & \ldots & B(\gamma) & A(\gamma) \end{pmatrix} \tag{2.2}$$

of dimensions $(n(r+1) \times nr)$, and

$$\mathcal{D}_r\left[A(\gamma), B(\gamma)\right] = \Big(\ \mathcal{B}_r\left[B(\gamma)\right] \Big| \ \mathcal{A}_r\left[A(\gamma), B(\gamma)\right] \Big\| \ \Lambda_r\left[A(\gamma), B(\gamma)\right] \ \Big) \tag{2.3}$$

of dimensions $(n(r+1) \times n(r+2))$.

Assume that

$$\operatorname{rank}\Lambda_r\left[A(\gamma), B(\gamma)\right] = c = \text{const} \quad \forall\gamma\in\Gamma \tag{2.4}$$

for some r $(0 \leqslant r \leqslant n)$ and the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$ $\forall\gamma\in\Gamma$ has a nonsingular minor of order $n(r+1)$ containing c columns of the matrix $\Lambda_r\left[A(\gamma), B(\gamma)\right]$ and all columns of the matrix $\mathcal{A}_r\left[A(\gamma), B(\gamma)\right]$. It will be called *the resolving minor*.

We denote by $\mathcal{M}_r\left[A(\gamma), B(\gamma)\right]$ a square submatrix of the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$ that has order $n(r+1)$ and the resolving minor as its determinant.

Definition 1. *The unsolvability index* of the parametric family (1.1) is the smallest value r for which condition (2.4) holds and there exists the resolving minor in the matrix $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$.

According to Lemma 1 below, the definition of the unsolvability index implies the permanent internal structure of system (1.1) for all $\gamma\in\Gamma$.

Let us introduce the notation

$$\begin{pmatrix} A_1(\gamma) & A_2(\gamma) \end{pmatrix} = A(\gamma)Q, \quad \begin{pmatrix} B_1(\gamma) & B_2(\gamma) \end{pmatrix} = B(\gamma)Q, \tag{2.5}$$

where Q is a column permutation matrix such that all columns of the matrix

$$\mathcal{B}_{2,r}(\gamma) = \operatorname{col}\left(B_2(\gamma), O, \ldots, O\right) \tag{2.6}$$

belong to the resolving minor and those of $\operatorname{col}\left(B_1(\gamma), O, \ldots, O\right)$ do not. The blocks $B_2(\gamma)$ and $A_2(\gamma)$ from (2.5), (2.6) have dimensions $n\times d$, where $d = nr - c$. The construction of the matrix Q was described in [22]. Thus, d is the number of columns of the matrix $\mathcal{B}_r[B(\gamma)]$ that belong to the resolving minor of the matrix (2.3).

Lemma 1. *Assume that:*

1) $A(\gamma),\ B(\gamma)\in\mathbf{C}^1(\Gamma)$.

2) *Condition* (2.4) *holds.*

3) *The matrix* $\mathcal{D}_r\left[A(\gamma), B(\gamma)\right]$ *has the resolving minor.*

Then there exists an operator

$$\mathcal{R}_\gamma = R_0(\gamma) + R_1(\gamma)\frac{d}{dt} + \ldots + R_r(\gamma)\left(\frac{d}{dt}\right)^r, \tag{2.7}$$

where $R_j(\gamma)\in\mathbf{C}^1(\Gamma)$ $(j=\overline{0,r})$ *are matrices of dimensions* $(n\times n)$, *such that*

$$\mathcal{R}_\gamma\left[A(\gamma)Q\xi'(t) + B(\gamma)Q\xi(t)\right] = \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix}\xi'(t) + \begin{pmatrix} J_1(\gamma) & E_d \\ J_2(\gamma) & O \end{pmatrix}\xi(t) \tag{2.8}$$

for all $t\in T$ *and* $\gamma\in\Gamma$ *and any* n*-dimensional vector function* $\xi(t)\in\mathbf{C}^{r+1}(T)$. *Here,* E_d *stands for an identity matrix of order* d; Q *is the permutation matrix given by* (2.5); *finally,* $J_1(\gamma),\ J_2(\gamma)\in\mathbf{C}^1(\Gamma)$ *are some matrices of compatible dimensions.*

In addition,

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) \end{pmatrix} = \begin{pmatrix} E_n & O & \ldots & O \end{pmatrix}\mathcal{M}_r^{-1}\left[A(\gamma), B(\gamma)\right]. \tag{2.9}$$

Lemma 2. *Consider the DAEs* (1.1) *and assume that:*

1) *All the hypotheses of Lemma 1 are satisfied.*

2) *The matrix* $\mathcal{D}_{r+1}\left[A(\gamma), B(\gamma)\right]$ *has an invertible submatrix* $\mathcal{M}_{r+1}\left[A(\gamma), B(\gamma)\right]$ *of order* $n(r+2)$ *for all* $\gamma\in\Gamma$ *that includes the matrix* $\mathcal{M}_r\left[A(\gamma), B(\gamma)\right]$ *and also* n *columns of the matrix* $\Lambda_{r+1}\left[A(\gamma), B(\gamma)\right]$.

Then the operator $\mathcal{R}_\gamma$ *possesses the left inverse operator* $\mathcal{L}_\gamma = L_0(\gamma) + L_1(\gamma)\frac{d}{dt}$, *where* $L_0(\gamma),\ L_1(\gamma)\in\mathbf{C}^1(\Gamma)$ *are matrices of dimensions* $(n\times n)$.

The proofs of these lemmas are provided in the Appendix.

2.2. Robust Stability Conditions

Due to Lemma 1, the operator $\mathcal{R}_\gamma$ reduces the family (1.1) to the form

$$\tilde{A}(\gamma)\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \tilde{B}(\gamma)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{2.10}$$

where

$$Q\operatorname{col}(x_1(t), x_2(t)) = x(t), \tag{2.11}$$

Q is the row permutation matrix (see (2.5) and (2.6)), and

$$\tilde{A}(\gamma) = \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} = (R_0(\gamma)A(\gamma) + R_1(\gamma)B(\gamma))\,Q, \tag{2.12}$$

$$\tilde{B}(\gamma) = \begin{pmatrix} J_1(\gamma) & E_d \\ J_2(\gamma) & O \end{pmatrix} = R_0(\gamma)B(\gamma)Q. \tag{2.13}$$

The component $x_1(t)$ of the solution of the DAEs (2.10) satisfies the equation

$$x_1'(t) + J_2(\gamma)x_1(t) = 0. \tag{2.14}$$

In turn, $x_2(t) = -J_1(\gamma)x_1(t)$. The matrix $J_1(\gamma)$ is constant for each fixed value $\gamma \in \Gamma$. Therefore, the family (2.10), (2.12), (2.13) is asymptotically stable if and only if the same property holds for system (2.14).

Under the hypotheses of Lemma 2, the operator $\mathcal{R}_\gamma$ has the left inverse operator. Moreover, the solutions of systems (1.1) and (2.10), (2.12), (2.13) are related through the row permutation matrix Q (see (2.11)). Based on these facts, we conclude that the DAE family (1.1) is asymptotically stable if and only if system (2.14) is asymptotically stable.

Let the entire family (2.14) have a general quadratic Lyapunov function

$$W(x_1) = x_1^\top V x_1 \tag{2.15}$$

with a positive definite time derivative along the trajectories of system (2.14). Here, the matrix V of dimensions $(n-d)\times(n-d)$ is symmetric and positive definite. As is known [2, p. 198; 21, p. 210], the family (2.14) is asymptotically stable if there exists a solution of the system of linear matrix inequalities (LMIs)

$$J_2(\gamma)^\top V + VJ_2(\gamma) > 0, \quad \gamma \in \Gamma. \tag{2.16}$$

In this case, the DAE family (1.1) is asymptotically stable as well. Thus, we arrive at the following result.

Theorem 1. *Under the hypotheses of Lemma 2, assume the existence of a symmetric and positive definite constant matrix V satisfying the LMIs* (2.16).

Then the parametric DAE family (1.1) *is asymptotically stable.*

We formulate another useful fact based on a known result from perturbation theory [2, p. 198].

Theorem 2. *Under the hypotheses Lemma 2, assume that all eigenvalues $\lambda_i(0)$ of the matrix $J_2(0)$ in system* (2.10)–(2.13) *are different and let α_i and β_i be the corresponding right and left eigenvectors:*

$$J_2(0)\alpha_i = \lambda_i(0)\alpha_i, \quad \beta_i^* J_2(0) = \lambda_i(0)\beta_i^*,$$
$$\|\alpha_i\|_{\mathbf{R}^{n-d}} = \|\beta_i\|_{\mathbf{R}^{n-d}} = 1, \quad i = \overline{1, n-d}.$$

Then the eigenvalues of the matrix $J_2(\gamma)$ can be written as

$$\lambda_i(\gamma) = \lambda_i(0) + \sum_{j=1}^{l} \frac{\beta_i^* \Theta_j \alpha_i}{\beta_i^* \alpha_i} \gamma_j + o(\gamma),$$

where

$$\Theta_j = \frac{\partial J_2(\gamma)}{\partial \gamma_j}\Big|_{\gamma=0}.$$

Example. Consider the DAEs

$$\begin{pmatrix} 1 & -\gamma_2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -\varphi(\gamma_3) & 0 & 0 & \varphi(\gamma_3) \\ 0 & 0 & \psi(\gamma_3) & 0 \end{pmatrix} x'(t) + \begin{pmatrix} 2 & 0 & 0 & \gamma_1 - 1 \\ 0 & 2-\gamma_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} x(t) = 0, \tag{2.17}$$

where

$$\Gamma = \left\{ \operatorname{col}(\gamma_1, \gamma_2, \gamma_3) : \ \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2} \leqslant 1/2 \right\}.$$

The functions $\varphi(\gamma_3)$ and $\psi(\gamma_3)$ are infinitely differentiable with respect to the parameter $\gamma_3 \in [-1/2,\ 1/2]$ and given by the rule $\psi(\gamma_3) = 0$ if $\varphi(\gamma_3) \neq 0$. Since $\phi(\gamma_3)$ and $\psi(\gamma_3)$ vanish either simultaneously or in turn, the matrix at the derivative has variable rank in the domain Γ.

The matrix $\mathcal{D}_2\,[A(\gamma),\ B(\gamma)]$ has the resolving minor indicated by the dashed line:

$$\left(\begin{array}{cc|cc|cccc||cccc|cc|cc}
2 & 0 & 0 & f_1 & 1 & -\gamma_2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & f_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -\varphi & 0 & 0 & \varphi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & \psi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 2 & 0 & 0 & f_1 & 1 & -\gamma_2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & f_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\varphi & 0 & 0 & \varphi & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & \psi & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & f_1 & 1 & -\gamma_2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f_2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -\varphi & 0 & 0 & \varphi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & \psi & 0
\end{array}\right).$$

Here, the dependence of φ and ψ on the parameter γ_3 is omitted, $f_1 = \gamma_1 - 1$, and $f_2 = 2 - \gamma_2$. The matrix $\Lambda_2[A(\gamma), B(\gamma)]$ is located to the right of the double vertical line and has rank 6.

We construct the matrix $\mathcal{D}_3\,[A(\gamma),\ B(\gamma)]$ by supplementing $\mathcal{D}_2$ with four zero columns on the right and four rows at the bottom:

$$\left(\begin{array}{ccc|ccc||ccc|cccc|cccc}
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 2 & 0 & 0 & f_1 & \boxed{1} & -\gamma_2 & 1 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & f_2 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 0 & \boxed{1} & 0 & -\varphi & 0 & 0 & \varphi \\
0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & -1 & 0 & 0 & \boxed{1} & 0 & 0 & \psi & 0
\end{array}\right).$$

There is the submatrix $\mathcal{M}_3\,[A(\gamma),\ B(\gamma)]$ in $\mathcal{D}_3$: it includes all columns corresponding to the resolving minor and four more columns with the framed units. Thus, all the hypotheses of Lemma 2 are satisfied, $d = 2$, and $Q = E_4$.

The operator

$$\mathcal{R}_\gamma = R_0(\gamma) + R_1(\gamma)\frac{d}{dt} + R_2(\gamma)\left(\frac{d}{dt}\right)^2,$$

where

$$R_0(\gamma) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & \gamma_2 & 0 & 1-\gamma_1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad R_1(\gamma) = \begin{pmatrix} 0 & 0 & 0 & -\varphi(\gamma_3) \\ 0 & 0 & -\psi(\gamma_3) & 0 \\ 0 & 0 & (\gamma_1-1)\psi(\gamma_3)-1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_2(\gamma) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \varphi(\gamma_3) \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

transforms system (2.17) into

$$\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) x'(t) + \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ \hline 1+\gamma_1 & \gamma_2\,(2-\gamma_2) & 0 & 0 \\ 0 & 2-\gamma_2 & 0 & 0 \end{array}\right) x(t) = 0.$$

In addition, $J_2(\gamma) = \begin{pmatrix} 1+\gamma_1 & \gamma_2\,(2-\gamma_2) \\ 0 & 2-\gamma_2 \end{pmatrix}$.

Let $V = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ in the matrix inequality (2.16). Then

$$J_2(\gamma)^\top V + V J_2(\gamma) = \begin{pmatrix} 4\,(1+\gamma_1) & 2\gamma_2\,(2-\gamma_2) \\ 2\gamma_2\,(2-\gamma_2) & 4\,(2-\gamma_2) \end{pmatrix}.$$

This matrix is positive definite due to the positive definiteness of all its principal minors for any γ_1 and γ_2 from the segment $[-1/2, 1/2]$. In particular, the determinant of this matrix is $4(2-\gamma_2)\{4(1+\gamma_1) - \gamma_2^2(2-\gamma_2)\}$. Obviously, $2-\gamma_2 > 0 \ \ \forall \gamma_2 \in [-1/2, 1/2]$. The expression in curly brackets achieves a minimum of $11/8 > 0$ for $\gamma_1 = -1/2$ and $\gamma_2 = -1/2$.

By Theorem 1, this means that the family (2.17) is asymptotically stable.

3. ROBUST STABILITY CONDITIONS FOR DAES WITH A SCALAR PARAMETER

Consider the DAEs

$$A(\gamma_0)x'(t) + B(\gamma_0)x(t) = 0, \quad t \in T, \tag{3.1}$$

where $\gamma_0 \in G_0$ denotes a scalar parameter and $G_0 = \{\gamma_0 \in \mathbf{R} : \ |\gamma_0| \leqslant a\}$ is a given admissible set.

Theorem 3. *Assume that:*

1) $A(\gamma_0), B(\gamma_0) \in \mathbf{C}^A(G_0)$.

2) $\operatorname{rank} \Lambda_r[A(\gamma_0), B(\gamma_0)] = \text{const} \ \ \forall \gamma_0 \in G_0$.

3) *The matrix* $\mathcal{D}_r[A(\gamma_0), B(\gamma_0)]$ *has a resolving minor.*

Then there exist matrices $P(\gamma_0)$, $S(\gamma_0) \in \mathbf{C}^A(G_0)$ *invertible for all* $\gamma_0 \in G_0$ *such that, with multiplication on the left by* $P(\gamma_0)$ *and the change of variable*

$$x(t) = S(\gamma_0)\operatorname{col}(z_1(t), z_2(t)),$$

system (3.1) *is transformed into*

$$\begin{pmatrix} E_{n-d} & O \\ O & N(\gamma_0) \end{pmatrix} \begin{pmatrix} z_1'(t) \\ z_2'(t) \end{pmatrix} + \begin{pmatrix} J(\gamma_0) & O \\ O & E_d \end{pmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = 0, \quad t \in T, \tag{3.2}$$

where $N(\gamma_0)$ *is an upper triangular matrix with* r *square zero blocks on the principal diagonal,* $N^r(\gamma_0) \equiv O$, *and* $J(\gamma_0)$ *is some matrix of dimensions* $(n-d)\times(n-d)$.

The proof of this theorem is omitted. It involves the step-by-step zeroing of linearly dependent rows in the matrix at $x'(t)$ and the differentiation operator applied to the corresponding rows of the matrix at $x(t)$.

The structural form (3.2) is an analog of the strong standard canonical form introduced in [25] for the system $A(t)x'(t) + B(t)x(t) = f(t)$ under the analytical solvability assumption.

Under the hypotheses of Theorem 3, the solutions of systems (3.1) and (3.2) are related by the identity

$$x(t) = S(\gamma_0)\mathrm{col}\,(z_1(t), z_2(t)) \tag{3.3}$$

with the matrix $S(\gamma_0)$ invertible for all $\gamma_0 \in G_0$.

Consider the DAEs (3.2). Due to the structure of the matrix $N(\gamma_0)$, the subsystem

$$N(\gamma_0)z_2'(t) + z_2(t) = 0$$

has only the trivial solution $z_2(t) \equiv 0$ for all $\gamma_0 \in G_0$.

In turn, the component $z_1(t)$ satisfies the subsystem

$$z_1'(t) + J(\gamma_0)z_1(t) = 0. \tag{3.4}$$

Therefore, the DAEs (3.2) are asymptotically stable if and only if the same property holds for system (3.4).

For each fixed value γ_0, the matrix $S(\gamma_0)$ in (3.3) is constant. Hence, the DAE family (3.1) possesses asymptotic stability if and only if system (3.4) does so.

Let

$$W_0(z_1) = z_1^\top V_0 z_1$$

be a general Lyapunov function for system (3.4) with a symmetric and positive definite matrix V_0 of dimensions $(n-d) \times (n-d)$ that satisfies the matrix inequality

$$J^\top(\gamma_0)V_0 + V_0 J(\gamma_0) > 0, \quad \gamma_0 \in G_0. \tag{3.5}$$

In this case, the family (3.4) is asymptotically stable, and hence the parametric family (3.1) possesses the same property.

Thus, the following result holds.

Theorem 4. *Under the hypotheses of Theorem 3, the DAE family* (3.1) *is asymptotically stable if there exists a constant, symmetric, and positive definite matrix* V_0 *that satisfies the matrix inequality* (3.5) *for all* $\gamma_0 \in G_0$.

4. CONDITIONS FOR PRESERVING THE TYPE OF FUNCTIONAL DEPENDENCE ON UNCERTAIN PARAMETERS

In the previous sections, we have considered the reduction of parametric families of DAEs to some structural forms. As it has been demonstrated above, under certain conditions, the robust stability of the original family is equivalent to the robust stability of its differential subsystem. Unfortunately, in the process of such transformations, the original type of the functional dependence of the system coefficients on the uncertain parameters generally becomes more complicated.

Consider the family of DAEs with an affine uncertainty:

$$\left(A_0 + \sum_{j=1}^{l} \gamma_j A_j\right) x'(t) + \left(B_0 + \sum_{j=1}^{l} \gamma_j B_j\right) x(t) = 0, \quad t \in T, \tag{4.1}$$

where A_j and B_j $(j = \overline{0,l})$ are given real matrices of dimensions $(n \times n)$; $\det A_0 = 0$ and $\gamma = \operatorname{col}(\gamma_1, \ldots, \gamma_l) \in \Gamma_a = \{\gamma \in \mathbf{R}^l : |\gamma_j| \leqslant a, \ j = \overline{1,l}\}$.

This section presents sufficient conditions on the structure of the coefficients A_j and B_j $(j = \overline{1,l})$ under which the robust stability of the family (4.1) is equivalent to the robust stability of a system of ordinary differential equations resolvable with respect to the derivative, with affine uncertainty.

4.1. Conditions Based on the Canonical Kronecker–Weierstrass Form

Definition 2. A matrix pencil $\mu A_0 + B_0$ is said to be *regular* if there exists $\mu \in \mathbf{R}$ such that $\det(\mu A_0 + B_0) \neq 0$.

Lemma 3 [26, p. 313]. *Assume that a matrix pencil $\mu A_0 + B_0$ is regular. Then there exist invertible matrices P and S of dimensions $(n \times n)$ such that*

$$PA_0S = \begin{pmatrix} E_{n-d} & O \\ O & N \end{pmatrix}, \quad PB_0S = \begin{pmatrix} J_0 & O \\ O & E_d \end{pmatrix}, \tag{4.2}$$

where J_0 is some square matrix of order $n-d$ and N is an upper triangular matrix with r square zero blocks on the principal diagonal such that $N^r = O$.

The system $PA_0Sz'(t) + PB_0Sz(t) = 0$, $z(t) = S^{-1}x(t)$, with property (4.2) is called *the canonical Kronecker–Weierstrass form* for the DAEs $A_0x'(t) + B_0x(t) = 0$.

Let the matrix pencil $\mu A_0 + B_0$ in (4.1) be regular. Then, by Lemma 4, there exist invertible matrices P and S with property (4.2). Assume that

$$PA_jS = \begin{pmatrix} O & O \\ A_{j,1} & O \end{pmatrix}, \quad PB_jS = \begin{pmatrix} B_{j,1} & O \\ B_{j,2} & O \end{pmatrix}, \quad j = \overline{1,l}, \tag{4.3}$$

where $A_{j,1}$, $B_{j,1}$, and $B_{j,2}$ are some matrices with possibly nonzero elements. Note that $A_{j,1}$ and $B_{j,2}$ have dimensions $d \times (n-d)$, whereas the square block $B_{j,1}$ is of order $(n-d)$.

We multiply (4.1) on the left by the matrix P and change the variable: $x(t) = S\operatorname{col}(x_1(t), x_2(t))$. In view of (4.2), (4.3), the resulting system has the form

$$\begin{pmatrix} E_{n-d} & O \\ \sum\limits_{j=1}^{l} \gamma_j A_{j,1} & N \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_0 + \sum\limits_{j=1}^{l} \gamma_j B_{j,1} & O \\ \sum\limits_{j=1}^{l} \gamma_j B_{j,2} & E_d \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0. \tag{4.4}$$

Denoting

$$Y_1(\gamma) = J_0 + \sum_{j=1}^{l} \gamma_j B_{j,1}, \tag{4.5}$$

$$Y_2(\gamma) = \sum_{j=1}^{l} \gamma_j \left(B_{j,2} - A_{j,1}Y_1(\gamma)\right), \tag{4.6}$$

from the first equation of system (4.4) we find

$$x_1'(t) = -Y_1(\gamma)x_1(t) \tag{4.7}$$

and consequently,

$$\left(\frac{d}{dt}\right)^i x_1(t) = (-1)^i Y_1^i(\gamma)x_1(t), \quad i = \overline{2,l}. \tag{4.8}$$

Considering (4.5)–(4.7), the second equation of system (4.4) can be written as

$$Nx_2'(t) + x_2(t) + Y_2(\gamma)x_1(t) = 0. \tag{4.9}$$

Applying the operator

$$E_d + \sum_{k=1}^{r-1}(-1)^k N^k \left(\frac{d}{dt}\right)^k \tag{4.10}$$

to (4.9) and using formulas (4.7) and (4.8), we obtain

$$x_2(t) = \left(Y_2(\gamma) + \sum_{k=1}^{r-1} N^k Y_2(\gamma)Y_1^k(\gamma)\right) x_1(t). \tag{4.11}$$

Obviously, the inverse of the operator (4.10) has the form $E_d + N\frac{d}{dt}$.

Following the same considerations as in the previous sections, we can demonstrate that the DAE family (4.1) is asymptotically stable for all $\gamma \in \Gamma_a$ if and only if the same property holds for the family (4.7) or, equivalently, for the DAEs

$$x_1'(t) + \left(J_0 + \sum_{j=1}^{l} \gamma_j B_{j,1}\right) x_1(t) = 0. \tag{4.12}$$

In system (4.12), the affine structure of uncertainty is preserved, being violated in equation (4.11).

Remark. If the dependence on the parameters in the original family were not affine (e.g., multilinear or polynomial), conditions (4.3) would ensure the same type of uncertainty in an equation analogous to (4.12).

The family (4.12) is asymptotically stable for all values $\gamma \in \Gamma_a$ if there exists a general Lyapunov function (2.15) with a positive definite time derivative along the trajectories of system (4.12).

Theorem 5. *Consider system* (4.1) *under the assumptions that the matrix pencil* $\mu A_0 + B_0$ *is regular and equalities* (4.3) *hold. If there exists a symmetric and positive definite matrix* V *of dimensions* $(n-d) \times (n-d)$ *that satisfies the inequality*

$$Y_1^\top(\gamma)V + VY_1(\gamma) > 0 \tag{4.13}$$

for all $\gamma \in \Gamma_a$, *then the family* (4.1) *is asymptotically stable for all* $\gamma \in \Gamma_a$. *In this case, the matrix* $Y_1(\gamma)$ *is calculated by formula* (4.5).

Since the affine structure of the dependence on the uncertain parameters is preserved in (4.12), it suffices to solve inequalities (4.13) only at a finite number of points for which $|\gamma_j| = a$ $(j = \overline{1,l})$ [2, p. 199].

For the same reason, Theorem 2 can be used to estimate the stability radius of system (4.12).

Suppose that all eigenvalues λ_k of the matrix J_0 have positive real parts: $\mathrm{Re}\,(\lambda_k) > 0$, $k = \overline{1, n-d}$. We denote by λ the eigenvalue of this matrix with the least real part. Let α and β be the corresponding right and left eigenvectors. By Theorem 2, λ turns into the eigenvalue $\lambda(\gamma)$ of the matrix $J_0 + \sum\limits_{j=1}^{l} \gamma_j B_{j,1}$:

$$\lambda(\gamma) \approx \lambda + \sum_{j=1}^{l} \frac{\beta^* B_{j,1}\alpha}{\beta^*\alpha}\gamma_j$$

for small γ. Therefore, for

$$\bar{a} = \mathrm{Re}\,\lambda / \sum_{j=1}^{l} \left| \mathrm{Re}\,\frac{\beta^* B_{j,1}\alpha}{\beta^*\alpha} \right|$$

at least one eigenvalue of the matrix (4.5) has zero real part. In other words, the value $\bar{a}$ provides an estimate for the stability radius of system (4.12) [2, p. 198].

4.2. Conditions Based on the Differential Operator

Note that for a regular pencil $\mu A_0 + B_0$, the construction of matrices P and S with the property (4.2) is generally a nontrivial problem. Therefore, we will obtain other, more constructive, conditions for preserving the type of functional dependence on the parameters for the family (4.1). In this case, some constraints will be imposed on the coefficients of the system under consideration, different from those adopted in Section 4.1.

Assume that the matrices A_0 and B_0 in system (4.1) satisfy the following conditions:

A1) There is a resolving minor in the matrix $\mathcal{D}_r[A_0, B_0]$.

A2) $\mathrm{rank}\,\Lambda_{r+1}[A_0, B_0] = \mathrm{rank}\,\Lambda_r[A_0, B_0] + n$.

Then, by Lemma 2, there exist an operator

$$\mathcal{R} = R_0 + R_1\frac{d}{dt} + \ldots + R_r\left(\frac{d}{dt}\right)^r$$

such that

$$\mathcal{R}\left[A_0 x'(t) + B_0 x(t)\right] = \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix}\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 & E_d \\ J_2 & O \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

where $Q\mathrm{col}\,(x_1(t), x_2(t)) = x(t)$ and Q is the corresponding row permutation matrix (see (2.5) and (2.6)). In addition, $\mathcal{R}$ has the left inverse operator $\mathcal{L} = L_0 + L_1\frac{d}{dt}$ and

$$(\, R_0\; R_1\; \ldots\; R_r \,) = (\, E_n\; O\; \ldots\; O \,)\, M_r^{-1}[A_0, B_0],$$

where the determinant of the matrix $M_r[A_0, B_0]$ is the resolving minor.

Applying the operator $\mathcal{R}$ to (4.1) yields the system

$$\begin{gathered} \begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix}\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 & E_d \\ J_2 & O \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \\ + \sum_{i=0}^{r+1}\left[\sum_{j=1}^{l}\gamma_j R_{i,j}\right]\left(\frac{d}{dt}\right)^i\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \end{gathered} \tag{4.14}$$

where

$$\begin{gathered} R_{0,j} = R_0 B_j Q; \quad R_{i,j} = (R_{i-1}A_j + R_i B_j)\,Q, \quad i = \overline{1, r}; \\ R_{r+1,j} = R_r A_j Q, \quad j = \overline{1, l}. \end{gathered} \tag{4.15}$$

Suppose that the matrix coefficients (4.15) have the structure

$$R_{0,j} = \begin{pmatrix} B_{j,1} & O \\ B_{j,2} & O \end{pmatrix}, \quad R_{i,j} = \begin{pmatrix} A_j^{[i]} & O \\ O & O \end{pmatrix}, \quad j = \overline{1,l},\ i = \overline{1,r+1}, \tag{4.16}$$

where $B_{j,1}$, $B_{j,2}$, and $A_j^{[i]}$ are some matrices: $B_{j,1}$ and $A_j^{[i]}$ have dimensions $d \times (n-d)$, and the square block $B_{j,2}$ is of order $(n-d)$.

Due to (4.16), system (4.14) takes the form

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 + \sum\limits_{j=1}^{l} \gamma_j B_{j,1} & E_d \\ J_2 + \sum\limits_{j=1}^{l} \gamma_j B_{j,2} & O \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$+ \begin{pmatrix} \sum\limits_{i=1}^{r+1} \left[\sum\limits_{j=1}^{l} \gamma_j A_j^{[i]} \right] \left(\dfrac{d}{dt} \right)^i x_1(t) & O \\ O & O \end{pmatrix} = 0.$$

Following the same considerations as in the previous case, under Assumptions A1 and A2, we can demonstrate that system (4.1) is asymptotically stable for each $\gamma \in \Gamma_a$ if and only if the same property holds for the subsystem

$$x_1'(t) + \left(J_2 + \sum_{j=1}^{l} \gamma_j B_{j,2} \right) x_1(t) = 0.$$

Thus, the following analog of Theorem 5 is true.

Theorem 6. *Consider system* (4.1) *under Assumptions A*1 *and A*2 *and equalities* (4.16)*. If there exists a symmetric and positive definite matrix V of dimensions $(n-d) \times (n-d)$ that satisfies the inequality*

$$\left(J_2 + \sum_{j=1}^{l} \gamma_j B_{j,2} \right)^{\top} V + V \left(J_2 + \sum_{j=1}^{l} \gamma_j B_{j,2} \right) > 0$$

for all $\gamma \in \Gamma_a$, then the family (4.1) *is asymptotically stable for all $\gamma \in \Gamma_a$.*

5. CONCLUSIONS

In this paper, the robust stability of differential-algebraic equations (DAEs) with norm-bounded vector and scalar uncertain parameters has been analyzed by proposing algorithms for building structural forms with separated differential and algebraic subsystems (Lemma 1 and Theorem 3, respectively).

The approach based on a linear differential operator has a constructive nature: the coefficients of this operator are determined by inverting the matrix whose determinant is the resolving minor. Moreover, the system obtained by such a transformation is equivalent to the original system in the sense of solutions (Lemma 2).

When investigating the stability of DAEs, the main difficulty is that even in the simplest cases, the internal structure of the system (and, consequently, the form of the general solution) may change under an arbitrarily small perturbation of the coefficients. As a result, the structure and properties of the unperturbed system may lose any significance for analysis.

A distinctive feature of the results presented above is that the stability analysis involves no information about the internal structure of the nominal system. The structural forms are built for the entire family (Subsections 2.1 and 3.1). For this reason, there is no need to introduce additional structural constraints on the perturbations that would ensure the coinciding internal structures of the nominal and perturbed DAEs.

It has been demonstrated that the stability of the parametric family (1.1) is equivalent to the robust stability of its differential subsystem, which also depends on the uncertain parameters. The sufficient condition for robust stability follows from the existence of a general quadratic Lyapunov function (2.15) for the differential subsystem (Theorems 1 and 4).

On the other hand, reducing the system to one or another structural form may appreciably complicate the type of functional dependence on the uncertain parameters of this subsystem compared to system (1.1). In this regard, for DAEs with affine uncertainty, sufficient conditions have been established under which the differential subsystem is also an affine family (Section 4). This approach is also applicable to systems with other types of parametric uncertainty.

APPENDIX

Proof of Lemma 1. Let Q_Λ be a column permutation matrix such that

$$\Lambda_r\left[A(\gamma),B(\gamma)\right]Q_\Lambda=\left(\Lambda_{r,1}(\gamma)\ \Lambda_{r,2}(\gamma)\right),$$

where the block $\Lambda_{r,1}(\gamma)$ belongs to the resolving minor and consists of c columns, $\operatorname{rank}\Lambda_{r,1}(\gamma)=c$ $\forall\gamma\in\Gamma$.

Consider the matrix $\mathcal{D}_r\left[A(\gamma),B(\gamma)\right]$. Multiplying it on the right by the column permutation matrix $Q_r=\operatorname{diag}\{Q,Q,Q_\Lambda\}$ [1] and on the left by $\mathcal{M}_r^{-1}\left[A(\gamma),B(\gamma)\right]$ yields

$$\mathcal{M}_r^{-1}\left[A(\gamma),B(\gamma)\right]\mathcal{D}_r\left[A(\gamma),B(\gamma)\right]Q_r=\left(\begin{array}{cc|cc||cc} J_1(\gamma) & E_d & O & O & O & \Phi_1(\gamma)\\ J_2(\gamma) & O & E_{n-d} & O & O & \Phi_2(\gamma)\\ \hline J_3(\gamma) & O & O & E_d & O & \Phi_3(\gamma)\\ J_4(\gamma) & O & O & O & E_c & \Phi_4(\gamma)\end{array}\right),$$

where $J_i(\gamma)$ and $\Phi_i(\gamma)$ $(i=\{1,2,3,4\})$ are some matrices of compatible dimensions. Due to (2.4),

$$\Phi_1(\gamma)\equiv O,\quad \Phi_2(\gamma)\equiv O,\quad \Phi_3(\gamma)\equiv O,\quad \gamma\in\Gamma.$$

In this case,

$$\begin{aligned}&\left(E_n\ \ O\ \ \ldots\ \ O\right)\mathcal{M}_r^{-1}\left[A(\gamma),B(\gamma)\right]\mathcal{D}_r\left[A(\gamma),B(\gamma)\right]Q_r\operatorname{col}\left(\xi(t),\ldots,\xi^{(r+1)}(t)\right)\\ &=\left(\begin{array}{cc|cc||ccc} J_1(\gamma) & E_d & O & O & O & \ldots & O\\ J_2(\gamma) & O & E_{n-d} & O & O & \ldots & O\end{array}\right)\operatorname{col}\left(\xi(t),\ldots,\xi^{(r+1)}(t)\right),\end{aligned}\tag{A.1}$$

which obviously implies formulas (2.9) and (2.8).

Proof of Lemma 2. Note that identity (A.1) remains in force for

$$Q_r=\operatorname{diag}\{Q,\ldots,Q\}.$$

[1] The matrix Q_r is quasi-diagonal: the blocks listed in curly brackets stand on the principal diagonal and all other elements are zero.

Let $\xi(t)\in\mathbf{C}^{r+2}(T)$ be an arbitrary n-dimensional vector function. In view of the representation (2.9), differentiating (A.1) with respect to the variable t gives

$$\begin{aligned}&(\,O\;\; R_0(\gamma)\;\; R_1(\gamma)\;\;\ldots\;\; R_r(\gamma)\,)\,\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]Q_{r+1}\mathrm{col}\left(\xi(t),\ldots,\xi^{(r+2)}(t)\right)\\&=\left(\begin{array}{cc|cc|cc|ccc} O & O & J_1(\gamma) & E_d & O & O & O & \ldots & O\\ O & O & J_2(\gamma) & O & E_{n-d} & O & O & \ldots & O\end{array}\right)\mathrm{col}\left(\xi(t),\ldots,\xi^{(r+2)}(t)\right).\end{aligned}\tag{A.2}$$

From (A.1) and (A.2) it follows that

$$\begin{aligned}&\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) & O\\ O & R_0(\gamma) & \ldots & R_{r-1}(\gamma) & R_r(\gamma)\end{pmatrix}\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]Q_{r+1}\\&=\left(\begin{array}{cc|cc|cc} J_1(\gamma) & E_d & O & O & O & O\\ J_2(\gamma) & O & E_{n-d} & O & O & O\\ \hline O & O & J_1(\gamma) & E_d & O & O\\ O & O & J_2(\gamma) & O & E_{n-d} & O\end{array}\right).\end{aligned}\tag{A.3}$$

According to assumption 2 of Lemma 2, the matrix

$$\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]=\Big(\;\mathcal{B}_{r+1}\left[B(\gamma)\right]\,\Big|\,\mathcal{A}_{r+1}\left[A(\gamma),B(\gamma)\right]\,\Big\|\,\Lambda_{r+1}\left[A(\gamma),B(\gamma)\right]\;\Big)$$

has an invertible submatrix $\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]$ of order $n(r+2)$ on Γ. In addition, d columns of the matrix $\mathcal{B}_{r+1}\left[B(\gamma)\right]$ are included in $\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]$, namely, those of the matrix

$$\mathcal{B}_{2,r+1}(\gamma)=\mathrm{col}\left(B_2(\gamma),O,\ldots,O\right)$$

(see (2.5) and (2.6)); $(c+n)$ columns of the matrix $\Lambda_{r+1}\left[A(\gamma),B(\gamma)\right]$ are included in $\mathcal{M}_{r+1}$.

Consider a matrix that eliminates in $\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]Q_{r+1}$ the columns not included in $\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]$. Multiplying both sides of equality (A.3) on the right by this matrix yields

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) & O\\ O & R_0(\gamma) & \ldots & R_{r-1}(\gamma) & R_r(\gamma)\end{pmatrix}\mathcal{M}_{r+1}\left[A(\gamma),B(\gamma)\right]=(\,\mathcal{E}(\gamma)\;\;O\,),\tag{A.4}$$

where

$$\mathcal{E}(\gamma)=\left(\begin{array}{c|cc||c} E_d & O & O & O\\ O & E_{n-d} & O & O\\ \hline O & J_1(\gamma) & E_d & O\\ O & J_2(\gamma) & O & E_{n-d}\end{array}\right).\tag{A.5}$$

In addition, the block column

$$\mathrm{col}\left(O,O,O,E_{n-d}\right)\tag{A.6}$$

of the matrix on the right-hand side of identity (A.3) completely enters the matrix $\mathcal{E}(\gamma)$. Indeed, $\mathcal{D}_{r+1}\left[A(\gamma),B(\gamma)\right]$ and the matrix on the right-hand side of (A.3) have full row rank for all $\gamma\in\Gamma$; therefore, due to (A.3), the same property holds for the matrix

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \ldots & R_r(\gamma) & O\\ O & R_0(\gamma) & \ldots & R_{r-1}(\gamma) & R_r(\gamma)\end{pmatrix}.\tag{A.7}$$

Assume on the contrary that the transformation of (A.3) into (A.4) eliminates some columns of the matrix (A.6). In this case, the matrix $\mathcal{E}(\gamma)$ will no longer be invertible, which contradicts the full row rank of the matrix (A.7).

From (A.4) it follows that

$$\begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \dots & R_r(\gamma) & O \\ O & R_0(\gamma) & \dots & R_{r-1}(\gamma) & R_r(\gamma) \end{pmatrix} = \left(\, \mathcal{E}(\gamma) \;\; O \,\right) \mathcal{M}_{r+1}^{-1} \left[A(\gamma), B(\gamma)\right]. \tag{A.8}$$

The desired result of this lemma will be established by demonstrating that the system

$$\left(\, L_0(\gamma) \; L_1(\gamma) \,\right) \begin{pmatrix} R_0(\gamma) & R_1(\gamma) & \dots & R_r(\gamma) & O \\ O & R_0(\gamma) & \dots & R_{r-1}(\gamma) & R_r(\gamma) \end{pmatrix} = \left(\, E_n \; O \; \dots O \,\right) \tag{A.9}$$

has a solution $(\, L_0(\gamma) \; L_1(\gamma) \,) \in \mathbf{C}^1(\Gamma)$.

Considering the representation (A.8), a necessary and sufficient condition for the point-wise solvability of equation (A.9) can be written as

$$\operatorname{rank} \mathcal{E}(\gamma) = \operatorname{rank} \left(\frac{(\, \mathcal{E}(\gamma) \; O \,)}{(\, E_n \; O \; \dots O \,)\, \mathcal{M}_{r+1} \left[A(\gamma), B(\gamma)\right]} \right). \tag{A.10}$$

Obviously, see (A.5), $\operatorname{rank} \mathcal{E}(\gamma) = 2n$. By construction,

$$\left(\, E_n \; O \; \dots O \,\right) \mathcal{M}_{r+1} \left[A(\gamma), B(\gamma)\right] = \left(\; B_2(\gamma) \;\middle|\; A_1(\gamma) \quad A_2(\gamma) \;\middle\|\; O \;\right). \tag{A.11}$$

In view of (A.5) and (A.11), it is straightforward to verify (A.10) for all $\gamma \in \Gamma$.

The solution $L_0(\gamma),\ L_1(\gamma) \in \mathbf{C}^1(\Gamma)$ of system (A.9) is given by the formula

$$\left(\, L_0(\gamma) \; L_1(\gamma) \,\right) = \left(\, E_n \; O \; \dots O \,\right) \mathcal{M}_{r+1} \left[A(\gamma), B(\gamma)\right] \begin{pmatrix} \mathcal{E}^{-1}(\gamma) \\ O \end{pmatrix}.$$

The proof of Lemma 2 is complete.

REFERENCES

1. Brenan, K.E., Campbell, S.L., and Petzold, L.R., *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, Classics in Applied Mathematics, vol. 14, Philadelphia: SIAM, 1996.

2. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.

3. Chistyakov, V.F. and Shcheglova, A.A., *Izbrannye glavy teorii algebro-differentsial'nykh sistem* (Selected Chapters of the Theory of Algebraic-Differential Systems), Novosibirsk: Nauka, 2003.

4. Byers, R. and Nichols, N.K., On the Stability Radius of a Generalized State-Space System, *Linear Algebra Appl.*, 1993, vol. 188–189, pp. 113–134.

5. Qiu, L. and Davison, E.J., The Stability Robustness of Generalized Eigenvalues, *IEEE Trans. Automat. Control*, 1992, vol. 37, pp. 886–891.

6. Chyan, C.J., Du, N.H., and Linh, V.H., On Data-Dependence of Exponential Stability and the Stability Radii for Linear Time-Varying Differential-Algebraic Systems, *J. Differ. Equat.*, 2008, vol. 245, pp. 2078–2102.

7. Du, N.H. and Linh, V.H., Stability Radii for Linear Time-Varying Differential-Algebraic Equations with Respect to Dynamics Perturbations, *J. Differ. Equat.*, 2006, vol. 230, pp. 579–599.

8. Fang, C.-H. and Chang, F.-R., Analysis of Stability Robustness for Generalized State-Space Systems with Structured Perturbations, *Systems Control Lett.*, 1993, vol. 21, pp. 109–114.

9. Lee, L., Fang, C.-H., and Hsieh, J.-G., Exact Unindirectional Perturbation Bounds for Robustness of Uncertain Generalized State-Space Systems: Continuous-Time Cases, *Automatica*, 1997, vol. 33, pp. 1923–1927.

10. De Teran, F., Dopico, F.M., and Moro, J., First Order Spectral Perturbation Theory of Square Singular Matrix Pencil, *Linear Algebra Appl.*, 2008, vol. 429, pp. 548–576.

11. Linh, V.H. and Mehrmann, V., Lyapunov, Bohl and Sacker-Sell Spectral Intervals for Differential-Algebraic Equations, *J. Dyn. Differ. Equat.*, 2009, vol. 21, pp. 153–194.

12. Lin, Ch., Lam, J., Wang, J., and Yang, G.-H., Analysis on Robust Stability for Interval Descriptor Systems, *Syst. Control Lett.*, 2001, vol. 42, pp. 267–278.

13. Berger, T., Robustness of Stability of Time-Varying Index-1 DAEs, *Math. Control Signals Syst.*, 2014, vol. 26, pp. 403–433.

14. Du, N.H., Linh, V.H., and Mehrmann, V., Robust Stability of Differential-Algebraic Equations, in *Surveys in Differential-Algebraic Equations I.*, Ilchmann, A. and Reis, T., Eds., Berlin–Heidelberg: Springer-Verlag, 2013.

15. Barbosa, K., de Sousa, C., and Coutinho, D., Robust Admissibility and H_∞ Performance of Time-Varying Descriptor Systems, *Proc. 10th IEEE Int. Conf. on Control Autom.*, Hangzhou, China, 2013, pp. 1138–1143.

16. Bara, G.I., Robust Analysis and Control of Parameter-Dependent Uncertain Descriptor Systems, *Syst. Control Lett.*, 2011, vol. 60, no. 5, pp. 356–364.

17. Gao, L., Chen, W., and Sun, Y., On Robust Admissibility Condition for Descriptor Systems with Convex Polytopic Uncertainty, *Proc. 2003 Amer. Control Conf.*, Denver, USA, 2003, vol. 6, pp. 5083–5088.

18. Xing, S., Zhang, Q., and Zhu, B., Mean-Square Admissibility for Stochastic T-S Fuzzy Singular Systems Based on Extended Quadratic Lyapunov Function Approach, *Fuzzy Sets Syst.*, 2017, vol. 307, pp. 99–114.

19. Chen, G., Zheng, M., Yang, Sh., and Li, L., Admissibility Analysis of a Sampled-Data Singular System Based on the Input Delay Approach, *Complexity*, 2022. https://doi.org/10.1155/2022/3151620

20. Taniguchi, T., Tanaka, K., Yamafuji, K., and Wang, H.O., Fuzzy Descriptor Systems: Stability Analysis and Design via LMIs, *Proc. 1999 Amer. Control Conf.*, San Diego, USA, 1999, vol. 3, pp. 1827–1831.

21. Dong, X.-Z., Admissibility Analysis of Linear Singular Systems via a Delta Operator Method, *Int. J. Syst. Sci.*, 2014, vol. 45, pp. 2366–2375.

22. Shcheglova, A.A. and Kononov, A.D., Robust Stability of Differential-Algebraic Equations with an Arbitrary Unsolvability Index, *Autom. Remote Control*, 2017, vol. 78, no. 5, pp. 798–814.

23. Shcheglova, A.A. and Kononov, A.D., Stability of Differential-Algebraic Equations under Uncertainty, *Differential Equations*, 2018, vol. 54, no. 7, pp. 860–869.

24. Demidovich, B.P., *Lektsii po matematicheskoi teorii ustoichivosti* (Lectures on the Mathematical Theory of Stability), Moscow: Nauka, 1967.

25. Campbell, S.L. and Petzold, L.R., Canonical Forms and Solvable Singular Systems of Differential Equations, *SIAM J. Alg. Discrete Methods*, 1983, vol. 4, pp. 517–521.

26. Gantmacher, F.R., *Theory of Matrices*, New York: Chelsea Publishing, 1959.

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NONLINEAR SYSTEMS

Pattern Bifurcation in a Nonlocal Erosion Equation

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Abstract—This paper considers a periodic boundary value problem for a nonlinear partial differential equation with a deviating spatial variable. It is called the nonlocal erosion equation and was proposed as a model for the formation of dynamic patterns on the semiconductor surface. As is demonstrated below, the formation of a spatially inhomogeneous relief is a self-organization process. An inhomogeneous relief appears due to local bifurcations in the neighborhood of homogeneous equilibria when they change their stability. The analysis of this problem is based on modern methods of the theory of infinite-dimensional dynamic systems, including such branches as the theory of invariant manifolds, the apparatus of normal forms, and asymptotic methods for studying dynamic systems.

Keywords: nonlocal erosion equation, attractors, stability, bifurcations, normal forms, asymptotics

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1. INTRODUCTION

Since the 1980s, describing the formation of an inhomogeneous (e.g., wavelike) relief on the surface of semiconductor materials bombarded by an ion flow has always been a topical problem of micro- and nanoelectronics. The nature of a wavelike relief continues to cause much discussion; for example, see [1–6]. According to experimental evidence, a wavelike nanorelief is formed on the surface of semiconductors and dielectrics in a certain range of ion incidence angles. Of course, it depends on the beam intensity, ion type, and the material of the sample subjected to ion bombardment. The process of nanostructure formation on the silicon surface has been most intensively investigated at the experimental level.

Almost immediately, mathematical models were proposed to explain the phenomenon of inhomogeneous micro- and nano-relief formation. Two approaches were used: stochastic and deterministic.

In terms of applications, a more attractive approach is to treat such a process as dynamic. The best-known model was introduced by Bradley and Harper [7]. This model involves one version of the well-known Kuramoto–Sivashinsky equation supplemented with boundary conditions, natural from a physical point of view. In principle, different variations and modifications of this model gave a sufficiently convincing description of the process of inhomogeneous (wavelike) relief formation. At the same time, this model suffers from several drawbacks in its initial formulation. One of them is that in many cases, the corresponding boundary value problem revealed the possible formation of an inhomogeneous relief with the leading role of the first possible mode; for example, see [8–11]. In many cases, such a conclusion contradicts the results of experiments.

A possible informal modification is a model known as the nonlocal erosion equation [8–11]. It covers several nonlocal effects, first of all, the fact that the points of ion penetration (entry) into the semiconductor material and its exit do not necessarily coincide. This led to the emergence of a mathematical model with a partial differential equation in which some terms contain an unknown function with a deviating spatial variable.

This paper considers a nonlocal erosion equation augmented with periodic boundary conditions. Other problem statements for the nonlocal erosion equation can be found in [11–16].

Below, we study a periodic boundary value problem (BVP) for the nonlocal erosion equation

$$u_\tau = du_{yy} - c_1 w_y + c_2 w_y^2 + c_3 w_y^3, \tag{1.1}$$

$$u(\tau, y + 2l) = u(\tau, y), \tag{1.2}$$

where $u = u(\tau, y)$, $w = u(\tau, y - h_0)$, and h_0 is a positive constant to consider nonlocal effects (this constant is set proportional to the average distance between the entry and exit points of the ion from the incident beam); $d > 0$, $c_1 > 0$, c_2, and $c_3 \in \mathbb{R}$ are constants characterizing the bombardment conditions. For example, d is the diffusion coefficient of the target material, and the coefficient c_1 specifies the intensity (energy) of the ion beam. The deviation $h_0 > 0$ is the main parameter distinguishing this model from others (e.g., from the Bradley–Harper model). The constant h_0 is proportional to the inclination angle between the beam direction and the normal to the surface, which is considered flat before the bombardment. From the very beginning, let us emphasize the following aspect. In principle, the relief deviation from an equilibrium, $u(\tau, y)$, must depend on the second spatial coordinate y_1: $u = u(\tau, y, y_1)$. But in most experiments, the dependence on y_1 is rather weak and, therefore, the approximation $u = u(\tau, y)$ is considered acceptable.

With the changes of variables

$$\tau = \frac{l}{\pi c_1} t, \quad y = \frac{l}{\pi} x,$$

the BVP (1.1), (1.2) can be written as

$$u_t = au_{xx} - w_x + b_2 w_x^2 + b_3 w_x^3, \tag{1.3}$$

$$u(t, x + 2\pi) = u(t, x), \tag{1.4}$$

where $u = u(t, x),\ w = u(t, x - h)$, and

$$h = \frac{h_0 \pi}{l}, \quad a = \frac{d\pi}{lc_1}, \quad b_2 = \frac{c_2 \pi}{c_1 l}, \quad b_3 = \frac{c_3 \pi^2}{l^2 c_1}.$$

Note that the BVP (1.3), (1.4) has the solution $u(t, x) = \alpha$, where $\alpha \in \mathbb{R}$. If the BVP (1.3), (1.4) is supplemented by the boundary conditions

$$u(0, x) = f(x) \ \ (w(0, x) = f(x - h)), \tag{1.5}$$

where $f(x) \in \mathbb{H}_2^1$, the resulting initial boundary value problem will be locally well-posed. This outcome follows from the results obtained in [17, 18]. Moreover, the initial boundary value problem (1.3)–(1.5) generates a local smooth semiflow $T^t : f(x) \to u(t, x),\ t \in (0, \delta), \delta > 0$.

Recall that $f(x) \in \mathbb{H}_2^1$ if:

1) $f(x + 2\pi) = f(x)$.

2) For $x \in [0, 2\pi]$, the inclusion $f(x) \in \mathbb{W}_2^1[0, 2\pi]$ holds, where $\mathbb{W}_2^1[0, 2\pi]$ is the space of functions $f(x)$ such that $f(x) \in \mathbb{L}_2(0, 2\pi)$ and their generalized derivative $f'(x) \in \mathbb{L}_2(0, 2\pi)$ (for example, see [19]).

The BVP (1.3), (1.4) has one peculiarity as follows. Let $u(t, x)$ be any solution of this problem; then $\alpha + u(t, x)$ is also its solution. Below, we will investigate the structure of the neighborhood of all solutions $u(t, x) = \alpha$ (spatially homogeneous equilibria). In particular, it is supposed to study the formation mechanism of local attractors containing spatially inhomogeneous solutions.

2. SOME PRELIMINARIES

Consider the nonlinear BVP (1.3), (1.4). We denote by

$$M(u) = \frac{1}{2\pi}\int\limits_0^{2\pi} u(t,x)dx$$

the space mean of the function $u(t,x)$. Representing the solution $u(t,x)$ as a Fourier series with respect to the spatial variable x gives

$$u(t,x) = u_0(t) + \sum_{n\neq 0} u_n(t)\exp(inx),$$

where

$$u_0(t) = M(u), \quad u_n(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} u(t,x)\exp(-inx)dx.$$

Hence, any solution solution of the nonlinear BVP (1.3), (1.4) can be written as

$$u(t,x) = u_0(t) + v(t,x),\ v(t,x) = \sum_{n\neq 0} u_n(t)\exp(inx),\ \ M(v) = 0.$$

Therefore, the BVP (1.3), (1.4) takes the form

$$u_{0t}(t) = b_2 M(w_x^2) + b_3 M(w_x^3), \tag{2.1}$$

$$v_t = Av + F_2(w_x) + F_3(w_x), \tag{2.2}$$

$$v(t, x+2\pi) = v(t,x),\ M(v) = 0. \tag{2.3}$$

Equations (2.1) and (2.2) employ the following notations:

$$Av = av_{xx} - w_x, \quad w = v(t, x-h),$$
$$F_2(w_x) = b_2 w_x^2 - b_2 M(w_x^2), \quad F_3(w_x) = b_3 w_x^3 - b_3 M(w_x^3).$$

When forming the right-hand side of the differential equation (2.2), we take into account that $Au_0 = 0$ and the right-hand side of the original partial differential equation (1.3) is independent of $u_0(t)$.

The BVP (1.3), (1.4) can be analyzed in two stages. The first stage consists in studying the BVP (2.2), (2.3). After that, the second stage is to reconstruct $u_0(t)$ using equation (2.1). Note that the function $u_0(t)$ is reconstructed within an arbitrary constant without additional conditions from equation (2.1).

Thus, the main point in investigating the BVP (1.3), (1.4) is to study the auxiliary nonlinear BVP (2.2), (2.3). We emphasize that it has the unique spatially homogeneous equilibrium $v(t,x) = 0$.

3. STABILITY OF THE TRIVIAL SOLUTION OF THE AUXILIARY NONLINEAR BVP

To analyze the stability of the trivial equilibrium of the nonlinear BVP (2.2), (2.3), we first examine its linearized version, i.e., the linear BVP

$$v_t = Av,\ Av = av_{xx} - w_x, \tag{3.1}$$

$$v(t, x+2\pi) = v(t,x),\ M(v) = 0,\ w = v(t, x-h). \tag{3.2}$$

Consider the linear differential operator (LDO)

$$Ap = A(a,h)p = ap_{xx}(x) - p_x(x-h),$$

where a sufficiently smooth function $p(x)$ satisfies the periodic boundary conditions $p(x+2\pi) = p(x)$ and has the zero mean. This operator possesses a countable set of eigenvalues

$$\lambda_n = \lambda_n(a,h) = -an^2 - in\exp(-inh), \quad n = \pm 1, \pm 2, \ldots;$$

the corresponding eigenfunctions $\{\exp(inx)\}$ form a complete orthogonal system of functions in the space $\mathbb{L}_{2,0}(0,2\pi)$, i.e., $f(x) \in \mathbb{L}_{2,0}(0,2\pi)$ if $f(x) \in \mathbb{L}_2(0,2\pi)$ and $M(f) = 0$.

The following result is true.

Lemma 1. *If*

$$Re\lambda_n(a,h) < 0$$

for given values a and h, then all solutions of the linear BVP (3.1), (3.2) *are asymptotically stable in the metric of the initial condition space of the BVP* (3.1), (3.2).

A natural choice of the initial condition space is the functional space $\mathbb{H}^1_{2,0}$: $f(x) \in \mathbb{H}^1_{2,0}$ if $f(x) \in \mathbb{H}^1_2$ and $M(f) = 0$. Indeed, let us consider the initial boundary value problem (3.1), (3.2), (1.5) for $f(x) \in \mathbb{H}^1_{2,0}$. Its explicit-form solution is given by

$$v(t,x) = \sum_{n \neq 0} f_n \exp(\lambda_n t)\exp(inx), \tag{3.3}$$

where λ_n denote the eigenvalues of the LDO A and $\{f_n\}$ are the Fourier coefficients of the function $f(x)$ ($f_0 = 0$ since $M(f) = 0$).

It is straightforward to verify that:

1) $v(t,x) \to f(x)$ in the metric of $\mathbb{H}^1_2$ as $t \to +0$.

2) For $t \geqslant t_0 > 0$, the solution (3.3) is an infinitely differentiable function.

Property 2) is immediate from the following result, which can be checked in a fairly standard way: for $t \geqslant t_0 > 0$ the series on the right-hand side of (3.3) converges uniformly together with its partial derivatives of any order. This result is proved using the fact that

$$\lim_{|n|\to\infty} \frac{\lambda_n(a,h)}{n^2} = -a.$$

Note also that $\lim_{|n|\to\infty}(Im(\lambda_n(a,h)))/n^2 = 0$. Hence, $|Im\lambda_n(a,h)| \leqslant K|Re\lambda_n(a,h)|$ if $|n| \geqslant n_0$ ($n_0 \in \mathbb{N}$, the set of natural numbers) and K is some positive constant.

In particular, these results allow stating that the LDO A is the generator of the analytic semigroup of linear bounded operators, and the linear BVP (3.1), (3.2) can be included in the class of abstract parabolic equations in the sense of the definitions from [17, 18, 20].

If $Re\lambda_m(a,h) > 0$ for some $n = m$, the solutions of the linear BVP (3.1), (3.2) are, of course, unstable.

The considerations presented above lead to another result as follows.

Lemma 2. *Assume that*

$$Re\lambda_n \leqslant -\gamma_0 < 0$$

for all $n \in \mathbb{Z}_*$ (the set of integers $n \neq 0$). Then the trivial solution of the nonlinear BVP (2.2), (2.3) *is asymptotically stable. At the same time, if there exists an integer $m \in \mathbb{Z}_*$ such that $Re\lambda_m > 0$, then this solution is unstable.*

Note that the conditions $Re\lambda_n \leqslant 0$, $Re\lambda_m = 0$ for some $m \in Z_*$ select a critical case in the stability problem of the trivial solution of the BVP (2.2), (2.3).

The remainder of this section focuses on the following question: under what conditions is the critical case implemented in the stability problem of the trivial solution of the BVP (2.2), (2.3). Let us emphasize that, for $h = 0$,

$$\lambda_n(a, 0) = -an^2 - in$$

and, consequently, $Re\lambda_n(a, 0) < 0$ for all $n \in \mathbb{Z}_*$ ($\mathbb{Z}_* = \mathbb{Z}\backslash\{0\}$). Therefore, the critical case is possible only if $h > 0$ ($h \geqslant 0$ by the problem statement). For all $a > 0$ we will determine the least positive value $h = h_*(a)$ implementing the critical case.

First, it is necessary to find h_n satisfying

$$Re\lambda_n(a, h_n) = 0.$$

Such values h_n should be obtained as the solutions of the equation $-an^2 - n\sin nh = 0$ or

$$\sin nh = -an. \tag{3.4}$$

Equation (3.4) has solutions if $|an| \leqslant 1$. Without loss of generality, assume that $n \in \mathbb{N}$ (the set of natural numbers) since replacing n with $-n$ does not change equation (3.4).

Thus, $an \leqslant 1$. Then equation (3.4) possesses two groups of solutions:

1) $h_n(m) = \frac{1}{n}(2\pi m - \arcsin(an)), m \in \mathbb{Z}$,
2) $h_n(k) = \frac{1}{n}(\pi + \arcsin(an) + 2\pi k), k \in \mathbb{Z}$.

In the first group of solutions of the trigonometric equation, the least positive root is $h_n(1) = (2\pi - \arcsin(na))/n$; in the second group of solutions, $h_n(0) = (\pi + \arcsin(na))/n$. Furthermore, we have the inequality

$$\frac{1}{n}(\pi + \arcsin(na)) \leqslant \frac{1}{n}(2\pi - \arcsin(na)) \tag{3.5}$$

for any natural number n (of course, if $na \leqslant 1$). Inequality (3.5) is equivalent to

$$2\arcsin(na) \leqslant \pi \quad \text{or} \quad \arcsin(na) \leqslant \frac{\pi}{2}.$$

The resulting conclusion is that h_* should be determined as the least element of the sequence

$$d_n = d_n(a) = \frac{1}{n}(\pi + \arcsin(an)) \text{ if } n \leqslant \frac{1}{a}.$$

We underline that $d_n = h_n(0)$. Clearly, in principle, the least value d_n can be chosen by linear search. For example, if $a = 1$, this sequence contains one element $d_1 = 3\pi/2$ and $h_* = 3\pi/2$. In the case $a = 1/2$, we obtain $d_1 = 7\pi/6$ and $d_2 = 3\pi/4$; hence, $h_* = 3\pi/4$. At the same time, the number of elements in the sequence $d_n(a)$ grows when decreasing a. Therefore, linear search should be improved to select h_* faster.

For example, suppose that for all $a \geqslant a_0$, $d_k(a) \leqslant d_{k-1}(a)$; in other words, the sequence $d_k(a)$ decreases with increasing k. (An appropriate positive constant a_0 will be specified below.) This property of the sequence $d_k(a)$ can be checked by analyzing the inequalities

$$g_k(a) = (k-1)\arcsin(ka) - k\arcsin((k-1)a) \leqslant \pi;$$

for $k = 1, 2, 3$, they are trivial and hold for all admissible a. As is easily established,

$$\frac{dg_k(a)}{da} = k(k-1)\left(\frac{1}{\sqrt{1-(ak)^2}} - \frac{1}{\sqrt{1-a^2(k-1)^2}}\right) > 0, \quad ak \in [0, 1).$$

Hence, $d_k(a) \leqslant d_{k-1}(a)$ for all admissible a if this inequality is valid for the maximum possible value $a\,(a = 1/k)$. The issue under consideration is thereby reduced to checking the inequalities

$$\frac{1}{k-1}\left(\pi + \arcsin\frac{k-1}{k}\right) \geqslant \frac{3\pi}{2k} \quad \text{or} \quad \arcsin\frac{k-1}{k} \geqslant \frac{\pi(k-3)}{2k}.$$

As it turns out, the latter inequalities hold for $k = 1, \ldots, 10$.

Well, let $a > 1/11$. Then the minimum value h_* is $h(a) = (\pi + \arcsin(ka))/k$, where $k = [1/a]$, since this choice of a makes the sequence d_k decreasing. For the other numbers k, i.e., $k \geqslant 11$ $(a \leqslant 1/11)$, the choice procedure of h_* can be alternatively simplified as follows. Consider the auxiliary function

$$B(z) = \frac{1}{z}(\pi + \arcsin(z)),$$

which is defined for $z \in (0, 1]$. It is straightforward to verify that this function decreases for $z \in (0, z_*)$ and increases for $z \in (z_*, 1)$; naturally, z_* is its minimum point, i.e., $B'(z_*) = 0$. We determine the corresponding value z_* as the least positive root of the equation $B'(z) = 0$. As it turns out, $z = z_* \approx 0.9761$ and $d_n(a) = aB(na)$. Therefore, $h_* = \min\{d_m(a), d_{m+1}(a)\}$, where $ma \leqslant z_*$ and $(m+1)a > z_*$.

The possibility $d_m = d_{m+1}$ will be eliminated from consideration as a special (exceptional) case not discussed here. This case leads to another bifurcation problem with a codimension of 2. Further analysis will be restricted to the general case $d_m \neq d_{m+1}$.

Consider now the LDO depending on a small parameter, i.e.,

$$A(\varepsilon)y = ay'' - y'(x - h(\varepsilon)),$$

where $h(\varepsilon) = h_*(1 + \nu\varepsilon)$, $\nu = \pm 1$ or $0, \varepsilon \in (0, \varepsilon_0)$, $h_* = (\pi + \arcsin(ma))/m$. The definitional domain of this operator consists of sufficiently smooth functions satisfying the condition $y(x + 2\pi) = y(x)$, $M(y) = 0$. For such $h = h(\varepsilon)$, the LDO has a countable set of eigenvalues

$$\lambda_k(\varepsilon) = -ak^2 - ik\exp(-ikh(\varepsilon)), \quad k = \pm 1, \pm 2, \ldots.$$

In addition, given $k \neq \pm m$,

$$Re\lambda_k \leqslant -\gamma_0 < 0,$$

$\lambda_{\pm m}(\varepsilon) = -am^2 \mp im\exp(-i(\pi + \mu_m)(1 + \nu\varepsilon))$, where $\mu_m = \arcsin(ma)$, and consequently,

$$\lambda_{\pm m}(0) = \pm i\sigma_m, \quad \sigma_m = m\cos(\mu_m) = m\sqrt{1 - (ma)^2}.$$

In other words, for $(ma)^2 \neq 1$, the LDO A has a pair of simple pure imaginary eigenvalues; for $(ma)^2 = 1$, it has double zero eigenvalue. Therefore, the case $(ma)^2 = 1$ needs deeper analysis. The next section will be restricted to the general case $\sigma_m \neq 0$.

Note also that

$$\lambda'_m(\varepsilon)|_{\varepsilon=0} = \tau'_m + i\sigma'_m,$$

where $\tau'_m = \nu m(\pi + \arcsin(ma))\sqrt{1 - (ma)^2} \neq 0$ and $\sigma'_m = -\nu(\pi + \arcsin(ma))m^2 a \neq 0$. For $\nu = 1$, we obtain the inequality $\tau'_m > 0$, i.e., stability is lost when exceeding the critical value $h = h_*$; if $\nu = -1$, then $\tau'_m < 0$ and the trivial solution of the BVP (2.2), (2.3) remains stable. Finally, $\nu = 0$ corresponds to the critical case of a pair of pure imaginary eigenvalues.

4. LOCAL BIFURCATIONS

This section is devoted to the nonlinear BVP (2.2), (2.3) for $h = h(\varepsilon) = h_*(1+\nu\varepsilon)$. With such a choice of h, the BVP (2.2), (2.3) can be written as

$$v_t = A(\varepsilon)v + F_2(w_x,\varepsilon) + F_3(w_x,\varepsilon), \tag{4.1}$$

$$v(t,x+2\pi) = v(x),\ \ M(v) = 0, \tag{4.2}$$

where

$$w = v(t,x-h(\varepsilon)),\quad A(\varepsilon)v = av_{xx}(t,x) - v_x(t,x-h(\varepsilon)),$$
$$F_j(w_x,\varepsilon) = F_j(w_x),\quad j = 2,3.$$

The functions $F_j(w_x)$ have been introduced above. Recall that the deviation value h_* has been chosen in Section 3. As a result, an almost critical case is implemented for the stability spectrum of the trivial solution of the BVP (4.1), (4.2) (the spectrum of the LDO $A(\varepsilon)$). The BVP (4.1), (4.2) has a two-dimensional invariant manifold $M_2(\varepsilon)$ attracting all solutions from a sufficiently small neighborhood $Q(r_0)$ of the trivial solution of this BVP. In addition, the radius r_0 of the ball in the space $\mathbb{H}_2^1$ is sufficiently small but independent of ε; for example, see [21–23]. As is well known (e.g., see [23]), bifurcations can be analyzed by studying a system of two differential equations, commonly called the normal form (NF) according to the terminology originally proposed by A. Poincaré [23]. In the general case, such an equation can be written in the complex-valued form

$$z' = (\tau_m' + i\sigma_m')z + (l_1 + il_2)z|z|^2, \tag{4.3}$$

where $z = z(s)$, $s = \varepsilon t$ denotes "slow" time, the prime indicates the derivative with respect to s, and $l_1, l_2 \in \mathbb{R}$. By an a priori assumption, $l_1 \neq 0$: the first Lyapunov value is nonzero. In equation (4.3), all vanishing terms as $\varepsilon \to 0$ are discarded. Equation (4.3) represents the principal part of the NF or the truncated NF. Under $l_1 \neq 0$, equation (4.3) plays a determinative role in the analysis of the BVP (2.2), (2.3), (4.1), (4.2).

The solutions of the BVP (4.1), (4.2) belonging to $M_2(\varepsilon)$ will be found as the sum

$$v(t,x,z,\overline{z},\varepsilon) = \varepsilon^{1/2}v_1(t,x,z,\overline{z}) + \varepsilon v_2(t,x,z,\overline{z}) + \varepsilon^{3/2}v_3(t,x,z,\overline{z}) + O(\varepsilon^2). \tag{4.4}$$

In addition, of course,

$$w(t,x,z,\overline{z},\varepsilon) = \varepsilon^{1/2}w_1(t,x,z,\overline{z}) + \varepsilon w_2(t,x,z,\overline{z}) + \varepsilon^{3/2}w_3(t,x,z,\overline{z}) + O(\varepsilon^2), \tag{4.5}$$

where

$$w(t,x,z,\overline{z},\varepsilon) = v(t,x-h_*(\varepsilon),z,\overline{z},\varepsilon),$$
$$w_j(t,x,z,\overline{z}) = v_j(t,x-h_*,z,\overline{z}),\quad j = 1,2,3.$$

Finally,

$$v_1(t,x) = zq + \overline{zq},\quad q = q(t,x) = \exp(i\sigma_m t)\exp(imx).$$

The functions v_2 and v_3 are $v_2(t,x,z,\overline{z}), v_3(t,x,z,\overline{z}) \in \Phi$. The symbol Φ denotes the class of functions defined above.

We have $\varphi = \varphi(t,x,z,\overline{z}) \in \Phi$ if this function satisfies the following conditions:

1) It smoothly depends on the variables for all $t,\ x \in \mathbb{R}$, and $|z| < \delta$, where δ is some positive constant.

2) $\varphi(t,x,0,0) = 0$.

3) It has periods of $2\pi/\sigma_m$ and 2π in the variables t and x, respectively.

4) a) $M(\varphi) = 0$ for all $t, z, \overline{z}$ under consideration.

b) $M_\pm(\varphi) = \dfrac{\sigma_m}{(2\pi)^2} \int\limits_0^{2\pi} \left(\int\limits_0^{2\pi/\sigma_m} \varphi q_\mp dt \right) dx = 0,\ q_+ = q,\ q_- = \overline{q}.$

To analyze the BVP (4.1), (4.2), we formulate an auxiliary statement, often called the solvability conditions in branches of differential equations. Let $A_0 = A(0)$.

Remark 1. Consider the linear inhomogeneous BVP

$$A_0 v = g(t, x),\ v = v(t, x),$$
$$v(t, x + 2\pi) = v(t, x),\ M(v) = 0.$$

Here, $g(t, x)$ is a sufficiently smooth function with periods of $2\pi/\sigma_m$ and 2π in the variables t and x, respectively. In addition, suppose that $M(g(t, x)) = 0$. Then this BVP has periodic solutions in the variable t if

$$M_\pm(g(t, x)) = 0.$$

The conditions $M_\pm(v) = 0$ highlight one suitable solution of the linear inhomogeneous BVP considered in Remark 1.

Substituting the sum (4.4) and the related sum (4.5) into the nonlinear BVP (4.1), (4.2) yields linear inhomogeneous BVPs for determining v_2 and v_3.

Extracting the terms at ε, we obtain the inhomogeneous BVP

$$v_{2t} - A_0 v_2 = \Phi_2(t, x, z, \overline{z}), \tag{4.6}$$
$$v_2(t, x + 2\pi) = v_2(t, x),\ M(v_2) = M_\pm(v_2) = 0. \tag{4.7}$$

When extracting the terms proportional to $\varepsilon^{3/2}$, a similar BVP has the form

$$v_{3t} - A_0 v_3 = \Phi_3(t, x, z, \overline{z}), \tag{4.8}$$
$$v_3(t, x + 2\pi) = v_3(t, x),\ M(v_3) = M_\pm(v_3) = 0, \tag{4.9}$$

where $A_0 v_j = a v_{jxx} - w_{jx}$, $w_j = v_j(t, x - h_*)$, $j = 2, 3$,

$$\Phi_2(t, x, z, \overline{z}) = b_2 w_{1x}^2 - b_2 M(w_{1x}^2),$$
$$\Phi_3(t, x, z, \overline{z}) = b_3 w_{1x}^3 - b_3 M(w_{1x}^3) + 2b_2(w_{1x}w_{2x} - M(w_{1x}w_{2x})) + A_1 u_1 - (z' q + \overline{z}'\overline{q}),$$
$$w_{1x} = im(Qzq - \overline{Q}\overline{zq}),\ Q = \exp(-imh_*).$$

Note that $A_1 = A'(\varepsilon)|_{\varepsilon=0}$ and

$$Q = Q_1 + iQ_2,\ Q_1 = -\sqrt{1 - (ma)^2},\ Q_2 = ma.$$

The solutions $v_2(t, x, z, \overline{z}) \in \Phi$ of the BVP (4.6), (4.7) can (and should) be found in the form

$$v_2(t, x, z, \overline{z}) = \eta_m z^2 q^2 + \overline{\eta}_m \overline{z}^2 \overline{q}^2.$$

In our case,

$$\Phi_2(t, x, z, \overline{z}) = b_2 m^2 (Q^2 z^2 q^2 - \overline{Q}^2 \overline{z}^2 \overline{q}^2),$$

and quite easy calculations give

$$\eta_m = -\frac{b_2 m^2 Q^2 \overline{p}_m}{|p_m|^2},\ p_{m_1} = 4mQ_2(1 - Q_1),\ p_{m_2} = 2m(Q_1^2 - Q_2^2 - Q_1).$$

Now, we pass to the BVP (4.8), (4.9). It has a solution $v_3(t,x,z,\overline{z})\in\Phi$ under the solvability conditions (see Remark 1), i.e.,

$$M_{\pm}(\Phi_3)=0. \tag{4.10}$$

Conditions (4.10) serve to determine the coefficients of the NF (4.3). As it turns out,

$$l_1=l_1^{(2)}+l_1^{(3)},\quad l_2=l_2^{(2)}+l_2^{(3)},$$

where

$$l_1^{(3)}=-3b_3m^3Q_2,\quad l_2^{(3)}=3b_3m^3Q_1,$$
$$l_1^{(2)}=-\frac{4b_2^2m^4}{p_{m_1}^2+p_{m_2}^2}\Big(p_{m_1}Q_1(Q_1^2-3Q_2^2)+p_{m_2}Q_2(3Q_1^2-Q_2^2)\Big),$$
$$l_2^{(2)}=-\frac{4b_2^2m^4}{p_{m_1}^2+p_{m_2}^2}\Big(p_{m_1}Q_2(3Q_1^2-Q_2^2)-p_{m_2}Q_1(Q_1^2-3Q_2^2)\Big),$$
$$\tau_m'=\nu m(\pi+\mu_m)\sqrt{1-(ma)^2},\ \sigma_m'=-\nu(\pi+\mu_m)m^2a,\ \mu_m=\arcsin(ma).$$

Note that $\tau_m'>0$ if $\nu=1$, and $\tau_m'<0$ if $\nu=-1$. Thus, the coefficients of the NF (4.3) have been calculated explicitly.

Let us emphasize that, after clear transformations,

$$l_1^{(2)}=-\frac{8b_2^2m^6a}{p_{m_1}^2+p_{m_2}^2}\Big(\sqrt{1-(ma)^2}(1+4(ma)^2)+1\Big).$$

Hence,

$$l_1=-3b_3m^4a-\frac{8b_2^2m^6a}{p_{m_1}^2+p_{m_2}^2}\Big(\sqrt{1-(ma)^2}(1+4(ma)^2)+1\Big).$$

According to this formula, $l_1<0$ for $b_3>0$. Obviously, the case $l_1<0$ is implemented "more frequently" compared to the one $l_1>0$.

To proceed, we investigate the NF (4.3) by letting

$$z(s)=\rho(s)\exp(i\varphi(s)).$$

Transition to the trigonometric form leads to the two real-valued differential equations

$$\rho'=\tau_m'\rho+l_1\rho^3, \tag{4.11}$$
$$\varphi'=\sigma_m'+l_2\rho^2. \tag{4.12}$$

Besides the trivial equilibrium $\rho=0$, equation (4.11) may also have the nonzero one

$$\rho(s)=\xi=\sqrt{-\frac{\tau_m'}{l_1}},$$

which exists if $\tau_m'/l_1<0$.

The standard analysis using Lyapunov's theorem on the first (linear) approximation shows that the nonzero equilibrium $\rho(s)=\xi$ is asymptotically stable if $l_1<0$ ($\tau_m'>0$ or, equivalently, $\nu=1$) and unstable if $l_1>0$ ($\tau_m'<0$ or, equivalently, $\nu=-1$). The trivial equilibrium of the differential equation (4.11) is asymptotically stable if $\tau_m'<0$ and unstable if $\tau_m'>0$.

As is easily observed, for $\rho(s)=\xi$, equation (4.12) has the solution

$$\varphi(s)=(\sigma_m'+l_2\xi^2)s+\varphi_0,\quad \varphi_0\in\mathbb{R}.$$

The considerations presented above establish the following result.

Lemma 3. *The differential equation* (4.3) *has the limit cycle C_0 generated by the one-parameter family of periodic solutions*

$$z(s) = \xi \exp(i(\sigma'_m + l_2\xi^2)s + i\varphi_0), \tag{4.13}$$

where $\xi = \sqrt{-\tau'_m/l_1}$. These periodic solutions are stable (orbitally asymptotically stable) if $l_1 < 0$ ($\tau'_m > 0$) and unstable if $l_1 > 0$ ($\tau'_m < 0$).

Lemma 3 was proved in many publications. According to [11–16], we arrive at the following fact.

Theorem 1. *There exists a value $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the BVP* (2.2), (2.3) *with $h = h(\varepsilon) = h_*(1 + \nu\varepsilon)$ has a limit cycle $C(\varepsilon)$ corresponding to C_0 that inherits the stability of C_0. The solutions forming $C(\varepsilon)$ satisfy the asymptotic representation*

$$v(t, x, \varepsilon, \gamma) = \varepsilon^{1/2}\xi\Big(\exp(imx + i(\sigma_m + \varepsilon\beta_m)t + i\gamma) + \exp(-imx - i(\sigma_m + \varepsilon\beta_m)t - i\gamma)\Big)$$
$$+ \varepsilon\xi^2\Big(\eta_m \exp(2imx + 2i(\sigma_m + \varepsilon\beta_m)t + 2i\gamma) + \overline{\eta}_m \exp(-2imx - 2i(\sigma_m + \varepsilon\beta_m)t - 2i\gamma)\Big) + O(\varepsilon^{3/2}),$$

where $\xi = \sqrt{-\tau'_m/l_1}$, $\beta_m = \sigma'_m - \tau'_m l_2/l_1$, γ is an arbitrary real constant, and the constant η_m is chosen when constructing the NF (see the algorithm above).

5. A SPECIAL CASE OF THE BIFURCATION PROBLEM

As has been mentioned in Section 3, the eigenvalues $\lambda_k(\varepsilon)$ of the LDO $A(\varepsilon)$ are given by

$$\lambda_k(\varepsilon) = -ak^2 - ik\exp(-ikh(\varepsilon)), \quad k = \pm 1, \pm 2, \ldots.$$

In the special case $a = 1/m$, it is easy to verify $Re\lambda_k(\varepsilon) \leqslant -\gamma < 0$ for $k \neq \pm m$ and $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, where $0 < \varepsilon_0 << 1$.

For $a = 1/m$, where $m < 11$ is some natural number, the following assertions are true:

1) $h_* = 3\pi/(2m)$.

2) If $h(\varepsilon) = h_* + \varepsilon$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, then $\lambda_m(\varepsilon) = \tau_m(\varepsilon) + i\sigma_m(\varepsilon)$, where $\tau_m(\varepsilon) = -m(1 - \cos m\varepsilon)$ and $\sigma_m(\varepsilon) = -m\sin m\varepsilon$.

3) The analytical function $\tau_m(\varepsilon)$ is an even function of the variable ε.

4) The analytical function $\sigma_m(\varepsilon)$ is an odd function of the variable ε.

5) $\lambda_m(0) = \lambda_{-m}(0) = 0$. For $\varepsilon = 0$, the LDO A_0 has the double zero eigenvalue. The corresponding eigenfunctions are $\exp(\pm imx)$.

Hence, in the BVP (4.1), (4.2) with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, there exists a two-dimensional invariant manifold $M_2(\varepsilon)$ in the neighborhood of the trivial equilibrium (as before, see Section 4). This manifold will be a local attractor for the solutions of the BVP (4.1), (4.2) with sufficiently small initial conditions. By analogy with the previous section, the dynamics of the solutions of the BVP (4.1), (4.2) can be analyzed by investigating the complex-valued differential equation (NF)

$$z' = (\tau(\varepsilon) + i\sigma(\varepsilon))z + \psi(z, \overline{z}, \varepsilon) \tag{5.1}$$

with a sufficiently smooth function $\psi(z, \overline{z}, \varepsilon)$ such that

$$\psi(0, 0, \varepsilon) = \frac{\partial\psi}{\partial z}|_{z=0} = \frac{\partial\psi}{\partial\overline{z}}|_{\overline{z}=0} = 0.$$

Once again, when analyzing the behavior of the solutions of the BVP (4.1), (4.2), we emphasize the determinative role of both equation (5.1) and its truncated version

$$z' = (\tau(\varepsilon) + i\sigma(\varepsilon))z + \psi_0(z, \overline{z}), \tag{5.2}$$

where $\psi_0(z,\overline{z}) = \psi(z,\overline{z},0)$ and a sufficiently smooth function $\psi_0(z,\overline{z})$ has an infinitesimal order above 1 at the zero point.

To find the principal part of the function $\psi_0(z,\overline{z})$, it suffices to consider the BVP (4.1), (4.2) with $\varepsilon = 0$ and construct the NF for it. The solutions of the BVP (4.1), (4.2) with $\varepsilon = 0$ will be obtained in the form

$$v(t,x,z,\overline{z}) = (qz + \overline{qz}) + p_2(x)z^2 + p_0(x)z\overline{z} + \overline{p}_2(x)\overline{z}^2 + r_3(x)z^3 + r_1(x)z^2\overline{z} + \overline{r}_1(x)z\overline{z}^2 + \overline{r}_3(x)\overline{z}^3 + \ldots, \tag{5.3}$$

where the ellipsis indicates the terms of a higher infinitesimal order in the variables $z,\overline{z}$. Finally, $q(x) = \exp(imx)$, the functions $p_2(x), p_0(x), r_1(x)$, and $r_3(x)$ have a period of 2π in the variable x and the zero means:

$$M(p_j) = 0,\ \ M_\pm(p_j) = 0,\ \ M(r_k) = 0,\ \ M_\pm(r_k) = 0,$$

where $j = 0,2$, $k = 1,3$, $q_+ = \exp(imx)$, $q_- = \exp(-imx)$, $q = q_+$, $\overline{q} = q_-$, and

$$M(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi dx,\ \ M_\pm(\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \varphi q_\mp dx,\ \ \varphi = \varphi(x).$$

Substituting the sum (5.3) into the BVP (4.1), (4.2) with $\varepsilon = 0$ and sequentially extracting the terms proportional to z^2, $z\overline{z}$, $\overline{z}^2$, z^3, $z^2\overline{z}$, $z\overline{z}^2$, and $\overline{z}^3$, we get a system of linear inhomogeneous equations. This system will be analyzed to determine the principal part of the complex-valued function $\psi_0(z,\overline{z})$. When forming the BVP, it is necessary to consider the formula

$$\psi_0(z,\overline{z}) = \psi_2 z^2 + \psi_0 z\overline{z} + \overline{\psi}_2\overline{z}^2 + \psi_3 z^3 + \psi_1 z^2\overline{z} + \overline{\psi}_1 z\overline{z}^2 + \overline{\psi}_3\overline{z}^3 + \ldots,$$

where the ellipsis indicates the terms of a higher infinitesimal order in the variables z and $\overline{z}$.

As a result, we arrive at the following inhomogeneous BVPs for determining the periodic functions with the zero means. For example, $p_2(x)$ and $p_0(x)$ are obtained from the two linear inhomogeneous BVPs

$$A_0 p_2(x) = -b_2 m^2 q^2 + \psi_2 q, \tag{5.4}$$

$$p_2(x+2\pi) = p_2(x),\ \ M(p_2) = M_\pm(p_2) = 0, \tag{5.5}$$

$$A_0 p_0(x) = \psi_0, \tag{5.6}$$

$$p_0(x+2\pi) = p_0(x),\ \ M(p_0) = M_\pm(p_0) = 0. \tag{5.7}$$

In the required class of functions, the solutions of the auxiliary linear inhomogeneous BVPs (5.4), (5.5) and (5.6), (5.7) should be found with $\psi_2 = 0$ and $\psi_0 = 0$, and consequently,

$$p_0(x) = 0,\ p_2(x) = \eta_2 Q^2 q^2,\ \overline{p}_2(x) = \overline{\eta}_2\overline{Q}^2\overline{q}^2,$$

where (in this case) $Q = \exp(-ih_* m)$, i.e., $Q = \exp(-i3\pi/2) = i$ and $Q^2 = -1$. Finally,

$$\eta_2 = \frac{b_2 m}{10}(2+i).$$

At the third step of the algorithm, we have two BVPs for determining $r_1(x)$ and $r_3(x)$:

$$A_0 r_3(x) = \psi_3 q + b_3 m^3 q^3 - 4b_2\eta_2 i m^2 q^3, \tag{5.8}$$

$$r_3(x+2\pi) = r_3(x),\ M(r_3) = M_\pm(r_3) = 0, \tag{5.9}$$

$$A_0 r_1(x) = \psi_1 q + 3b_3 m^3 q - 4b_2\eta_2 i m^2 q, \tag{5.10}$$

$$r_1(x+2\pi) = r_1(x),\ M(r_1) = M_\pm(r_1) = 0. \tag{5.11}$$

From the solvability conditions of the BVPs (5.8), (5.9) and (5.10), (5.11) it follows that $\psi_3 = 0$ and $\psi_1 = -3b_3m^3 + \frac{2}{5}b_2^2m^3(-1+2i)$.

Thus, the principal part of the NF (5.1) is

$$z' = \left(-\frac{\varepsilon^2}{2}m^3 - im^2\varepsilon\right) z + (l_1 + il_2)z|z|^2, \tag{5.12}$$

where $l_1 = -3b_3m^3 - \frac{2}{5}b_2^2m^3$ and $l_2 = \frac{4}{5}b_2^2m^3$.

For analysis, we write this differential equation in the trigonometric form and let

$$z(t) = \rho(t)\exp(i\varphi(t)). \tag{5.13}$$

Then the complex-valued equation (5.12) is replaced by the two real-valued differential equations

$$\rho' = -\frac{\varepsilon^2}{2}m^3\rho + l_1\rho^3, \tag{5.14}$$

$$\varphi' = -\varepsilon m^2 + l_2\rho^2. \tag{5.15}$$

By analogy with the previous section, we start the analysis of system (5.14), (5.15) with the differential equation (5.14) for the amplitude $\rho(t)$.

Lemma 4. *In addition to the trivial equilibrium S_0: $\rho = 0$, the differential equation* (5.14) *may have the equilibrium S_*: $\rho_* = \sqrt{\varepsilon^2m^3/(2l_1)}$. The equilibrium ρ_* of the differential equation* (5.14) *exists if $l_1 > 0$. This equilibrium is unstable. The asymptotically stable equilibrium is $\rho = 0$.*

The stability of these equilibria is analyzed using Lyapunov's theorem on the first (linear) approximation. Note that the equilibrium S_* is associated with the solution of the differential equation (5.15) of the form

$$\varphi_*(t) = \left(-\varepsilon m^2 + \frac{l_2\varepsilon^2m^3}{2l_1}\right)t + \varphi_0,$$

where φ_0 is an arbitrary real constant. Equality (5.13) allows finding the periodic solution of the NF (5.12) in the variable t:

$$z(t) = z(t,\varepsilon) = \sqrt{\frac{\varepsilon^2m^3}{2l_1}}\exp(i\varphi_*(t)).$$

According to [11–16], we obtain the following result.

Theorem 2. *Let $am = 1$ ($h_* = 3\pi/(2m)$), where $m = 1,\ldots,10$. There exists a positive constant ε_0 such that, for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $h = h_* + \varepsilon$, and $\varepsilon \neq 0$, the BVP* (4.1), (4.2) *has the one-parameter family of unstable periodic solutions in the variable t :*

$$\begin{aligned} v_*(t,x,\varepsilon,\varphi_0) &= \rho_*(q_m(t,x,\varphi_0) + \overline{q}_m(t,x,\varphi_0)) \\ &+ \rho_*^2(\eta_2q_m^2(t,x,\varphi_0) + \overline{\eta}_2\overline{q}_m^2(t,x,\varphi_0)) + o(\varepsilon^2), \end{aligned} \tag{5.16}$$

where $q_m(t,x,\varphi_0) = \exp(i\omega(\varepsilon)t + imx + i\varphi_0)$ and $\omega(\varepsilon) = -\varepsilon m^2 + l_2m^3\varepsilon^2/(2l_1) + o(\varepsilon^2)$. The constants l_1 and l_2 have been specified above. The family of solutions (5.16) *exists if the first Lyapunov value is $l_1 > 0$ (see Lemma 4).*

In the case $l_1 < 0$, the trivial solution of the BVP (4.1), (4.2) *is asymptotically stable.*

Remark 2. The existence of periodic solutions in the special case of the BVP (4.1), (4.2) occurs rather rarely; if it does, they are unstable. Obviously, the case $l_1 > 0$ is rare as well. The dominating situation is when $l_1 > 0$. Then the differential equation (5.14), and hence the BVP (4.1), (4.2), has no small periodic solutions in the variable t.

6. THE MAIN RESULT

We revert to the analysis of the primary nonlinear BVP (1.3), (1.4) with $h = h(\varepsilon) = h_*(1+\nu\varepsilon)$ provided that $am \neq 1$.

Let $v(t,x,\varepsilon,\gamma)$ be a periodic solution obtained for the BVP (2.2), (2.3). Then $u_0(t)$ is determined from equation (2.1): $v(t,x,\varepsilon,\gamma)$ is substituted into its right-hand side and the resulting equation is integrated. In this case, we have

$$u_0(t,\varepsilon,\gamma_0) = \Big(2b_2\xi^2 m^2\varepsilon + o(\varepsilon)\Big)t + \gamma_0,$$

where γ_0 is an arbitrary real constant and $\xi^2 = -(\tau_m'/l_1) > 0$.

Theorem 3. *There exists a positive constant ε_0 such that, for all $\varepsilon \in (0,\varepsilon_0)$ and $h = h_*(1+\nu\varepsilon)$, the nonlinear BVP (1.3), (1.4) has the two-parameter family $V_2(\varepsilon,\gamma_0,\gamma)$ of solutions*

$$u(t,x,\varepsilon) = u_0(t,\varepsilon,\gamma_0) + v(t,x,\varepsilon,\gamma)$$

if the BVP (2.2), (2.3) possesses the limit cycle $C(\varepsilon)$. This family forms the integral manifold of the nonlinear BVP (1.3), (1.4).

The family $V_2(\varepsilon,\gamma_0,\gamma)$ is a local attractor if the limit cycle $C(\varepsilon)$ of the auxiliary BVP (2.2), (2.3) is a local attractor. This family is unstable (saddle) if the same property holds for $C(\varepsilon)$.

Assume that the special case is implemented: $am = 1$ and $h_* = 3\pi/(2m)$. Of course, the main result needs an appropriate correction. In this case, Theorem 2 implies the following result.

Theorem 4. *There exists a positive constant ε_0 such that, for $\varepsilon \in (-\varepsilon_0, 0) \cup (0,\varepsilon_0)$ and $h = h_* + \varepsilon$, the nonlinear BVP (1.3), (1.4) has the two-parameter family $V_*(\varepsilon,\gamma_0,\varphi_0)$ of solutions*

$$u(t,x,\varepsilon) = u_0(t,\varepsilon,\gamma_0) + v_*(t,x,\varepsilon,\varphi_0),$$

where $v_*(t,x,\varepsilon,\varphi_0)$ is the solution (5.16) of the auxiliary BVP (4.1), (4.2) (see Theorem 2) and

$$u_0(t,\varepsilon,\gamma_0) = \left(b_2\frac{\varepsilon^2 m^2}{l_{1,0}} + o(\varepsilon^2)\right)t + \gamma_0,$$

where γ_0 is an arbitrary constant and $l_{1,0} = -3b_3 - 2b_2^2/5$. Recall that in this case, the solution exists if $l_{1,0} > 0$.

Note that the solution family $V_*(\varepsilon,\gamma_0,\varphi_0)$ is always unstable.

7. CONCLUSIONS

In this paper, we have studied local bifurcations in a periodic boundary value problem for a nonlocal erosion equation. It represents a partial differential equation with a deviating spatial variable. It has been demonstrated that proper consideration of the deviating variable is an essential factor in bifurcation analysis. For the deviation value $h = 0$, an inhomogeneous relief is not formed. Increasing h to some threshold values causes nanorelief formation.

In most cases, such a relief is formed as the result of Andronov–Hopf bifurcations with an appropriate choice of $h \approx h_*$ and a. The special case $ak \approx 1$ leads to another type of bifurcations and unstable patterns.

The above analysis of nanorelief formation, a topical physical problem, has turned out quite effective due to applying modern methods of the theory of dynamic systems, namely, the methods of invariant manifolds and the theory of normal Poincaré forms, extended to the class of problems with an infinite-dimensional phase space. We emphasize that the use and development of the method of integral (invariant) manifolds is quite productive in the analysis of many problems of mathematical physics: in many cases, this method reduces an original infinite-dimensional problem to the analysis of a finite-dimensional dynamic system. Another approach to the analysis of infinite-dimensional dynamic systems was shown in [24, 25].

Note that the inclusion (consideration) of nonlocal terms in a partial differential equation often significantly changes the dynamics of its solution towards higher complexity and richness. For example, bifurcations may arise on higher modes, which has been repeatedly observed in experiments.

Moreover, considering nonlocal terms in mathematical models reveals new effects in nanoelectronics and in other nonlinear models of physics (for example, see [26–29]).

REFERENCES

1. Sigmund, P., Theory of Sputtering. I. Sputtering Yield of Amorphous and Polycrystalline Targets, *Phys. Rev.*, 1969, vol. 184, no. 2, pp. 383–416.
2. Yamamura, Y. and Shindo, S., An Empirical Formula for Angular Dependence of Sputtering Yields, *Radiat. Effect.*, 1984, vol. 80, no. 1–2, pp. 57–72.
3. Elst, K. and Vandervorst, W., Influence of the Composition of the Altered Layer on the Ripple Formation, *J. Vacuum Sci. Tech. A.*, 1994, vol. 12, no. 2, pp. 3205–3216.
4. Sigmund, P., A Mechanism of Surface Micro-Roughening by Ion Bombardment, *J. Mater. Sci.*, 1973, vol. 8, no. 2, pp. 1545–1553.
5. Smirnov, V.K., Kibalov, D.S., Lepshin, P.A., and Bachurin, V.I., The Influence of Topographic Irregularities on the Formation of Wavelike Microrelief on the Silicon Surface, *Izv. Ross. Akad. Nauk. Ser. Fiz.*, 2000, vol. 64, no. 4, pp. 626–630.
6. Rudyi, A.S., Kulikov, A.N., and Metlitskaya, A.V., Self-organization of Nanostructures within the Spatially Nonlocal Model of Silicon Surface Erosion by Ion Bombardment, in *Kremnievye nanostruktury. Fizika. Tekhnologiya. Modelirovanie* (Silicon Nanostructures. Physics. Technology. Modeling), Rudakov, V.I., Ed., Yaroslavl: Indigo, 2014, pp. 8–57.
7. Bradley, R.M. and Harper, M.E., Theory of Ripple Topography Induced by Ion Bombardment, *J. Vacuum Sci. Tech. A.*, 1988, vol. 6, no. 4, pp. 2390–2395.
8. Rudy, A.S. and Bachurin, V.I., Spatially Nonlocal Model of Surface Erosion by Ion Bombardment, *Bull. Russ. Acad. Sci. Phys.*, 2008, vol. 72, pp. 586–591.
9. Metlitskaya, A.V., Kulikov, A.N., and Rudy, A.S., Formation of the Wave Nanorelief at Surface Erosion by Ion Bombardment within the Bradley–Harper Model, *Russ. Microelectron.*, 2013, vol. 42, pp. 238–245.
10. Kulikov, A.N. and Kulikov, D.A., Nonlocal Model for the Formation of Ripple Topography Induced by Ion Bombardment. Nonhomogeneous Nanostructures, *Matem. Mod.*, 2016, vol. 28, no. 3, pp. 33–50.
11. Rudyi, A.S., Kulikov, A.N., Kulikov, D.A., et al., High-mode Wave Reliefs in a Spatially Nonlocal Erosion Model, *Russ. Microelectron.*, 2014, vol. 43, pp. 277–283.
12. Kulikov, D.A. and Rudy, A.S., Formation of a Warped Nanomodular Surface under Ion Bombardment. A Nanoscale Model of Surface Erosion, *Model. Anal. Inform. Sist.*, 2012, vol. 19, no. 5, pp. 40–49.
13. Kulikov, D.A., Spatially Nonhomogeneous Dissipative Structures of a Periodic Boundary-Value Problem for a Nonlocal Erosion Equation, *Nonlinear Oscillations*, 2014, vol. 17, no. 1, pp. 72–86.
14. Kovaleva, A.M., Kulikov, A.N., and Kulikov, D.A., Stability and Bifurcations of Undulate Solutions for One Functional-Differential Equation, *Izv. Inst. Mat. Inform. Udmurt. Gos. Univ.*, 2015, no. 2(46), pp. 60–68.
15. Kovaleva, A.M. and Kulikov, D.A., Bifurcations of Spatially Inhomogeneous Solutions in Two Versions of the Nonlocal Erosion Equation, *J. Math. Sci.*, 2020, vol. 248, no. 4, pp. 438–447.

16. Kulikov, D.A., Inhomogeneous Dissipative Structures in the Problem of Nanorelief Formation, *Dynamical Systems*, 2012, vol. 2 (30), no. 3–4, pp. 259–272.

17. Sobolevskii, P.E., Equations of Parabolic Type in a Banach Space, *Tr. Mosk. Mat. Obshch.*, 1961, vol. 10, pp. 297–350.

18. Yakubov, S.Ya., Solvability of the Cauchy Problem for Abstract Quasilinear Second-Order Hyperbolic Equations and Their Applications, *Tr. Mosk. Mat. Obshch.*, 1970, vol. 23, pp. 37–60.

19. Sobolev, S.L., *Nekotorye primeneniya funktsional'nogo analiza v matematicheskoi fizike* (Some Applications of Functional Analysis in Mathematical Physics), Leningrad: Leningrad State University, 1950.

20. Krein, S.G., *Lineinye differentsial'nye uravneniya v banakhovom prostranstve* (Linear Differential Equations in a Banach Space), Moscow: Nauka, 1977.

21. Marsden, J.E. and McCracken, M., *The Hopf Bifurcation and Its Applications*, New York: Springer-Verlag, 1976.

22. Kulikov, A.N., Inertial Invariant Manifolds of a Nonlinear Semigroup of Operators in a Hilbert Space, *Proceedings of the All-Russian Scientific Conference "Differential Equations and Their Applications" Dedicated to the 85th Anniversary of Prof. M.T. Terekhin*, Yesenin Ryazan State University, Ryazan, May 17–18, 2019, part 2, *Itogi Nauki i Tekhniki. Ser. Sovrem. Mat. Pril. Temat. Obz.*, 2020, vol 186, Moscow: VINITI, pp. 57–66.

23. Arnol'd, V.I., *Dopolnitel'nye glavy obyknovennykh differentsial'nykh uravnenii* (Additional Chapters of Ordinary Differential Equations), Moscow: Nauka, 1978.

24. Akhmetzyanov, A.V., Kushner, A.G., and Lychagin, V.V., Attractors in Models of Porous Media Flow, *Doklady Mathematics*, 2017, vol. 95, no. 1, pp. 72–75.

25. Kushner, A., Lychagin, V., and Rubtsov, V., *Contact Geometry and Non-linear Differential Equations*, Cambridge: Cambridge Univ. Press, 2007.

26. Kulikov, A.N. and Kulikov, D.A., A Possibility of Realizing the Landau–Hopf Scenario in the Problem of Tube Oscillations under the Action of a Fluid Flow, *Theor. Math. Phys.*, 2020, vol. 203, pp. 501–511.

27. Kulikov, A.N. and Kulikov, D.A., Invariant Varieties of the Periodic Boundary Value Problem of the Nonlocal Ginzburg–Landau Equation, *Math. Meth. Appl. Sci.*, 2021, vol. 44, no. 3, pp. 11985–11997.

28. Kulikov, A.N. and Kulikov, D.A., Invariant Manifolds and Global Attractor of the Ginzburg–Landau Integro-Differential Equation, *Diff. Equat.*, 2022, vol. 58, pp. 1499–1513.

29. Kulikov, A.N. and Kulikov, D.A., Invariant Manifolds of a Weakly Dissipative Version of the Nonlocal Ginzburg–Landau Equation, *Autom. Remote Control*, 2021, vol. 82, no. 2, pp. 264–277.

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Density Systems: Analysis and Control

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Abstract—This paper considers a class of systems called density systems. For such systems, the derivative of a quadratic function depends on some function termed the density function. The latter function is used to define the properties of the space affecting the behavior of the systems under consideration. The role of density systems in control law design is shown. Control systems are constructed for plants with known and unknown parameters. The theoretical results are illustrated by numerical simulation.

Keywords: dynamic system, quadratic function, stability, control

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1. INTRODUCTION

We study a class of dynamic normal-from systems whose right-hand side depends on some function defining the properties of the space and affecting the system behavior. This function will be called the density function. All relevant definitions will be provided in the main part of the paper.

A particular class of such systems was considered in [1–8]. The (in)stability of a system $\dot{x} = f(x)$ in the plane was first analyzed by introducing a new system $\dot{x} = \rho(x)f(x)$ with an auxiliary function $\rho(x) > 0$ for all x in the pioneering book [1]. Then, the (in)stability of such systems was studied using the properties of the divergence and phase velocity vector flow; see [2–8]. The function $\rho(x)$ was called the density function [4]. In [5–8], a connection was established between the obtained results and the continuity equation [9], which arises in electromagnetism, wave theory, fluid dynamics, mechanics of deformable solids, and quantum mechanics.

Several control methods proposed in [10–14] ensure that controlled signals are in given sets. This goal is achieved by introducing an auxiliary function through an appropriate control law; the form of this function determines the corresponding properties in the closed loop system. For example, funnel control and control with prescribed performance were presented in [10, 12] and in [11], respectively; under these control laws, transients belong to a pipe converging to the neighborhood of zero. The method proposed in [13, 14] generalizes the results of [10–12], ensuring that the output variables will be in a given pipe (possibly asymmetric with respect to the equilibrium and without convergence to a given constant).

This paper is devoted to a class of systems depending, explicitly or implicitly, on a density function. This function will be used to define the density of the space in the sense of distinguishing (in)stability domains and forbidden domains (where the system has no solutions). The behavior of the system under consideration will depend on the value of the density function.

The presentation has the following distinctive features:

(1) In contrast to [1–8], the density function is not necessarily multiplied by the entire right-hand side of the system.

(2) In contrast to [10–14], the density function may be present implicitly on the right-hand side of the system.

(3) In contrast to [10–14], the density function can ensure system solutions in an unbounded set with forbidden domains and the boundaries of these sets can be defined by continuous (under some assumptions, even discontinuous) functions in all arguments.

The remainder of this paper is organized as follows. Section 2 presents motivating examples as well as the definitions of a density function and a density system. Some properties of these systems are also demonstrated. In Section 3, the theoretical results are applied to design control laws for plants with known and unknown parameters. In addition, the control schemes are numerically simulated to confirm theoretical conclusions.

We adopt the following *notations*: $\mathbb{R}^n$ is the n-dimensional Euclidean space with the norm $|\cdot|$; $\mathbb{R}_+$ ($\mathbb{R}_-$) is the set of positive (negative, respectively) real numbers; $p = d/dt$ indicates the differentiation operator; finally, λ stands for the complex variable.

2. MOTIVATING EXAMPLES. DEFINITIONS

Prior to introducing the main definitions, we consider two examples as follows.

Example 1. As is well known, the solutions of the system

$$\dot{x} = -x, \quad x \in \mathbb{R}, \tag{1}$$

asymptotically vanish. Multiplying the right-hand side of this system by a function $\rho(x,t)$: $\mathbb{R} \times [0,+\infty) \to \mathbb{R}$ that is continuous in t and locally Lipschitz in x, we write (1) as

$$\dot{x} = -\rho(x,t)x. \tag{2}$$

Obviously, the behavior of system (2) depends on the properties of $\rho(x,t)$. The function $\rho(x,t)$ can be used to define some properties and constraints in the space (x,t), thereby affecting the quality of transients of the original system (1) and changing them qualitatively. In this context, $\rho(x,t)$ will be called a *density function*. Here are several examples of this function and the corresponding behavior of the new system (2).

1. The density function $\rho(x,t) = \alpha > 0$ allows preserving the single equilibrium $x = 0$, takes the same positive value for any x and t, and hence has no qualitative effect on the exponential stability of the trajectories of the original system (1) (see Fig. 1 on the left) except the rate of convergence of the solution of (2) to the equilibrium depending on the value α. Indeed, choosing the Lyapunov function $V = 0.5x^2$ yields $\dot{V} = -\alpha x^2 < 0$ in the domain $D_S = \mathbb{R} \setminus \{0\}$.

2. Consider the density function $\rho(x,t) = \frac{\alpha}{w(t)-|x(t)|}$ with a continuous function $w(t) > 0$. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R} : -w < x < w\}$, and $\rho(x,t) \to +\infty$ as $|x - w| \to 0$ in D_S. These properties ensure the uniform asymptotic stability of the equilibrium $x = 0$ under the initial conditions $x(0) \in (-w(0), w(0))$. In addition, the system trajectories will never leave this domain (see Fig. 1 on the right). Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = -\frac{\alpha}{w-|x|}x^2 < 0$ in the domain $x \in D_S \setminus \{0\}$, which confirms the conclusions drawn.

3. Consider the density function $\rho(x,t) = \alpha[x(t) - w(t)]\arctan(x/\epsilon)$ with a continuous function $w(t)$ and a sufficiently large positive number ϵ. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R} : x \in (-\infty, 0) \cup (w, +\infty)$ for $w > 0$ and $x \in (-\infty, w) \cup (0, +\infty)$ for $w < 0\}$ and negative values in the domain $D_U = \{x \in \mathbb{R} : x \in (0, w)$ for $w > 0$ and $x \in (w, 0)$ for $w < 0\}$, which

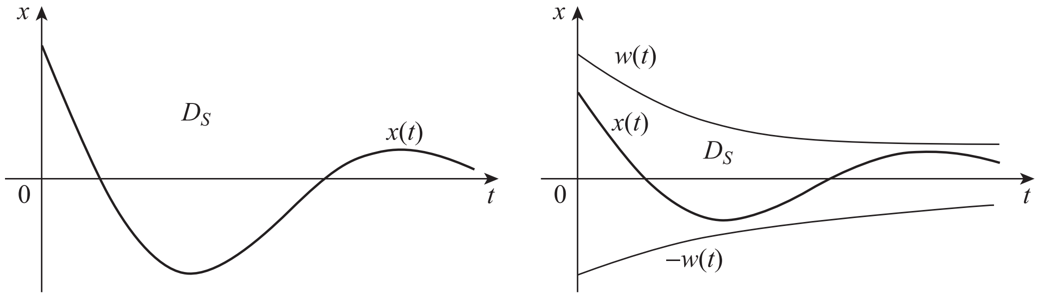

Fig. 1. Transients in system (2) with the density functions $\rho(x,t)=\alpha$ (left) and $\rho(x,t)=\frac{\alpha}{w(t)-|x(t)|}$ (right).

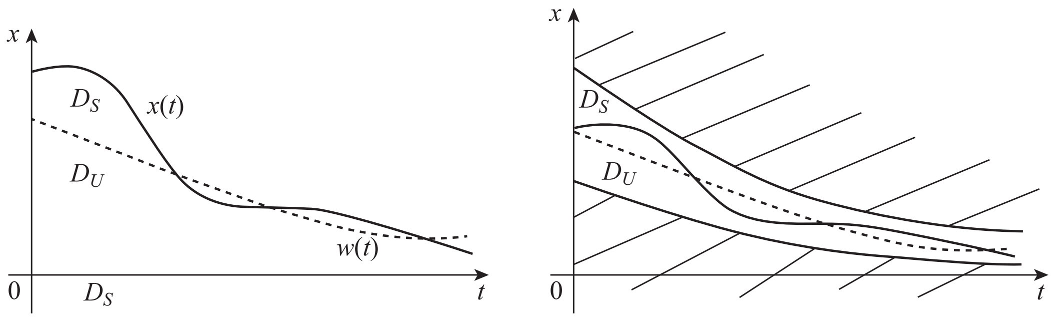

Fig. 2. Transients in system (2) with the density functions $\rho(x,t)=\alpha[x(t)-w(t)]\arctan(x/\epsilon)$ (left) and $\rho(x,t)=\alpha\ln\frac{\overline{w}(t)-x(t)}{x(t)-\underline{w}(t)}$ (right).

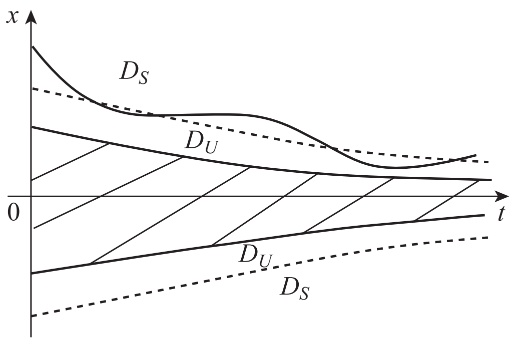

Fig. 3. Transients in system (2) with the density function $\rho(x,t)=\alpha\ln(x(t)-g(t))$.

ensures tracking of the trajectory $w(t)$ by $x(t)$ (see Fig. 2 on the left). Choosing the quadratic function $V=0.5x^2$ yields $\dot V=\alpha[x-w]\arctan(x/\epsilon)\,x^2<0$ for $x\in D_S$ and $\dot V>0$ for $x\in D_U$.

4. Consider the density function $\rho(x,t)=-\alpha\ln\frac{\overline{w}(t)-x(t)}{x(t)-\underline{w}(t)}$ with continuous functions $\overline{w}(t)>\underline{w}(t)>0$. Let us denote $w=0.5[\overline{w}+\underline{w}]$, where $\rho(x,t)=0$ for $x=w$ and any t. The function $\rho(x,t)$ takes positive values in the domain $D_S=\{x\in\mathbb{R}: w<x<\overline{w}\}$ and negative values in the domain $D_U=\{x\in\mathbb{R}:\underline{w}<x<w\}$, which ensures tracking of the trajectory $w(t)$ by $x(t)$. In the shaded domain, system (2) has no solutions (see Fig. 2 on the right). In addition, the system trajectories will never leave the domain $D_S\cup D_U$ since $|\rho(x,t)|\to+\infty$ as x approaches the boundaries $\underline{w}$ and $\overline{w}$. Choosing the quadratic function $V=0.5x^2$ yields $\dot V=\alpha\ln\frac{\overline{w}-x}{x-\underline{w}}x^2<0$ for $x\in D_S$ and $\dot V>0$ for $x\in D_U$, which confirms tracking of the trajectory $w(t)$ by $x(t)$.

5. Consider now $\rho(x,t) = \alpha \ln(x(t) - g(t))$ with a continuous function $g(t) > 0$. Let us denote $w = 1 + g$, where $\rho(x,t) = 0$ for $x = w$ and any t. The function $\rho(x,t)$ takes positive values in the domain $D_S = \{x \in \mathbb{R}_+ \colon w < x < +\infty\}$ and negative values in the domain $D_U = \{x \in \mathbb{R}_+ \colon g < x < w\}$, which ensures tracking of the trajectory $w(t)$ by $x(t)$ (see Fig. 3). In addition, the system trajectories will never enter the shaded domain: when approaching the boundary $g(t)$, we have $\rho(x,t) \to -\infty$ and, consequently, $x(t)$ moves along the surface (see Fig. 3). Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = -\alpha \ln(x - g)x^2 < 0$ for $x \in D_S$ and $\dot{V} > 0$ for $x \in D_U$.

Remark 1. Here, we study the possibility of analyzing dynamic systems with a discontinuous right-hand side in t and x, including the density function $\rho(x,t)$.

Consider first a nonautonomous system of the general form $\dot{x} = f(x,t)$ with $x \in \mathbb{R}^n$. Let the function $f(x,t)$ be defined in some open domain D of the variables (x,t). The function $f(x,t)$ is said to satisfy the Carathéodory condition [15] if it is continuous in x for almost all t and piecewise continuous in t for all x (assuming measurability in t is sufficient) and, for any compact set $G \subset D$, there exists a nonnegative integrable function $m(t)$ such that $|f(x,t)| \leqslant m(t)$ for all $(x,t) \in G$.

If the function $f(x,t)$ satisfies the Carathéodory condition, then by Theorems 1.1.1.1 and 1.1.4 of [15], for any initial conditions from the domain D, there exists a locally absolutely continuous solution $x(t)$ of the system $\dot{x} = f(x,t)$. The equation $\dot{x}(t) = f(x(t),t)$ holds for almost all t. The derivative of $x(t)$ may not exist for those t at which the function $f(x,t)$ suffers a jump in t. Furthermore, either the solution $x(t)$ is defined on $[0,+\infty)$, or for some finite t_0, the solution $x(t)$ tends to the boundary of the domain D as $t \to t_0$. If the function $f(x,t)$ is locally Lipschitz in x, then by Theorem 1.1.2 of [15] the solution is unique.

Consider now system (2) from Example 1. If the function $\rho(x,t)$ satisfies the Carathéodory condition, then this system has a locally absolutely continuous solution; if the function $\rho(x,t)$ is also locally Lipschitz in x, then this solution is unique.

We proceed to case 2 of Example 1. As the domain D, we define the set of all those (x,t) not belonging to the closures of the graphs of the functions $w(t)$ and $-w(t)$. For example, let $w(t) = 2$ for $t \in [0,1]$ and $w(t) = 1$ for $t > 1$; in this case, then not only the graph of $w(t)$ but also the point $(1,1)$ must be excluded from the domain D. The function $\rho(x,t)$ satisfies the Carathéodory condition and is locally Lipschitz in x. Therefore, the equation $\dot{x} = -\rho(x,t)x$ has a unique locally absolutely continuous solution that is either defined on the entire axis or, for some finite t_0, the distance from $(x(t),t)$ to the boundary of D will vanish as $t \to t_0$.

Consider the Lyapunov function $V(x) = 0.5x^2$. We take the solution $x(t)$ and examine the function $V(x(t))$ that is locally absolutely continuous. By the differentiability theorem of a complex function, $\dot{V} = -\rho(x(t),t)x(t)^2 \leqslant 0$ for almost all t. Due to absolute continuity, $V(t) - V(0) = \int_0^t \dot{V}(s)ds \leqslant 0$ for any t. (An absolutely continuous function can be reconstructed through the integral over its derivative; see the proof of this result in [16].) The inequality $V(t) \leqslant V(0)$ implies that the equilibrium is stable. Its asymptotic stability can be established using LaSalle's theorem for nonautonomous systems (Theorem 1 of [17]) if the set $\{x : |x| < w(t)\}$ contains a pipe of constant nonzero width and the function $w(t)$ is not infinitely increasing. If $w(t) \to 0$ as $t \to \infty$, then asymptotic stability simply follows from the fact that the solution remains in the domain D_S.

Here, however, a reserve concerning the piecewise continuity of $w(t)$ is required. It may happen that the solutions from the domain D_S will leave this domain. For example, let $w(t) = 2$ for $t \in [0,t_0]$ and $w(t) = 1$ for $t > t_0$. If t_0 is small enough and the initial condition $x(0)$ is close to 2 or -2, as $t \to t_0$, the trajectory $x(t)$ will simply smash into the wall formed by the jump of the function $w(t)$. The existence theorem guarantees that this solution is continuable further, but the stability analysis is not applicable in this case. Such a solution will jump out of the domain D_S and increase infinitely.

Under a continuous function $w(t)$, it is easy to show that the solutions with initial conditions from D_S will not leave the domain D_S. However, for a piecewise continuous function $w(t)$, this property generally fails. If the function $w(t)$ has many jumps, it may turn out that some solutions starting in D_S will jump out of this domain when smashing into the walls formed by jumps of the function $w(t)$. If the solution remains in the domain D_S, it will tend to the equilibrium.

When considering differential equations with a discontinuous right-hand side in x (or in t as well [18, 19]), it is necessary to understand the solutions of such systems in the Filippov sense [15]. In case 2 of Example 1, we can then consider the function $\rho(x,t) = \alpha[x(t) - w(t)]\mathrm{sgn}(x)$, where $\mathrm{sgn}(\cdot)$ is the sign function. The stability of discontinuous nonautonomous systems was studied in detail, e.g., in [18–21].

Thus, it is possible to examine below dynamic systems with a discontinuous right-hand side, including discontinuous functions $\rho(x,t)$. However, such systems complicate the analysis due to justifying the choice of initial conditions, the frequency and magnitude of jumps of the function, etc. Recall that this paper is devoted to studying the behavior of dynamic systems depending on the properties of the density function. For the sake of simplicity, all theoretical results will be therefore formulated for dynamic systems with a right-hand side continuous in t and locally Lipschitz in x. Still, for illustration purposes, some examples may contain discontinuous right-hand sides.

Well then, Example 1 has demonstrated how a density function $\rho(x,t)$ defined in the space (x,t) can qualitatively affect the transients of the original system (1). The next example shows that the density function need not be multiplied by the entire right-hand side, as in Example 1 (see (1) and (2)) but can be explicitly or implicitly present on the right-hand side of the system.

Example 2. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 - \rho_1(x,t)x_1, \\ \dot{x}_2 &= -x_1 - \rho_2(x,t)x_2, \end{aligned} \tag{3}$$

where $\rho_1(x,t)$ and $\rho_2(x,t)$ are continuous functions in t and x on $\mathbb{R}^2 \times [0, +\infty)$. We choose the quadratic function

$$V = 0.5(x_1^2 + x_2^2). \tag{4}$$

Taking the total time derivative of this function along the solutions of (3) yields

$$\dot{V} = -\rho_1 x_1^2 - \rho_2 x_2^2. \tag{5}$$

1. Let $\rho_1 = \rho_2 = \rho = \ln \frac{\overline{g} - |x_1|^\beta - |x_2|^\beta}{|x_1|^\beta + |x_2|^\beta - \underline{g}}$, where $\beta > 0$ (for $0 < \beta < 1$, see Remark 1) and $\overline{g}(t) > \underline{g}(t) > 0$ are continuous functions. We have $\rho = 0$ for $|x_1|^\beta + |x_2|^\beta = g$, where $g = 0.5(\underline{g} + \overline{g})$. Then $\dot{V} < 0$ for $\rho(x,t) > 0$, i.e., in the domain $D_S = \{x \in \mathbb{R}^2 : g < |x_1|^\beta + |x_2|^\beta < \overline{g}\}$, and $\dot{V} > 0$ for $\rho(x,t) < 0$, i.e., in the domain $D_U = \{x \in \mathbb{R}^2 : \overline{g} < |x_1|^\beta + |x_2|^\beta < g\}$. In this case, the density function $\rho(x,t)$ is explicitly present in system (3) but is not multiplied by the entire right-hand side, as in Example 1. Figure 4 shows the simulation results for $\beta = 1$, $\overline{g} = 3$, $\underline{g} = 2$ (left) and $\beta = 0.6$, $\overline{g} = 3^{0.6}$, $\underline{g} = 1$ (right) under $x(0) = col\{0,\ 2, 5\}$.

Hereinafter:

- The gray domains in the figures mean that the density function is chosen so that there are no solutions of the system in these domains. (At the boundary of such a domain, the density value increases to infinity.)
- The point curve corresponds to the zero value of the density function and, accordingly, this curve is the boundary separating the stable D_S and unstable D_U domains.

2. Let $\rho_1 = 1 - x_1^2$ and $\rho_2 = -x_1^2$. Then $\dot{V} = -\rho(x,t)x_1^2 < 0$ for $\rho(x,t) = x_1^2 + x_2^2 - 1 > 0$ in the domain $D_S = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 > 1 \text{ and } x_1 \neq 0\}$, and $\dot{V} > 0$ for $\rho(x,t) < 0$ in the domain

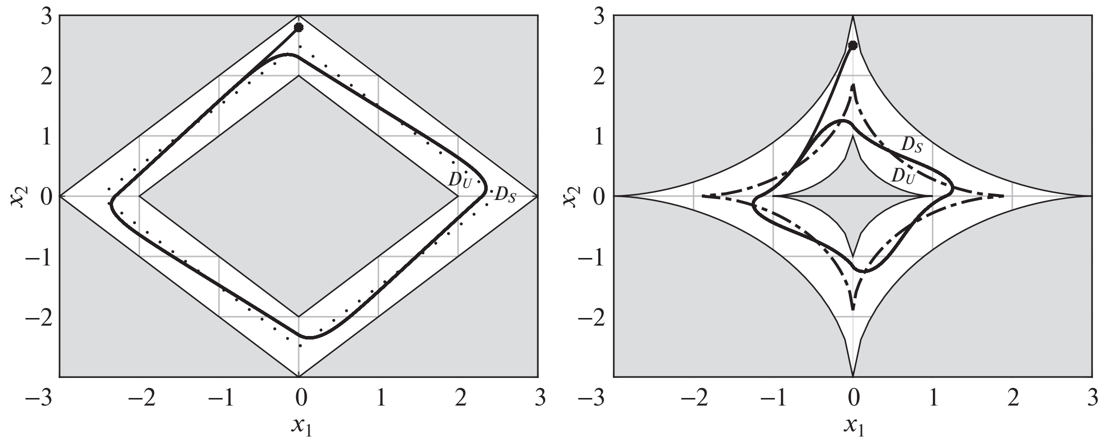

Fig. 4. The phase trajectory of system (3) with the density functions $\rho(x,t) = \ln \frac{3-|x_1|-|x_2|}{|x_1|+|x_2|-2}$ (left) and $\rho(x,t) = \ln \frac{3^{0.6}-|x_1|^{0.6}-|x_2|^{0.6}}{|x_1|^{0.6}+|x_2|^{0.6}-1}$ (right).

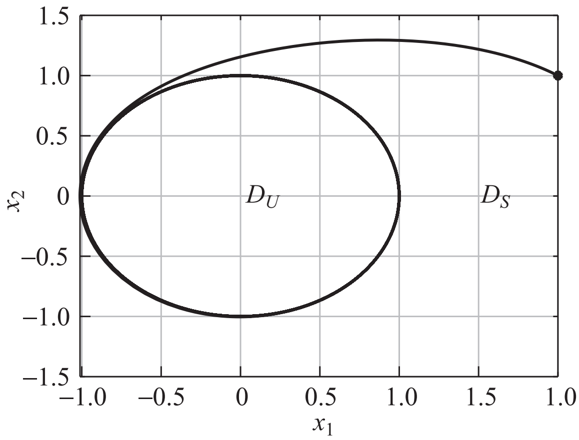

Fig. 5. The phase trajectory of system (3) with the density function $\rho(x,t) = x_1^2 + x_2^2 - 1$.

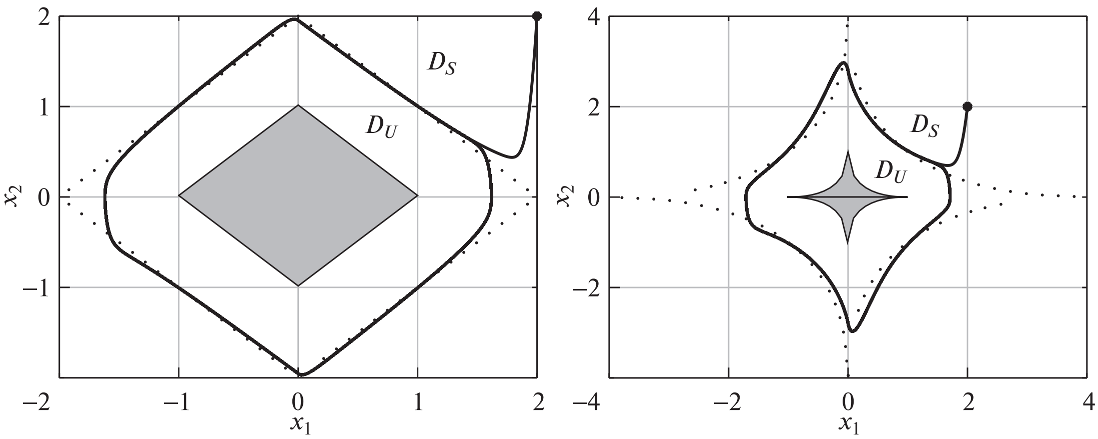

Fig. 6. The phase trajectory of system (3) with the density functions $\rho(x,t) = 20\ln(|x_1| + |x_2| - 1)$ (left) and $\rho(x,t) = 20\ln(|x_1|^{0.5} + |x_2|^{0.5} - 1)$ (right).

$D_U = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1 \text{ and } x_1 \neq 0\}$. In this case, unlike the previous one, the density function $\rho(x,t)$ is implicitly present in (3). However, in contrast to the previous case, the value of the density function does not affect system (3) everywhere (i.e., $\dot{V} = 0$ for $x_1 = 0$ regardless of the value of ρ). Figure 5 demonstrates the simulation results under $x(0) = col\{2, 1\}$.

3. Let $\rho_1 = \alpha \ln(|x_1|^\beta + |x_2|^\beta - 1)$, $\alpha > 0$, $\beta > 0$ (for $0 < \beta < 1$, see Remark 1), and $\rho_2 = 0$. Then $\dot{V} = -\rho(x,t)x_1^2 < 0$ for $\rho(x,t) = \alpha \ln(|x_1|^\beta + |x_2|^\beta - 1) > 0$ in the domain $D_S = \{x \in \mathbb{R}^2 : |x_1|^\beta + |x_2|^\beta > 2 \text{ and } x_1 \neq 0\}$, and $\dot{V} > 0$ for $\rho(x,t) < 0$ in the domain $D_U = \{x \in \mathbb{R}^2 : 1 < |x_1|^\beta + |x_2|^\beta < 2 \text{ and } x_1 \neq 0\}$. In this case, the density function $\rho(x,t)$ is present in just one of equations (3), in contrast to cases 1 and 2. However, as in case 2, the density function has no effect on system (3) everywhere. Figure 6 demonstrates the simulation results for $\beta = 1$ (left) and $\beta = 0.5$ (right) under $\alpha = 20$ and $x(0) = col\{2, 2\}$.

Consider now the dynamic system

$$\dot{x} = f(x,t), \tag{6}$$

where $t \geqslant 0$, $x \in D \subset \mathbb{R}^n$ denotes the state vector, and $f : D \times [0, +\infty) \to \mathbb{R}^n$ is a function continuous in t and locally Lipschitz in x on $D \times [0, +\infty)$. The possibility of studying systems (6) with a discontinuous right-hand side has been discussed in Remark 1.

Definition 1. System (6) is called a density system with a density function $\rho(x,t) : D \times [0, +\infty) \to \mathbb{R}$ if there exists a continuously differentiable function $V(x,t) : D \times [0, +\infty) \to \mathbb{R}$ such that:

(a) $w_1(x) \leqslant V(x,t) \leqslant w_2(x)$,

(b) $\dot{V} \leqslant \rho(x,t)W_1(x) \leqslant 0$ or $\dot{V} \geqslant \rho(x,t)W_2(x) \geqslant 0$

for any $t \geqslant 0$ and $x \in D$. Here, $\rho(x,t)$ is a function continuous in t and locally Lipschitz in x, $w_1(x)$ and $w_2(x)$ are positive definite functions, and $W_1(x)$ and $W_2(x)$ are continuous nonzero (except the equilibrium) functions in D.

Definition 2. If the functions $W_1(x)$ and $W_2(x)$ in Definition 1 are continuous in D, then system (6) is called a weak density system.

Definition 3. If $\dot{V} \leqslant \rho(x,t)W_1(x) <$ or $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in condition (b) of Definition 1, then system (6) is called a strict density system.

Definition 4. If $\dot{V} \leqslant \rho(x,t)W_1(x) \leqslant 0$ in the domain $D_S \times [0, +\infty)$, then the density function $\rho(x,t)$ and the domain D_S are said to be stable. If $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in the domain $D_U \times [0, +\infty)$, then the density function $\rho(x,t)$ and the domain D_U are said to be unstable.

Proposition 1. *Assume that system* (6) *is a strict density system in the domains D_S and D_U. If for each t, $V(x_s,t) - V(x_u,t) > 0$, where $x_s \in D_S$ and $x_u \in D_U$, then the system trajectories are attracted to the separation boundary of the domains D_S and D_U. If system* (6) *satisfies the condition $V(x_s,t) - V(x_u,t) < 0$ for each t, then the system trajectories move away from the separation boundary of the domains D_S and D_U.*

Proof. Let $V(x_s,t) - V(x_u,t) > 0$ for each $t \geqslant 0$, where $x_s \in D_S$ and $x_u \in D_U$. Since the system is a strict density system, by Definition 3 we have $\dot{V} \leqslant \rho(x,t)W_1(x) < 0$ in the domain D_S and $\dot{V} \geqslant \rho(x,t)W_2(x) > 0$ in the domain D_U. Hence, the separation boundary of the domains D_S and D_U is a set attracting the system trajectories.

Now let $V(x_s,t) - V(x_u,t) < 0$ for each $t \geqslant 0$, where $x_s \in D_S$ and $x_u \in D_U$. According to Definition 3, we have $\dot{V} \leqslant \rho(x,t)W_1(x) < 0$ in the domain D_S and $\dot{V} \geqslant -\rho(x,t)W_2(x) > 0$ in the domain D_U. Hence, the separation boundary of the domains D_S and D_U is a set left by the system trajectories.

Remark 2. In Proposition 1 and its proof, the attraction of trajectories to some set covers the cases where trajectories approach this set over time or belong to some neighborhood of this set.

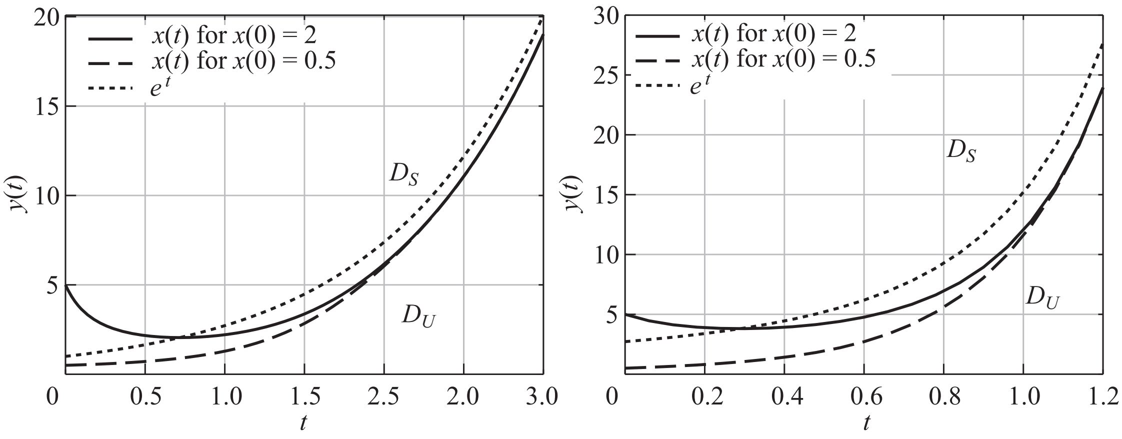

Fig. 7. $x(t)$ tracks the unbounded signals $w(t) = e^t$ (left) and $w(t) = e^{e^t}$ (right).

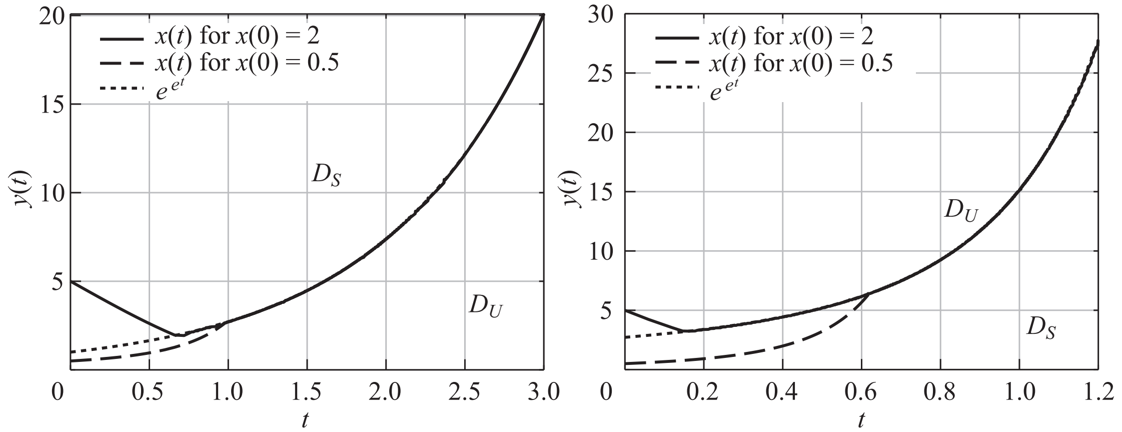

Fig. 8. $x(t)$ tracks the unbounded signals $w(t) = e^t$ (left) and $w(t) = e^{e^t}$ (right).

Note that the neighborhood size can remain constant or increase with time. It depends on the value of the space density. Here are some limiting cases:

- If in the neighborhood of the boundary separating the domains D_S and D_U the value of the density function decreases to zero, the system trajectories will not approach this boundary. Therefore, they can be in the neighborhood of this boundary or move far from it.
- If in the neighborhood of the boundary separating the domains D_S and D_U the value of the density function increases infinitely, the system trajectories will approach this boundary.

To illustrate Proposition 1 and Remark 2, we give another example as follows.

Example 3. Consider system (2) again but with $x \in \mathbb{R}_+$.

Choosing the quadratic function $V = 0.5x^2$ yields $\dot{V} = -\rho(x,t)x^2$. Hence, $\dot{V} < 0$ in the domain $D_S = \{x, t \in \mathbb{R}_+ : \rho(x,t) > 0\}$ and $\dot{V} > 0$ in the domain $D_U = \{x, t \in \mathbb{R}_+ : \rho(x,t) < 0\}$.

By Definition 3, system (2) is a strict density system for $x \in \mathbb{R}_+$. Analyzing Proposition 1, we fix an arbitrary time instant $t = t_1$. Then $V(x_s(t_1)) - V(x_u(t_1)) > 0$, and this inequality is obviously valid for any fixed t. According to Proposition 1, the system trajectories will be attracted to the separation boundary of the domains D_S and D_U.

Let the density function be $\rho(x,t) = x - w$, where $w(t) = e^t$. In this case, $\lim_{t\to\infty}(w(t) - x(t)) =$ const (see Fig. 7 on the left). If $w(t) = e^{e^t}$, the difference between $w(t)$ and $x(t)$ increases with time (see Fig. 7 on the right). This fact can be explained as follows: $w(t)$ is an unbounded function,

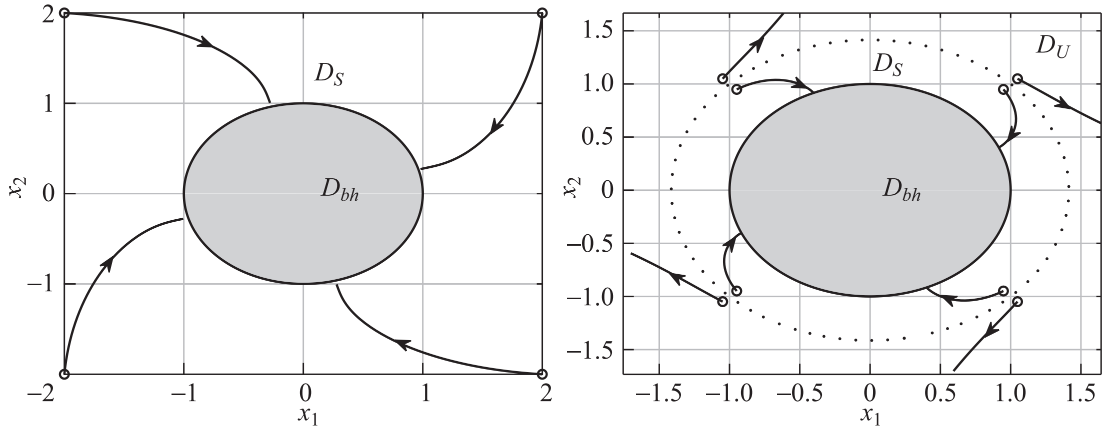

Fig. 9. The phase portrait of system (3) with the density functions $\rho(x) = e^{(x_1^2+x_2^2-1)^{-0.98}}$ (left) and $\rho(x) = -\ln(x_1^2 + x_2^2 - 1)$ (right).

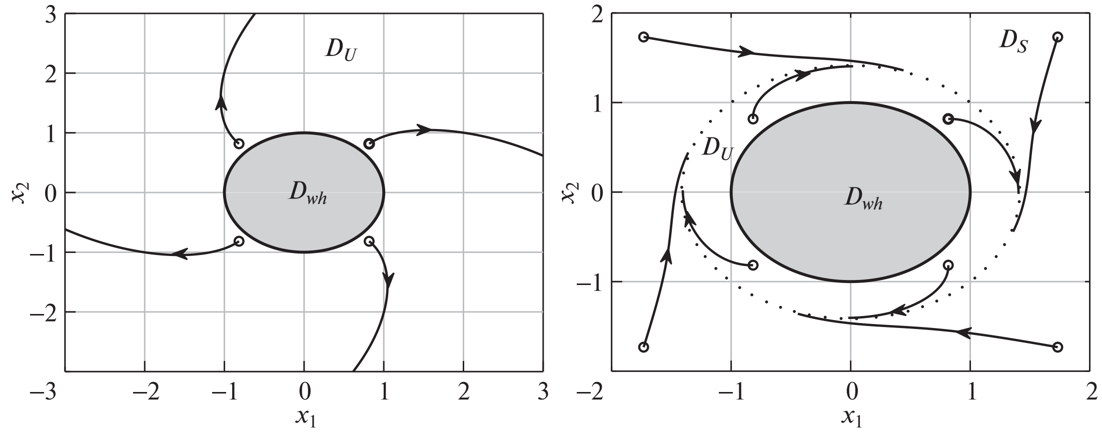

Fig. 10. The phase portrait of system (3) with the density functions $\rho(x) = -e^{(x_1^2+x_2^2-1)^{-0.98}}$ (left) and $\rho(x) = \ln(x_1^2 + x_2^2 - 1)$ (right).

whereas the density $|\rho(x,t)|$ decreases for x approaching w. As a result, $x(t)$ "tries to approach" $w(t)$ but fails due to the low density of the space in the neighborhood of $w(t)$ and a high rate of change of $w(t)$.

Let the density function be $\rho = w \arctan\frac{x-w}{\epsilon}$ (or $\rho = w\,\mathrm{sgn}(x-w)$ due to Remark 1), $\epsilon > 0$ be a sufficiently large number, and $w(t) = e^t$ or $w(t) = e^{e^t}$. In this case, the density of the space in the neighborhood of $w(t)$ grows with increasing $w(t)$, which ensures the approach of x to w as $t \to \infty$ (see Fig. 8).

When studying density systems, we will also distinguish special domains. They have been considered earlier as gray domains in the figures. Here is a rigorous definition.

Definition 5. If $\dot V \leqslant \rho(x,t)W_1(x) < 0$ in the neighborhood of a domain $D_{bh} \times [0, +\infty)$, there are no solutions of (6) in this domain, and the value of the density function grows infinitely when approaching it, then the domain D_{bh} is said to be absolutely stable.

Definition 6. If $\dot V \geqslant \rho(x,t)W_2(x) > 0$ in the neighborhood of a domain $D_{wh} \times [0, +\infty)$, there are no solutions of (6) in this domain, and the value of the density function grows infinitely when approaching it, then the domain D_{wh} is said to be absolutely unstable.

Example 4. Consider system (3) with $\rho_1(x,t) = \rho_2(x,t) = \rho(x)$. Choosing the quadratic function (4) yields $\dot{V} = -\rho(x)(x_1^2 + x_2^2)$.

If $\rho(x) = e^{(x_1^2+x_2^2-1)^{-0.98}}$, then all trajectories tend to the domain $D_{bh} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ (see Fig. 9 on the left). That is, from any initial conditions, the system trajectories will be attracted to D_{bh}, where the value of the density function grows to infinity when approaching the boundary of this domain. If $\rho(x) = -\ln(x_1^2 + x_2^2 - 1)$, the trajectories with initial conditions from the domain $x_1^2 + x_2^2 \geqslant \sqrt{2}$ will remain in this domain, but all trajectories with initial conditions from the domain $1 \leqslant x_1^2 + x_2^2 \leqslant \sqrt{2}$ cannot leave this region and will be attracted to the domain $D_{bh} = \{x \in \mathbb{R}^2 : x_1^2+x_2^2 \leqslant 1\}$ (see Fig. 9 on the right), where the density function grows to infinity when approaching the boundary of D_{bh}.

If $\rho(x) = -e^{(x_1^2+x_2^2-1)^{-0.98}}$, all trajectories will move away from the domain $D_{wh} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ (see Fig. 10 on the left) without approaching its boundary, where the density function will grow to minus infinity when approaching the boundary of D_{wh}. If $\rho(x) = \ln(x_1^2 + x_2^2 - 1)$, the trajectories with initial conditions from the domain $x_1^2 + x_2^2 \geqslant \sqrt{2}$ will remain in this domain, but all trajectories with initial conditions from the domain $1 \leqslant x_1^2 + x_2^2 \leqslant \sqrt{2}$ will move away from the domain $D_{wh} = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$ (see Fig. 10 on the right).

Remark 3. Let us explain the physical meaning of the systems under consideration. If the density function is explicitly present on the right-hand side of the system equation, e.g., in the form $\dot{x} = \rho(x,t)f(x,t)$, then the value of the space density directly affects the phase flow velocity. For example, for $\rho(x,t) = 1$, we have the original system $\dot{x} = f(x,t)$. If $\rho(x,t) > 0$, the presence of the density function possibly does not qualitatively affect the equilibria, but it may affect the phase portrait. Under $0 < \rho(x,t) < 1$, the value of the phase velocity vector decreases because the space density does so. In the case $\rho(x,t) > 1$, on the contrary, the value of the phase velocity vector increases because the space density does so. When the density function changes its sign, the phase portrait varies qualitatively.

Condition (b) of Definition 1 can be interpreted as the rate of change of the phase volume given by the function $V(x,t)$ considering the space density.

We now provide some models of real processes:

- The pendulum equation has the form $\dot{x}_1 = x_2$, $\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2$, where x_1 is the deviation angle of the pendulum from the vertical axis, x_2 is the angular velocity of the pendulum, g is the acceleration of gravity, l is the pendulum length, and k is the friction coefficient [22]. Choosing the Lyapunov function $V = \frac{g}{l}(1 - \cos x_1) + 0.5x_2^2$ (the total energy of the system) yields $\dot{V} = -\frac{k}{m}x_2^2$. Introducing the density function $\rho(x,t) = k$, we obtain persistent oscillations in the case of no friction ($\rho(x,t) = 0$) and damped oscillations otherwise ($\rho(x,t) \neq 0$).

- The types of reproduction models in [25] can be written as $\dot{x} = \rho(x)x$, where x denotes the biological population size. For $\rho(x) = k > 0$, we have the normal reproduction model; for $\rho(x) = kx$, the explosion model; for $\rho(x) = 1 - x$, the logistic curve model.

- Absolutely stable and absolutely unstable domains from Definitions 5 and 6 can be found as the simplest models of black holes and white holes, respectively, [26], which possess high density and gravity. Therefore, density systems can also be treated as *gravitational systems*, where $\rho(x,t)$ is a *gravity function*. In other words, the behavior of systems can be affected not by the density of the space but by the gravitational field produced by a dense body. In this regard, the control design principles based on the density function in the following sections can be considered control using the gravity function. All the mathematical descriptions and conclusions remain valid regardless of the name of the function $\rho(x,t)$ and the corresponding systems.

Remark 4. As has been noted in the Introduction, the concepts of density functions and density systems proposed in Section 2 and their properties generalize the results of [1–8, 10–14]. Let us discuss these issues in detail.

- In [1–8], the stability of an original system $\dot{x} = f(x)$ was analyzed by introducing a new system of the form $\dot{x} = \rho(x)f(x)$, which was (in turn) investigated using either divergent methods or the method of Lyapunov functions. In this section, we analyze systems of the form $\dot{x} = f(x)$ using the dependencies $\dot{V} \leqslant \rho(x,t)W_1(x)$ or $\dot{V} \geqslant \rho(x,t)W_1(x)$ (see Definition 1 and Example 2), which does not require multiplying the entire right-hand side of the system by the density function.
- In [10–14], the stability of systems $\dot{x} = f(x, \rho(x,t))$ with the density function explicitly appearing on the right-hand side was analyzed. The results proposed in this paper allow for the implicit presence of the density function on the right-hand side of the system (see case 2 in Example 2).
- In [10–14], the solutions of a system were placed into a given bounded set without forbidden domains by a special choice of the density function. Note that the boundaries of such sets were specified by functions continuously differentiable in t and x. In this paper, we allow for unbounded sets (see case 5 in Example 1 and case 3 in Example 2) with forbidden domains (see cases 4 and 5 in Example 1 as well as cases 1 and 3 in Example 2). Moreover, the boundaries of these sets can be specified by functions continuous in t and locally Lipschitz in x. If the solutions of systems are understood in the Filippov sense, then we can consider systems even with discontinuous right-hand sides; see Remark 1.

Thus, the new concepts of a density function and a density system introduced above provide a novel look at a certain class of dynamic systems, which is broader than those considered in [1–8, 10–14]. Also, the evolution of dynamic systems can now be considered and influenced by the density of the space.

The results obtained in this section can be used to analyze dynamic systems and, moreover, to design control laws. The next section will be devoted to this issue.

3. DENSITY CONTROL

We present several examples of designing control laws to obtain closed loop systems described by density systems.

3.1. Plants with Known Parameters

Consider a plant of the form

$$Q(p)y(t) = R(p)u(t), \tag{7}$$

where $y \in \mathbb{R}$ and $u \in \mathbb{R}$ denote the output and control signals, respectively, $Q(p)$ and $R(p)$ are linear differential operators with known constant coefficients, and $R(\lambda)$ is a Hurwitz polynomial.

If the relative degree of the plant (7) is 1 (i.e., $\deg Q(p) - \deg R(p) = 1$), then the control law

$$u(t) = -\frac{Q(p)}{pR(p)}\rho(y,t)y(t) \tag{8}$$

transforms system (7) into

$$\dot{y}(t) = -\rho(y,t)y(t), \tag{9}$$

which is structurally a density system. In particular, some density functions $\rho(y,t)$ have been defined in Example 1.

If the relative degree of the plant (7) exceeds 1 (i.e., $\deg Q(p) - \deg R(p) = \gamma > 1$), we write the control law (8) as

$$u(t) = -\frac{Q(p)}{pR(p)(\mu p + 1)^{\gamma-1}}\rho(y,t)y(t) \tag{10}$$

where $\mu > 0$ is a sufficiently small number. The resulting system takes the form

$$\dot{y}(t) = -\frac{1}{(\mu p + 1)^{\gamma-1}}\rho(y,t)y(t). \tag{11}$$

For $\mu = 0$, system (11) has the density system structure (9). If the solutions of the density system (9) with an appropriately chosen density function $\rho(y,t)$ are asymptotically stable, then [22, 24] there exists a sufficiently small number $\overline{\mu} > 0$ such that, for $0 < \mu < \overline{\mu}$, the solutions of system (11) are sufficiently close to the solution of system (9).

3.2. Plants with Unknown Parameters

Consider the plant (7) with the unknown parameters of the operators $Q(p)$ and $R(p)$ but a known value k. Assume that the relative degree of this plant is 1. All the results obtained can be extended to the plants with a relative degree above 1, e.g., using the schemes of [23]. In this paper, we focus on the plants with a relative degree of 1 only to avoid cumbersome considerations for overcoming the high relative degree problem.

Let the operators $Q(p)$ and $R(p)$ be written as $Q(p) = Q_m(p) + \Delta Q(p)$ and $R(p) = R_m(p) + \Delta R(p)$, where $Q_m(\lambda)$ and $R_m(\lambda)$ are arbitrary Hurwitz polynomials of degrees n and $n-1$, respectively, and the polynomials $\Delta Q(p)$ and $\Delta R(p)$ have degrees $n-1$ and $n-2$, respectively. Choosing $Q_m(\lambda)/R_m(\lambda) = \lambda + a$ with a known value $a > 0$ and taking the integer part of $\frac{\Delta Q(\lambda)}{Q_m(\lambda)} = k_{0y} + \frac{\Delta\tilde{Q}(\lambda)}{R_m(\lambda)}$, we transform (7) into

$$\dot{y}(t) = -ay(t) + k\left(u(t) + \frac{\Delta R(p)}{R_m(p)}u - \frac{\Delta\tilde{Q}(p)}{R_m(p)}y - k_{0y}y\right). \tag{12}$$

We introduce $c_0 = col\{c_{0y}, c_{0u}, k_{0y}\}$ as the vector of unknown parameters, where $\Delta\tilde{Q}(p) = c_{0y}^{\mathrm{T}}[1\ p\ \ldots\ p^{n-2}]$ and $\Delta R(p) = c_{0u}^{\mathrm{T}}[1\ p\ \ldots\ p^{n-2}]$, and the regression vector $w = col\{V_y, V_u, y\}$ constructed using the filters

$$\begin{aligned}\dot{V}_y &= FV_y + by,\\ \dot{V}_u &= FV_u + bu.\end{aligned} \tag{13}$$

Here, F is the Frobenius matrix with the characteristic polynomial $R_m(\lambda)$ and $b = col\{0,\ldots,0,1\}$.

With these notations, equation (12) can be written as

$$\dot{y}(t) = -ay(t) + k[u(t) - c_0^{\mathrm{T}}w(t)]. \tag{14}$$

Let the control law be given by

$$u(t) = c^{\mathrm{T}}(t)w(t) + \frac{a}{k}y(t) + \rho(y,t). \tag{15}$$

Substituting (15) into (14) yields the closed loop system

$$\dot{y}(t) = \rho(y,t) + k(c(t) - c_0)^{\mathrm{T}}w(t). \tag{16}$$

Theorem 1. *The control law* (15) *with the adaptation algorithm*

$$\dot{c} = -\alpha y w \tag{17}$$

transforms the plant (7) *into a density system. If we have a stable density $\rho(y,t)$ with a stable limit set in the neighborhood of zero as $t \to \infty$, then all signals in the closed loop system are bounded.*

Proof. Let us choose the quadratic function

$$V = \frac{1}{2}y^2 + \frac{k}{2\alpha}(c(t) - c_0)^{\mathrm{T}}(c(t) - c_0). \tag{18}$$

Calculating the total time derivative of (18) along the solutions of (16), (17), we write the result as

$$\dot{V} = \rho(y,t)y. \tag{19}$$

This is a density system.

If we have a stable density $\rho(y,t)$ with a stable limit set in the neighborhood of zero as $t \to \infty$, then $\rho(y,t)$ is chosen so that $\rho(y,t)y < 0$. Hence, $\lim\limits_{t\to\infty} y(t) = 0$. From (16) it then follows that $\lim\limits_{t\to\infty}(c(t) - c_0)^{\mathrm{T}}(t)w(t) = 0$. The ultimate boundedness of $V_y(t)$ is immediate from the first equation of (13), the ultimate boundedness of $y(t)$, and the Hurwitz property of the matrix F. Substituting (15) into the second equation of (13) yields

$$\begin{aligned}\dot{V}_u &= FV_u + bc_0^{\mathrm{T}}w + b(c - c_0)^{\mathrm{T}}w + b\frac{a}{k}y(t) + b\rho(y,t)\\ &= (F + bc_{0u})V_u + b\left[c_{0y}^{\mathrm{T}}V_y + k_{0y}y + b(c - c_0)^{\mathrm{T}}w + \frac{a}{k}y(t) + \rho(y,t)\right].\end{aligned}$$

The matrix $F + bc_{0u}$ has a Hurwitz characteristic polynomial $R(\lambda)$ by the problem statement. So, under bounded terms in square brackets, the function $V_u(t)$ is ultimately bounded. Then the regression vector $w(t)$ is also ultimately bounded. The condition $\lim\limits_{t\to\infty} y(t) = 0$, the ultimate boundedness of $w(t)$, and (17) imply $\lim\limits_{t\to\infty}\dot{c}(t) = 0$. Hence, $c(t)$ is an ultimately bounded function. From (15) it then follows that the control law is bounded. As a result, all signals in the closed loop system are bounded.

Example 5. Consider the plant (7) with unknown parameters of the operators $Q(p) = (p-1)^3$ and $R(p) = (p+1)^2$, $k = 1$ (known value), and the unknown initial conditions $p^2y(0) = 1$, $py(0) = 1$, and $y(0) = 4$.

We define $F = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ for the filters (13). Also, we choose $\alpha = 0.1$ in the adaptation algorithm (17) and $a = 1$ in the control law (15).

Consider different density functions $\rho(y,t)$ in (15).

1. For $\rho(y,t) = -\alpha y$, the closed loop system (16) has the equilibrium $y = 0$. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = -\alpha y^2 \leqslant 0$ in the domain $D_S = \mathbb{R}$. This is an active stabilization problem, described in detail in [23]. Figure 11 shows the transient for $\alpha = 1$, $p^2y(0) = py(0) = 0$, and $y(0) = 4$ (see the curve intersecting the grey domain).

2. For $\rho(y,t) = \alpha\ln\frac{g-y}{g+y}$, where $g(t) > 0$, the closed loop system (16) has the equilibrium $y = 0$. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = \alpha\ln\frac{g-y}{g+y}y < 0$ in the domain $D_S = \{y \in \mathbb{R} : -g < y < g\}$. Moreover, $\rho(y,t) \to -\infty$ as $y \to g$ and $\rho(y,t) \to +\infty$ as $y \to -g$. This is a stabilization problem with the symmetric constraints $-g$ and g. Figure 11 demonstrates the transients for $\alpha = 1$ (the curve inside the pipe with dashed boundaries), $p^2y(0) = py(0) = 0$, $y(0) = 4$, and $g(t) =$

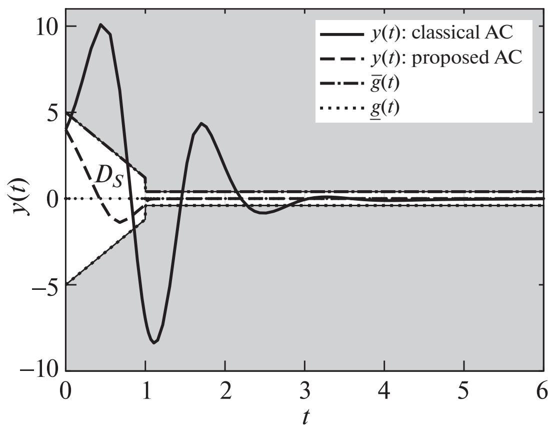

Fig. 11. Transients in the adaptive control system with the density functions $\rho(y,t) = -\alpha y$ [23] (the curve intersecting the grey domain) and $\rho(y,t) = \alpha \ln \frac{g-y}{g+y}$ (the curve inside the pipe with dashed boundaries).

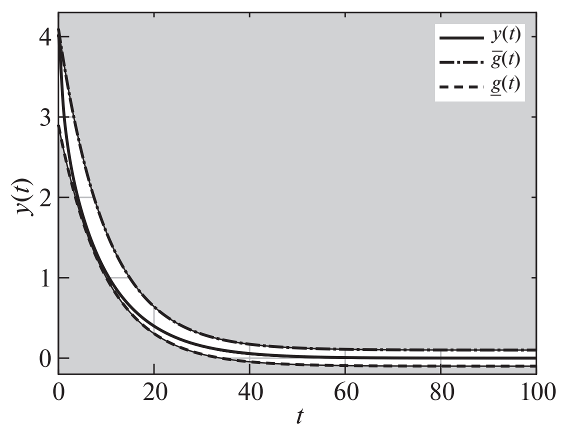

Fig. 12. Transients in the adaptive control system with the density function $\rho(y,t) = -\alpha \ln \frac{\overline{g}-y}{y-\underline{g}}$.

$\begin{cases} -4.6t+5, & t \leqslant 1 \\ 0.4, & t > 1. \end{cases}$ Clearly, in contrast to the classical adaptive control scheme [23] (the curve corresponding to $\rho(y,t) = -\alpha y$), the density function $\rho(y,t) = \alpha \ln \frac{g-y}{g+y}$ ensures a transient inside the pipe at any time instant.

3. Consider $\rho(y,t) = \alpha \ln \frac{\overline{g}-y}{y-\underline{g}}$, where $\overline{g}(t) > \underline{g}(t)$ for all t. Let us denote $y = w = \frac{\overline{g}+\underline{g}}{2}$. Then $\rho(y,t) = 0$ for $y = w$ and any t. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = \alpha \ln \frac{\overline{g}-y}{y-\underline{g}} y < 0$ in the domain $D_S = \{y \in \mathbb{R}_+ : w < y < \overline{g}\} \cup \left\{y \in \mathbb{R}_- : \underline{g} < y < w\right\}$ and $\dot{V} > 0$ in the domain $D_U = \left\{y \in \mathbb{R}_+ : \underline{g} < y < w\right\} \cup \{y \in \mathbb{R}_- : w < y < \overline{g}\}$. In addition, $\rho(y,t) \to -\infty$ as $y \to \overline{g}$ and $\rho(y,t) \to +\infty$ as $y \to \underline{g}$ if $y \in \mathbb{R}_+$; $\rho(y,t) \to +\infty$ as $y \to \overline{g}$ and $\rho(y,t) \to -\infty$ as $y \to \underline{g}$ if $y \in \mathbb{R}_-$. This is a stabilization problem with asymmetric constraints $\overline{g}$ and $\underline{g}$. Figure 12 shows the transient for $\alpha = 5$, $\overline{g} = 4e^{-0.1t} + 0.1$, $\underline{g} = 3e^{-0.1t} - 0.1$, $p^2 y(0) = py(0) = 0$, and $y(0) = 4$.

4. Let $\rho(y,t) = -\alpha(y - y_m)$. Then $\rho(y,t) = 0$ for $y = y_m$ and any t. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = -\alpha(y - y_m)y < 0$ in the domain $D_S = \{y \in \mathbb{R}_+ : y > y_m\} \cup \{y \in \mathbb{R}_- : y < y_m\}$ and $\dot{V} > 0$ in the domain $D_U = \{y \in \mathbb{R}_+ : y < y_m\} \cup \{y \in \mathbb{R}_- : y > y_m\}$. This is a tracking problem of y_m by y. Figure 13 demonstrates the transient for $\alpha = 50$, $y_m = e^{-0.1t}\sin(t)P(t)$, where

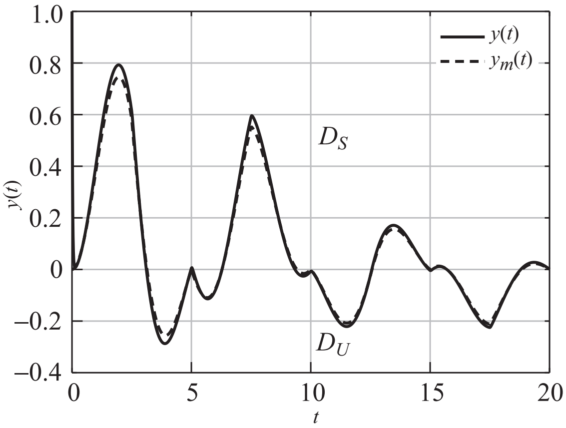

Fig. 13. Transients in the adaptive control system with the density function $\rho(y,t) = -\alpha(y - y_m)$.

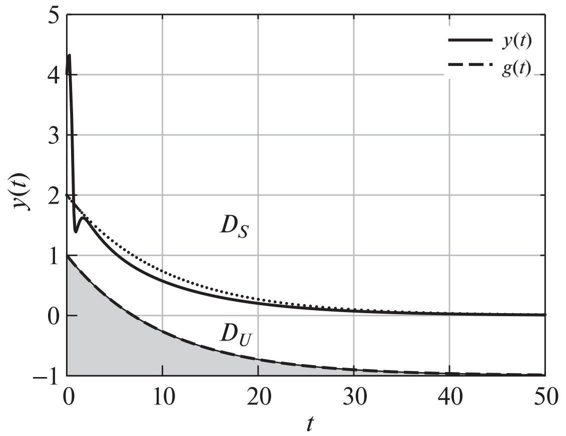

Fig. 14. Transients in the adaptive control system with the density function $\rho(y,t) = -\alpha \ln(y - g)$.

$P(t) \in [0, 1.25]$ is a generator of triangular pulses (with isosceles triangles) with a period of 5 s, $p^2y(0) = py(0) =0$, and $y(0) = 1$.

5. Consider $\rho(y,t) = -\alpha \ln(y - g)$, where $g(t) \geqslant -1$. Then $\rho(y,t) = 0$ for $y = g + 1$ and any t. Substituting $\rho(y,t)$ into (19) yields $\dot{V} = -\alpha \ln(y - g)y < 0$ in the domain $D_S = \{y \in \mathbb{R}_+ : y > g+1\}$ and $\dot{V} > 0$ in the domain $D_U = \{y \in \mathbb{R}_+ : y > g + 1\}$. In addition, $\rho(y,t) \to -\infty$ as $y \to g$. This is a descent problem of $x(t)$ long a surface with the boundary $g(t)$. Figure 14 shows the transient for $\alpha = 10$, $g = 2e^{-0.1t} - 1$, $p^2y(0) = py(0) = 0$, and $y(0) = 4$.

3.3. Comparing the Control Laws Proposed with Some Existing Ones

In this subsection, we compare the control laws (8), (10) and (13), (15), (17) with the control laws obtained using the method of barrier Lyapunov functions [27, 28], the method of funnel control [10, 12], and control methods with prescribed performance [11, 13, 14, 29].

- By the method and type of defining the target (admissible) set (a desired domain for the transients of the output signal in a closed loop system), the methods of [10–14, 27–29] and the proposed approach differ as follows:
  - — In [27, 28], the boundaries of the admissible set were constant whereas the reference signal was required to be smooth enough.

— In [10–12], the boundaries of the target set were defined by continuously differentiable functions symmetric with respect to the time axis. Only bounded target sets were considered therein.
— In [13, 14, 29], the boundaries of the target set were defined by continuously differentiable asymmetric functions. Only bounded target sets were considered therein.
— Within the approach proposed in this paper, target sets can be defined by continuous (or discontinuous, see Remark 1), asymmetric functions (see case 5 in Example 5). The target set may be unbounded (see cases 2, 3, and 5 in Example 5). The reference signal may be defined by continuous (see case 4 in Example 5) or piecewise continuous (see Remark 1) functions.

- By the control law design and stability analysis of the closed loop system, the methods of [10–14, 27–29] and the proposed approach differ as follows:
  — The methods [27, 28] involve special-form Lyapunov functions existing on a certain subset of the definitional domain of the system (an admissible set).
  — The methods [10–12] involve special-form density functions.
  — The approaches [11, 13, 14, 29] consider a nonlinear coordinate transformation reducing the original problem with constraints to an unconstrained problem. However, this transformation leads to the analysis of an extended system containing the variables of the original and new systems, which complicates the control law structure and the stability analysis of the closed loop system. Moreover, the nonlinear coordinate transformation leads to the study of the system $\dot{\varepsilon}(t) = \rho(\varepsilon, t) f(\varepsilon, t)$, where ε is a new variable and $\rho(\varepsilon, t) > 0$ is a density function depending on the derivative of the coordinate replacement function with respect to the variable ε. Hence, this system is a particular case of the systems considered in Section 2.
  — The approach proposed in the paper involves no coordinate transformation, which eliminates the need to consider additional variables and additional dynamic systems. The sign of the density function can be arbitrary. The Lyapunov function may exist in the entire definitional domain of the system.

4. CONCLUSIONS

This paper has considered a class of dynamic systems called density systems. Such systems contain on the right-hand side a density function defining the properties of the space. It is possible to affect the behavior of the system under study by determining the properties of the density function. This conclusion has been used to design control laws. As has been demonstrated above, different definitions of the density function lead to classical control laws and new ones with new target requirements for the system. In particular, an adaptive control law ensuring transients in a given set has been constructed as one example; classical adaptive control provides only the ultimate boundedness of system trajectories. In this case, the parameters of the set are specified using the density function, which defines the density of the space under consideration. The simulation results have confirmed the theoretical outcomes.

Also, it has been demonstrated how some existing control algorithms can be modified using the density function to obtain new transient quality. In the future, the properties of density systems can also be employed to design more complex control algorithms, e.g., output-feedback control with any relative degree of the plant, control using observers, sliding mode control, etc.

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REFERENCES

1. Krasnosel'skii, M.A., Perov, A.I., Povolotskii, A.I., and Zabreiko, P.P., *Plane Vector Fields*, New York: Academic Press, 1966.

2. Zhukov, V.P., Necessary and Sufficient Conditions for Instability of Nonlinear Autonomous Dynamic Systems, *Autom. Remote Control*, 1990, vol. 51, no. 12, pp. 1652–1657.

3. Zhukov, V.P., On the Divergence Conditions for the Asymptotic Stability of Second-Order Nonlinear Dynamical Systems, *Autom. Remote Control*, 1999, vol. 60, no. 7, pp. 934–940.

4. Rantzer, A., A Dual to Lyapunov's Stability Theorem, *Systems & Control Letters*, 2001, vol. 42, pp. 161–168.

5. Furtat, I.B., Divergent Stability Conditions of Dynamic Systems, *Autom. Remote Control*, 2020, vol. 81, no. 2, pp. 247–257.

6. Furtat, I.B. and Gushchin, P.A., Stability Study and Control of Nonautonomous Dynamical Systems Based on Divergence Conditions, *J. Franklin Institute*, 2020, vol. 357, no. 18, pp. 13753–13765.

7. Furtat, I.B. and Gushchin, P.A., Stability/Instability Study and Control of Autonomous Dynamical Systems: Divergence Method, *IEEE Access*, 2021, no. 9, pp. 49088–49094.

8. Furtat, I.B. and Gushchin, P.A., Divergence Method for Exponential Stability Study of Autonomous Dynamical Systems, *IEEE Access*, 2022, no. 10, pp. 49088–49094.

9. Landau, L.D. and Lifshitz, E.M., *Fluid Mechanics*, 2nd ed., vol. 6 of *Course of Theoretical Physics*, Butterworth-Heinemann, 1987.

10. Liberzon, D. and Trenn, S., The Bang-Bang Funnel Controller for Uncertain Nonlinear Systems with Arbitrary Relative Degree, *IEEE Trans. Autom. Control*, 2013, vol. 58, no. 12, pp. 3126–3141.

11. Bechlioulis, C. and Rovithakis, G., A Low-Complexity Global Approximation-Free Control Scheme with Prescribed Performance for Unknown Pure Feedback Systems, *Automatica*, 2014, vol. 50, no. 4, pp. 1217–1226.

12. Berger, T., Le, H., and Reis, T., Funnel Control for Nonlinear Systems with Known Strict Relative Degree, *Automatica*, 2018, vol. 87, pp. 345–357.

13. Furtat, I.B. and Gushchin, P.A., Control of Dynamical Plants with a Guarantee for the Controlled Signal to Stay in a Given Set, *Autom. Remote Control*, 2021, vol. 82, no. 4, pp. 654–669.

14. Furtat, I.B. and Gushchin, P.A., Nonlinear Feedback Control Providing Plant Output in Given Set, *Int. J. Control*, 2021. https://doi.org/10.1080/00207179.2020.1861336

15. Filippov, A.F., *Differential Equations with Discontinuous Righthand Sides*, Dordrecht: Springer, 1988.

16. Kolmogorov, A.N. and Fomin, S.V., *Elements of the Theory of Functions and Functional Analysis*, Dover, 1999.

17. LaSalle, J.P., Stability of Nonautonomous Systems, *Nonlinear Analysis, Theory, Methods & Applications*, 1976, vol. 1, no. 1, pp. 83–91.

18. Demyanov, V.F. and Rubinov, A.M., *Introduction to Constructive Nonsmooth Analysis*, Frankfurt a.M.–Bern–New York: Peter Lang Verlag, 1995.

19. Polyakov, A., Nonlinear Feedback Design for Fixed-Time Stabilization of Linear Control Systems, *IEEE Trans. Autom. Control*, 2012, vol. 57, no. 8, pp. 2106–2110.

20. Utkin, V.I., *Sliding Modes and Their Applications in Variable Structure Systems*, Moscow: Mir, 1978.

21. Dolgopolik, M.V. and Fradkov, A.L., Nonsmooth and Discontinuous Speed-Gradient Algorithms, *Nonlinear Anal. Hybrid Syst.*, 2017, vol. 25, pp. 99–113.

22. Khalil, H.K., *Nonlinear Systems*, 3rd ed., Pearson, 2001.

23. Miroshnik, I.V., Nikiforov, V.O., and Fradkov, A.L., *Nonlinear and Adaptive Control of Complex Systems*, Dordrecht–Boston–London: Kluwer Academic Publishers, 1999.

24. Vasil'eva, A.B. and Butuzov, V.F., *Asimptoticheskie razlozheniya reshenii singulyarno vozmushchennykh uravnenii* (Asymptotic Decompositions of Solutions of Singularly Perturbed Equations), Moscow: Nauka, 1973.

25. Arnold, V.I., *Ordinary Differential Equations*, Berlin–Heidelberg: Springer, 1992.

26. Carroll, S.M., *Spacetime and Geometry: An Introduction to General Relativity*, San Francisco: Addison Wesley, 2004.

27. Tee, K.P., Ge, S.S., and Tay, E.H., Barrier Lyapunov Functions for the Control of Output-Constrained Nonlinear Systems, *Automatica*, 2009, vol. 45, no. 4, pp. 918–927.

28. Azimi, V. and Hutchinson, S., Exponential Control Lyapunov-Barrier Function Using a Filtering-Based Concurrent Learning Adaptive Approach, *IEEE Trans. on Automatic Control*, 2022, vol. 67, no. 10, pp. 5376–5383.

29. Furtat, I.B., Gushchin, P.A., Nguyen, B.H., and Kolesnik, N.S., Adaptive Control with a Guarantee of a Given Performance, *Upravl. Bol'sh. Sist.*, 2023, no. 102, pp. 44–57.

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STOCHASTIC SYSTEMS

Approximate Method for Estimating Characteristics of Joint Service of Real-time Traffic and Elastic Data Traffic in Multiservice Access Nodes

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Abstract—A mathematical model of joint servicing of priority real-time traffic and elastic data traffic in multiservice access nodes has been constructed and studied. Definitions of quality indicators for joint servicing of incoming requests for information services are provided. A system of statistical equilibrium equations is formed and its use for calculating the exact values of the introduced characteristics is considered. A method for approximate calculation of characteristics is proposed, based on the construction of a system of simplified equilibrium equations. It has been established that the obtained estimates of customer service indicators are asymptotically accurate in the region of large and small losses. The use of the developed method is shown to solve the problem of estimating the volume of traffic offloaded in an overload situation to other access nodes or to other frequency ranges in order to achieve values of specified QoS indicators and to solve the problem of planning the volume of required transmission resource of a multiservice access node.

Keywords: multiservice traffic, real-time traffic, elastic traffic, recursive algorithms, asymptotically exact characteristics, offloading of excess traffic, transmission resource planning

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1. INTRODUCTION

The development of communication networks is moving towards expanding the number of services. The range of services received by the subscriber and their quality should not depend on where the subscriber is, how and at what speed he moves, and what access and information transfer technologies are used. This provision follows from the development trends of the telecommunications market and is enshrined in the recommendations of the International Telecommunication Union [1–6]. It is clear that each type of service, be it voice, video or data, has its own characteristics that should be taken into account when studying the conditions for their joint provision. In the simplest case, services differ only in the size of the required information transmission resource. In more complex situations, it is necessary to take into account the details of the formation of input streams of requests, the conditions of access to the network, the presence of priority and the possibility of resource redistribution during the service process, etc. The considered features of the functioning of current and future communication systems are analyzed in the family of so-called multiservice models. They are of great importance for practical applications and are the subject of intensive research by specialists in the field of communications [4–10].

The most important segment of a multiservice network of both fixed and mobile communications is the access node, which performs the function of concentrating subscriber traffic. Information flows entering access nodes can be divided into two categories: traffic of real-time services and traffic of

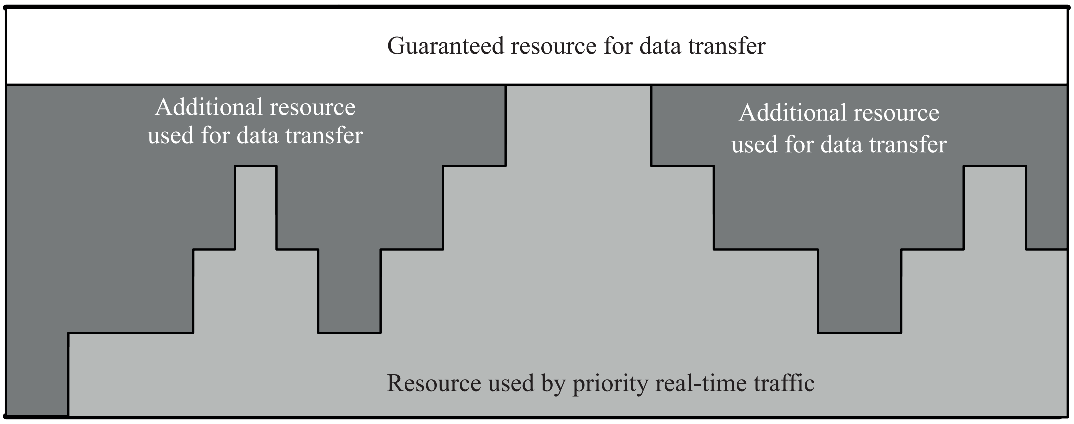

Fig. 1. Allocation of access node capacity during joint transmission of priority traffic of real-time services and elastic data.

elastic data transfer services. The transmission of real-time service traffic occurs with a preliminary reservation of a resource, which is retained for the entire duration of servicing the request [1–3, 6]. The quality of service is characterized by the proportion of lost requests and the average volume of occupied resources. To forward elastic traffic, a minimum guaranteed resource is allocated, as well as all or part of the resource that remains free from passing traffic from real-time services [8–11]. The quality of service of elastic data is measured by the average file transfer time and the average received throughput of the access node.

It is generally assumed that real-time traffic has resource occupancy advantage over elastic data. It is expressed in reducing the elastic data transmission rate to a predetermined minimum value if performing this procedure facilitates the acceptance of a request for a real-time service. When a free resource becomes available, the data transfer rate increases. Resource redistribution occurs at times determined by the mechanism used to adapt the speed of information transfer to the conditions of servicing requests. Usually these are the moments of receipt of requests and the end of their servicing. Let us call such a distribution procedure dynamic [7, 8, 10]. An example of its implementation is shown in Fig. 1.

The analyzed method of resource division allows to significantly increase its occupancy. This effect is especially important for cellular mobile communication networks due to the limited range of radio frequencies allocated for the formation of radio channels. According to experts, the gain can amount to several tens of percent of the total volume of the resource used. The noted effect is obtained as a result of the use of traffic control procedures that speed up the transmission of elastic data in situations where the number of telecommunications service users being served is decreasing. In mobile communication networks, dynamic resource allocation is performed by a packet scheduler and occurs under the control of the RRM (Radio Resource Management) software and hardware complex. Control decisions of the complex are made based on information about the state of the channel, the number and types of requests being serviced, the level of interference, etc. [12–15].

The relevance of the problem has attracted the attention of specialists of various professional backgrounds, which range from engineers involved in the design and operation of communication systems to mathematicians. Giving a general description of the published works, it should be noted that most of the studies are engineering developments based on the results of simulation modeling and full-scale experiments [1–3]. Theoretical analysis of the features of joint servicing of real-time traffic and elastic data usually ends with the construction of a mathematic model, determination of the characteristics of the quality of service of coming requests and the implementation of numerical algorithms for their assessment, based on solving a system of equilibrium equations using any standard linear algebra method [4, 5, 7–10, 13, 15, 16].

For practical applications, in particular for setting up procedures for dynamic resource allocation, it is necessary to construct approximate procedures that are easy to implement and have acceptable accuracy. Unfortunately, few such methods are presented in published works. We can only mention the studies [11, 14]. Of particular importance are approximate algorithms that

- are based on general principles and can be easily generalized to other models for the formation of input flows of requests and resource allocation procedures;
- have good accuracy for values of input parameters that correspond to practical applications, in particular for values of small losses, where the problem of planning the volume of information transmission resource required by the load is solved, and for values of large losses, where the problem is solved estimating the volume of traffic offloaded to other access nodes or to other frequency bands in order to achieve specified QoS indicators;
- are constructed using easy-to-implement analytical expressions and recursive procedures.

This work is devoted to solving these problems. Section 2 discusses the mathematical description of the analyzed access node model, using the example of which the principles of approximate assessment of the characteristics of the quality of joint service of real-time service traffic and elastic data traffic will be formulated. In Section 3, a system of equilibrium equations is constructed and an algorithm for its numerical solution is proposed. The results obtained are further used to estimate the error of approximate methods. In Sections 4 and 5, the principles of constructing approximate methods for estimating the characteristics of the considered access node model are formulated and their error is numerically studied, in particular, it is shown that the estimates are asymptotically accurate in the region of large and small losses. Section 6 discusses the possibility of using the model to solve the problem of determining the volume of offloaded traffic and the problem of estimating the required amount of resource. The last section formulates conclusions based on the results of the study.

The novelty of the results obtained is as follows:

- Methods are formulated for constructing approximate procedures for estimating the characteristics of joint servicing of priority traffic of real-time services and elastic data traffic. The proposed calculation procedures are based on the use of a system of simplified equilibrium equations.
- It is established that the obtained characteristics estimates are asymptotically accurate in the region of large and small losses.
- It is shown that the ideas underlying the approximate methods are of a general nature and can easily be generalized to other models for the formation of input flows of requests and resource allocation procedures.

2. MATHEMATICAL DESCRIPTION OF THE MODEL

Let us denote by C the capacity of a multiservice access node created by the fixed or wireless communication standard used and expressed in bits/s. The access node services n Poisson flows of requests for the transmission of real-time service traffic and one Poisson flow of requests for the transmission of elastic data. Requests of the kth flow for the transmission of real-time traffic arrive with an intensity of λ_k and require a capacity reservation of c_k bit/s for the entire service time, which has an exponential distribution with the parameter μ_k, $k = 1, \ldots, n$. Requests for the transfer of elastic data (files) are received with an intensity of λ_d. The file size has an exponential distribution with mean F expressed in bits.

Let us introduce the concept of a virtual channel, which is applicable to numerically estimate the size of the information transmission resource provided to users. Let us denote by c the information transmission rate of one channel, expressed in bits/s. The transition to virtual channels simplifies the modeling of the process of resource occupation by requests. The smaller the value of c, the more accurate the approximation of transmission rates. However, in this situation the number of model states increases. The choice of the value c depends on the formulation of the problem, for example, you can use the following expressions: $c = \min(c_1, \dots, c_n)$ or $c = \mathrm{GCD}(c_1, \dots, c_n)$, where abbreviation GCD means greatest common divisor. In the first case, the approximation of the bit rate requirements is rougher than in the second. The total number v of available virtual channels and the number b_k of virtual channels required to service the request of the kth flow are found from the relations:

$$v = \left\lfloor \frac{C}{c} \right\rfloor, \quad b_k = \left\lceil \frac{c_k}{c} \right\rceil.$$

Elastic data is maintained in accordance with the provisions of the Processor Sharing discipline. Let's consider the implementation of this procedure. For simplicity, let's assume that the minimum amount of bandwidth that can be used to transfer a file is one virtual channel. Let us denote by μ_d the parameter of the exponential distribution of file transmission time by one channel. Let i be the number of virtual channels used to service real-time traffic, and let $d > 0$—be the number of files being transmitted. According to the model description $(v - i)$ channels are used to serve them. Let $s = \left\lfloor \frac{v-i}{d} \right\rfloor$. Free channels are divided between d files according to the following rule. To serve each of the $(v - i - sd)$ files, $(s + 1)$ channels are used, and to serve each of the remaining $((s + 1)d - (v - i))$ files, s channels. As a result of this procedure, all $(v - i)$ channels are occupied. It is easy to show that in the situation under consideration, the time until the end of the transfer of one of the d files has an exponential distribution with the parameter $(v - i)\mu_d$.

An incoming request for real-time traffic transmission has priority in resource occupation, reducing, if necessary, the bandwidth used by one file to one channel. Let $i_k(t)$, $k = 1, \dots, n$—the number of requests of the kth flow for the transmission of traffic of real-time services at time t, and $d(t)$—the number of files being serviced at time t. The dynamics of changes in the number of serviced requests is described by the Markov process $r(t) = (i_1(t), \dots, i_n(t), d(t))$, defined on finite state space S, which includes states $(i_1, \dots, i_n, d)$, with components

$$\begin{aligned} &i_1 = 0, 1, \dots, \left\lfloor \frac{v}{b_1} \right\rfloor; \quad \dots \quad i_n = 0, 1, \dots, \left\lfloor \frac{v - i_1 b_1 - \dots - i_{n-1} b_{n-1}}{b_n} \right\rfloor; \\ &d = 0, 1, \dots, v - i_1 b_1 - \dots - i_n b_n. \end{aligned} \tag{1}$$

The quality of service for requests from the kth flow for the transmission of real-time traffic is determined by the values of the share of lost requests π_k and the average number of occupied virtual channels m_k. The value of the last characteristic makes it possible to calculate the average number of requests of the kth flow being serviced, $y_k = m_k/b_k$, and the average capacity of the access node they occupy $z_k = m_k c$. Let us denote by z_r the average amount of node capacity occupied by real-time traffic $z_r = \sum_{k=1}^{n} z_k$. The quality of service for requests for the transfer of elastic files is determined by the values of the share of lost files π_d, the average number of occupied virtual channels m_d, the average number of files being serviced y_d, the average value of used node bandwidth $z_d = m_d c$, average file transfer time h_d, average bitrate used for file transfer c_d, average number of virtual channels b used for file transfer.

The introduced indicators can be determined and calculated using the values of stationary probabilities $p(i_1, \dots, i_n, d)$ of states $(i_1, \dots, i_n, d) \in S$. For the state $(i_1, \dots, i_n, d)$, let i denote the number of virtual channels used to service real-time traffic $i = i_1 b_1 + \dots + i_n b_n$. Let us present the calculated expressions

$$\begin{aligned}
\pi_k &= \sum_{\{(i_1,\ldots,i_n,d)\in S \mid i+d+b_k>v\}} p(i_1,\ldots,i_n,d);\\
m_k &= \sum_{(i_1,\ldots,i_n,d)\in S} p(i_1,\ldots,i_n,d)i_k b_k; \quad k=1,\ldots,n;\\
\pi_d &= \sum_{\{(i_1,\ldots,i_n,d)\in S \mid i+d+1>v\}} p(i_1,\ldots,i_n,d);\\
m_d &= \sum_{\{(i_1,\ldots,i_n,d)\in S \mid d>0\}} p(i_1,\ldots,i_n,d)(v-i);\\
y_d &= \sum_{(i_1,\ldots,i_n,d)\in S} p(i_1,\ldots,i_n,d)d;\\
h_d &= \frac{y_d}{\lambda_d(1-\pi_d)}; \quad c_d = \frac{F}{h_d}; \quad b = \frac{m_d}{y_d}.
\end{aligned} \tag{2}$$

3. SYSTEM OF STATISTICAL EQUILIBRIUM EQUATIONS

To evaluate the characteristics specified by expressions (2), it is necessary find the values of $p(i_1,\ldots,i_n,d)\in S$[1]. To do this, it is enough to construct and solve a system of statistical equilibrium equations, connecting the unnormalized probabilities of the model. It looks like this:

$$\begin{aligned}
&P(i_1,\ldots,i_n,d)\left\{\sum_{k=1}^{n}\Big(\lambda_k\, I(i+d+b_k\leqslant v)+i_k\,\mu_k\Big)+\lambda_d\, I(i+d+1\leqslant v)+(v-i)\mu_d I(d>0)\right\} \quad (3)\\
&= \sum_{k=1}^{n} P(i_1,\ldots,i_k-1,\ldots,i_n,d)\,\lambda_k\, I(i_k>0)+P(i_1,\ldots,i_n,d-1)\,\lambda_d\, I(d>0)\\
&\quad+\sum_{k=1}^{n} P(i_1,\ldots,i_k+1,\ldots,i_n,d)\,(i_k+1)\,\mu_k\, I(i+d+b_k\leqslant v)\\
&\quad+P(i_1,\ldots,i_n,d+1)\,(v-i)\,\mu_d\, I(i+d+1\leqslant v), \qquad (i_1,\ldots,i_n,d)\in S.
\end{aligned}$$

Here and below $I(\cdot)$—indicator function defined by relation

$$I(\cdot)=\begin{cases}1, & \text{if the condition is met, formulated in brackets,}\\ 0, & \text{if this condition is not met.}\end{cases} \tag{4}$$

Resulting solutions of (3) $P(i_1,\ldots,i_n,d)$ needs to be normalized.

The system of equations (3) does not have any special properties that provide a recursive estimate of stationary probabilities. For this reason, standard linear algebra methods are used to solve (3). Based on the experience of solving similar systems, it is recommended to use the iterative Gauss–Seidel algorithm to estimate the probabilities of states. The use of this approach makes it possible to calculate stationary probabilities for models of communication systems with a number of states of up to several million. The standard implementation of Gauss–Seidel recursion for the solution (3) is not guaranteed to converge, but in most cases it does. Convergence is studied by indirect methods based on the analysis of the proximity of successive approximations and the fulfillment of known theoretical relationships connecting the values of the characteristics of the studied model of the

[1] We will use lowercase letters to denote the normalized values of the probabilities of states and characteristics and uppercase letters to denote their non-normalized values.

communication system. Such relations include Little's formula. Details of the implementation of this approach can be found in [10, 17–19]. To ensure convergence, it is enough to set one of the unknowns equal to unity, remove the corresponding equation (3) and proceed to solving an inhomogeneous system of linear equations. After the transformations performed, the Gauss–Seidel recursion always converges (due to the presence of weak diagonal dominance), but requires a significantly larger number of iterations for its implementation.

The Gauss–Seidel algorithm will be further used to estimate the error of the approximate method constructed in Section 4 for calculating the quality characteristics of joint servicing of incoming requests for the node model under study. Sufficiently substantiated approximate methods play a major role in the development of engineering techniques aimed at solving the problems of estimating the maximum sufficient load and the minimum required volume of information transmission resource of a multiservice access node. Subsequent sections of the work will be devoted to the formulation, analysis and examples of the use of approximate methods for calculating the characteristics of the constructed access node model.

4. ESTIMATION OF CHARACTERISTICS USING SIMPLIFIED EQUILIBRIUM EQUATIONS

The principle of simplified equilibrium equations was first formulated and used in constructing the boundaries of truncated state spaces, providing a given error in calculating the characteristics of models, taking into account the effect of repeated calls [20, 21]. The idea of the method is to form a system of relationships for the approximate calculation of stationary probabilities of model states based on the requirement that they comply with local conservation laws, which for the model under study are satisfied in individual macrostates, selected in accordance with the structure of the state space used and the physical meaning of individual state components.

For the constructed model, we refer to such macrostates as (i), where all states of the model are combined with the number of occupied virtual channels equal to i, $i = 0, 1, \ldots, v$, and (d), where all states of the model are combined with the number of transferred files equal to d, $d = 0, 1, \ldots, v$. Proceeding in a similar way, we also introduce the macrostate (i, d) and determine the stationary probability of this state from the expression

$$p(i,d) = \sum_{\left\{(i_1,\ldots,i_n,d)\in S \;|\; i_1b_1+\ldots+i_nb_n=i\right\}} p(i_1,\ldots,i_n,d),$$
$$i = 0,1,\ldots,v;\;\; d = 0,1,\ldots,v-i.$$

If the values of $p(i, d)$ are known, then the values of the characteristics specified by the expressions (2) can be calculated. To do this, it is enough to use the relations

$$\pi_k = \sum_{i=0}^{v}\sum_{d=0}^{v-i} p(i,d)I(i+d+b_k > v); \qquad m_k = \frac{\lambda_k}{\mu_k}\left(1-\pi_k\right)b_k; \tag{5}$$
$$\pi_d = \sum_{i=0}^{v}\sum_{d=0}^{v-i} p(i,d)I(i+d+1 > v); \qquad m_d = \sum_{i=0}^{v}\sum_{d=1}^{v-i} p(i,d)(v-i);$$
$$y_d = \sum_{i=0}^{v}\sum_{d=1}^{v-i} p(i,d)d.$$

The remaining elastic traffic service characteristics are determined from expressions (2).

Let us obtain the form of local conservation laws for the macrostates (i), (d), which will be further used for an approximate estimate of $p(i, d)$. Let's start with the macrostate (d). Equating the intensity of the exit $r(t)$ from (d) to the intensity of the transition to the macrostate (d), we obtain a system of relations of the following form:

$$\begin{gathered}\Big(P(0,d)+P(1,d)+\ldots+P(v-d-1,d)\Big)\lambda_d \\ +\Big(P(0,d)v+P(1,d)(v-1)+\ldots+P(v-d,d)d\Big)\mu_d I(d>0) \\ =\Big(P(0,d-1)+P(1,d-1)+\ldots+P(v-d,d-1)\Big)\lambda_d I(d>0) \\ +\Big(P(0,d+1)v+P(1,d+1)(v-1)+\ldots+P(v-d-1,d+1)(d+1)\Big)\mu_d, \\ d=0,1,\ldots,v-1.\end{gathered} \tag{6}$$

Having considered (6) sequentially for $d=0,1,\ldots,v-1$ and performing simple algebraic transformations, we reduce (6) to the following form:

$$\begin{gathered}\Big(P(0,d)+P(1,d)+\ldots+P(v-d-1,d)\Big)\lambda_d \\ =\Big(P(0,d+1)v+P(1,d+1)(v-1)+\ldots+P(v-d-1,d+1)(d+1)\Big)\mu_d, \\ d=0,1,\ldots,v-1.\end{gathered} \tag{7}$$

Now we obtain the form of local conservation laws for the macrostate (i). Equating the intensity of the exit $r(t)$ from (i) to the intensity of the transition to the macrostate (i), we obtain a system of relations of the following form:

$$\begin{gathered}\sum_{\{(i_1,\ldots,i_n,d)\in S\,|\,i_1b_1+\ldots+i_nb_n=i\}}\sum_{k=0}^{n}\Big(P(i_1,\ldots,i_k-1,\ldots,i_n,d)\lambda_k I(i_k>0) \\ +P(i_1,\ldots,i_k+1,\ldots,i_n,d)(i_k+1)\mu_k I(i+b_k+d\leqslant v)\Big) \\ =\sum_{\{(i_1,\ldots,i_n,d)\in S\,|\,i_1b_1+\ldots+i_nb_n=i\}}\sum_{k=0}^{n}\Big(P(i_1,\ldots,i_n,d)\lambda_k I(i+b_k+d\leqslant v) \\ +P(i_1,\ldots,i_n,d)i_k\mu_k I(i_k>0)\Big),\qquad i=0,1,\ldots,v.\end{gathered} \tag{8}$$

Let's move on to constructing a system of simplified equilibrium equations, which we will further use to approximately calculate the characteristics of the access node model under study. Let us denote by $\hat{P}(i_1,\ldots,i_n,d)$, $(i_1,\ldots,i_n,d)\in S$ the resulting estimates of the probabilities of stationary states of the considered model. Let us save for estimates of the introduced characteristics (2), probabilities $P(i,d)$, etc., obtained using $\hat{P}(i_1,\ldots,i_n,d)$, previously introduced notations, just add the symbol $\hat{}$ to their notation. The quantities $\hat{P}(i_1,\ldots,i_n,d)$ are found from the fulfillment requirement for $\hat{P}(i_1,\ldots,i_n,d)$ local conservation law (7)

$$\begin{gathered}\Big(\hat{P}(0,d)+\hat{P}(1,d)+\cdots+\hat{P}(v-d-1,d)\Big)\lambda_d \\ =\hat{P}(0,d+1)v\mu_d+\hat{P}(1,d+1)(v-1)\mu_d+\ldots+\hat{P}(v-d-1,d+1)(d+1)\mu_d, \\ d=0,1,\ldots,v-1,\end{gathered} \tag{9}$$

and relations

$$\begin{gathered}\hat{P}(i_1,\ldots,i_n,d)i_k=\hat{P}(i_1,\ldots,i_k-1,\ldots,i_n,d)I(i_k>1)a_k, \\ (i_1,\ldots,i_n,d)\in S,\qquad a_k=\frac{\lambda_k}{\mu_k},\ k=1,\ldots,n.\end{gathered} \tag{10}$$

The expressions (10) coincide in form with the detailed balance relations in the formation and servicing of requests for the transmission of real-time traffic in the Erlang multiservice model [7, 10, 22, 23]. They determine the approximate nature of the resulting estimates, since they are not satisfied for the original model.

Using equalities (10), we can show that for $\hat{P}(i_1, \ldots\ldots, i_n, d)$ the local conservation law (8) is satisfied, i.e. the following relations are valid

$$\sum_{\{(i_1,\ldots,i_n,d)\in S \mid i_1b_1+\ldots+i_nb_n=i\}} \sum_{k=1}^{n} \hat{P}(i_1,\ldots,i_k-1,\ldots,i_n,d)\lambda_k I(i_k>0) \tag{11}$$
$$+ \hat{P}(i_1,\ldots,i_k+1\ldots,i_n,d)(i_k+1)\mu_k I(i+b_k+d\leqslant v)$$
$$= \sum_{\{(i_1,\ldots,i_n,d)\in S \mid i_1b_1+\ldots+i_nb_n=i\}} \sum_{k=1}^{n} \hat{P}(i_1,\ldots,i_n,d)\lambda_k I(i+b_k+d\leqslant v)$$
$$+ \hat{P}(i_1,\ldots,i_n,d)i_k\mu_k I(i_k\geqslant v), \qquad i=0,1,\ldots,v.$$

Next, we will show that the relations (9), (10) allow us to uniquely determine the values of $\hat{P}(i_1,\ldots,i_n,d)$, $(i_1,\ldots,i_n,d)\in S$, and with them the introduced estimates of the characteristics of joint servicing of requests in the studied model of the access node. Thus, estimates of stationary probabilities $\hat{P}(i_1,\ldots,i_n,d)$, $(i_1,\ldots,i_n,d)\in S$, as well as their exact values $P(i_1,\ldots,i_n,d)$, $(i_1,\ldots,i_n,d)\in S$ satisfy local conservation laws of the same form (7), (8) and (9), (11), which allows us to expect good estimation accuracy. Based on the probabilistic interpretation (9), (10), let's call these relations as system of simplified equilibrium equations.

Let us construct a recursive algorithm for calculating the values of $\hat{P}(i,d)$ for a fixed d. Let us introduce an auxiliary characteristic

$$\hat{Y}_k(i,d) = \sum_{\{(i_1,\ldots,i_n,d)\in S \mid i_1b_1+\ldots+i_nb_n=i\}} \hat{P}(i_1,\ldots,i_n,d)i_k, \quad i=0,1,\ldots,v;\ d=0,1,\ldots,v-i.$$

Summing (10) over all $(i_1,\ldots,i_n,d)\in S$ satisfying the condition $i_1b_1+\ldots+i_nb_n=i$, we obtain the equality

$$\hat{Y}_k(i,d) = \hat{P}(i-b_k,d)a_k I(i\geqslant b_k). \tag{12}$$

Let's multiply (12) by b_k and sum over $k=1,2,\ldots,n$. Rearranging the order of summation on the left, we obtain a recursive relation connecting successive values $\hat{P}(i,d)$,

$$\sum_{\{(i_1,\ldots,i_n,d)\in S \mid i_1b_1+\ldots+i_nb_n=i\}} \hat{P}(i_1,\ldots,i_n,d)\sum_{k=1}^{n} i_kb_k = \hat{P}(i,d)\,i \tag{13}$$
$$= \sum_{k=1}^{n} a_k\, b_k\, \hat{P}(i-b_k,d)I(i\geqslant b_k), \quad d=0,1,\ldots,v-1; \quad i=1,\ldots,v-d.$$

Relations (7) and (13) allow us to construct a recursive algorithm for estimating the values of $\hat{P}(i,d)$, $d=0,1,\ldots,v$; $i=1,\ldots,v-d$. Let us list the sequence of actions for its implementation.

(1) Let's set $d=0$, and the value $\hat{P}(0,0)=1$.

(2) Let's express values $\hat{P}(i,0)$, $i=1,\ldots,v$, through $\hat{P}(0,0)$, using the relation (13) with $d=0$

$$\hat{P}(i,0) = \frac{1}{i}\times\sum_{k=1}^{n}\hat{P}(i-b_k,0)I(i\geqslant b_k)$$

and successively increasing i from 1 to v. For fixed i values of estimates $\hat{P}(i-b_k, 0)$, $k = 1, \ldots, n$, or already expressed through $\hat{P}(0,0)$ (for $i - b_k \geqslant 0$), or equal to 0 (for $i - b_k < 0$).

(3) Let's put $d = 1$ and the value $\hat{P}(0,1) = x$.

(4) Let us express values $\hat{P}(i,1)$, $i = 1, \ldots, v-1$, through x, using the relation (13) with $d = 1$

$$\hat{P}(i,1) = \frac{1}{i} \times \sum_{k=1}^{n} \hat{P}(i-b_k, 1) I(i \geqslant b_k)$$

and successively increasing i from 1 to $v-1$.

(5) Set in (9) $d = 0$. We get the ratio

$$\Big(\hat{P}(0,0) + \hat{P}(1,0) + \cdots + \hat{P}(v-1,0)\Big)\lambda_d \tag{14}$$
$$= \hat{P}(0,1)v\mu_d + \hat{P}(1,1)(v-1)\mu_d + \ldots + \hat{P}(v-1,1)\mu_d,$$

which allows us to express the value of x in terms of $\hat{P}(0,0)$. Using the obtained relation and the results of step 4, we obtain expressions for $\hat{P}(i,1)$, $i = 0, 1, \ldots, v-1$, through $\hat{P}(0,0)$.

(6) Next, we set $d = 2, 3, \ldots, v$ and, having implemented the sequence of actions formulated in the above stages of the proposed algorithm, we find expressions for $\hat{P}(i,d)$, $d = 0, 1, \ldots, v$; $i = 1, \ldots, v-d$ via $\hat{P}(0,0)$.

(7) We find an expression in terms of $\hat{P}(0,0)$ for the normalization constant

$$N = \sum_{d=0}^{v} \sum_{i=0}^{v-d} \hat{P}(i,d).$$

(8) We determine the normalized values of probability estimates $\hat{p}(i,d)$:

$$\hat{p}(i,d) = \frac{\hat{P}(i,d)}{N}, \quad d = 0, 1, \ldots, v; \quad i = 1, \ldots, v-d.$$

(9) Using $\hat{p}(i,d)$ and expressions (5), we find the values of estimates of the characteristics of joint servicing of requests in the studied model of a multiservice access node.

In Fig. 2, arrows indicate the sequence of calculating $\hat{p}(i,d)$ using the constructed algorithm for $v = 5$.

From a computational point of view, the implementation of the constructed algorithm does not cause any difficulties. The complexity of the procedure is comparable to the repeated use of a recursive algorithm for assessing the characteristics of a multiservice Erlang model [7, 10, 22, 23].

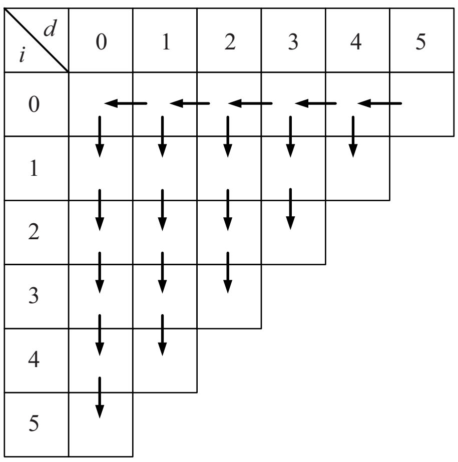

Fig. 2. Sequence of implementation of the algorithm for $v = 5$.

Let's compare the amount of calculations when estimating the characteristics of the model using the Gauss–Seidel method and the constructed approximate algorithm. At each step of the iterative solution method (3), the approximate values of the probabilities of all states of the model $(i_1, \ldots, i_n, d) \in S$ belonging to the space S are determined, given by the relations (1). As a result of applying the approximate method, probability estimates $P(i,d)$ are found for states $(i,d) \in \hat{S}$, where $i = 0, 1, \ldots, v$; $d = 0, 1, \ldots, v - i$ (see Fig. 2) and calculations are performed only once. From this follows an approximate assessment of the effectiveness of the developed method. The volume of calculations in comparison with the iterative method is reduced by a number of times equal to the number of iterations in the implementation of the Gauss–Seidel method, multiplied by the ratio of the number of states in the space S to the number of states in the space $\hat{S}$. Depending on the model parameters and calculation conditions, this value can significantly exceed several thousand times.

5. ERROR IN ESTIMATING THE CHARACTERISTICS OF AN ACCESS NODE

Let us conduct a numerical study of the accuracy of the approximate method for estimating the characteristics of an access node, constructed in the previous section. Let's choose the following values of the input parameters: $C = 100$ Mbit/s; $n = 2$; $c_1 = 2$ Mbit/s; $c_2 = 5$ Mbit/s. Based on the accepted assumptions, we obtain the structural parameters of the model: $c = 1$ Mbit/s; $v = 100$ virtual channels (v.c.); $b_1 = 2$ v.c.; $b_2 = 5$ v.c. Let's assume that $F = 80$ Mbit. The average file transfer time using one channel is 80 s. When performing calculations, this time will be taken as time unit. Hence $\mu_d = 1$. For real-time services, we select service time parameters from the expressions $\mu_1 = 0.5$ and $\mu_2 = 0.5$.

Let's introduce the ρ parameter, which we will use to estimate the potential load of one virtual channel. Let us determine ρ from the expression

$$\rho = \frac{a_1 b_1 + a_2 b_2 + \frac{\lambda_d}{\mu_d}}{v}. \tag{15}$$

For elastic data, we calculate the potential resource load from the condition of using one channel for file transfer[2]. We will assume that all three flows create the same potential resource load. From here follow the expressions for estimating the intensities of incoming requests:

$$\lambda_1 = \frac{v\rho\mu_1}{3b_1}; \quad \lambda_2 = \frac{v\rho\mu_2}{3b_2}; \quad \lambda_d = \frac{v\rho\mu_d}{3}. \tag{16}$$

For selected values of input parameters, Figs. 3–8 shows the dependence of the exact and approximate calculation of the main characteristics of the model from changes in ρ potential load of the virtual channel. The exact values of the characteristics are obtained from solving the system of equilibrium equations (3) using the iterative Gauss–Seidel method and using definitions (2). Approximate values were found from the relations (5) after substituting into them instead of $p(i,d)$ the estimates $\hat{p}(i,d)$ obtained as a result of implementing a recursive algorithm based on the use of simplified equilibrium equations (9), (10) (see Section 4). The corresponding curves are indicated by dotted lines. The number next to the curve indicates the number of the flow for real-time services. In Figs. 3–4 the error in calculating the share of lost requests for servicing real-time service traffic (Fig. 3) and the share of lost requests for servicing elastic data (Fig. 4) was estimated. In Figs. 5–6, the error in calculating the average value of the bitrate used for servicing traffic of real-time services (Fig. 5) and the average value of the bitrate used for servicing elastic data (Fig. 6) was estimated. In Figs. 7–8 the error in calculating the average file transfer time (Fig. 7) and the average number of channels used to transfer one file (Fig. 8) was estimated.

[2] Worst case scenario for servicing elastic data.

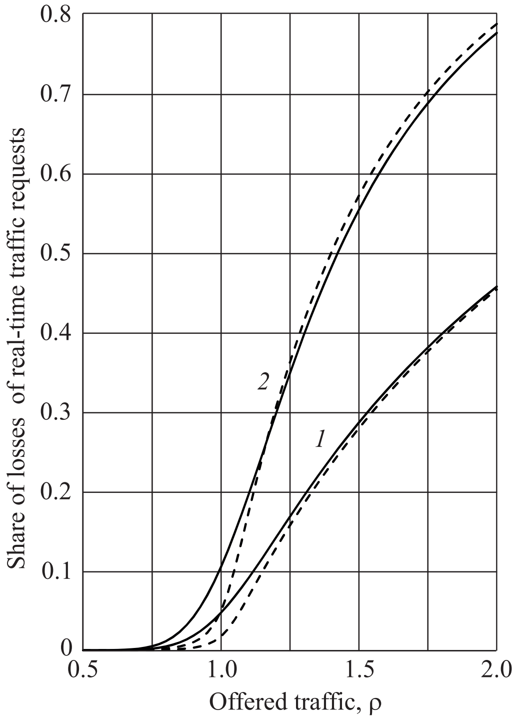

Fig. 3. Error in estimating the share of lost requests for servicing real-time traffic.

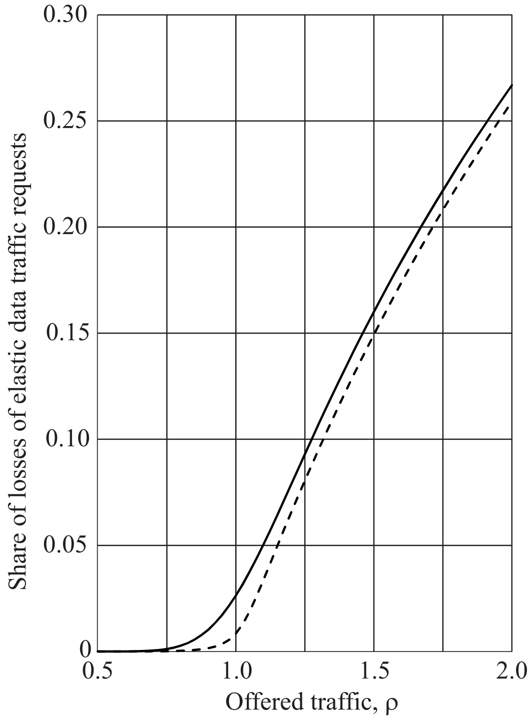

Fig. 4. Error in estimating the share of lost requests for servicing elastic data.

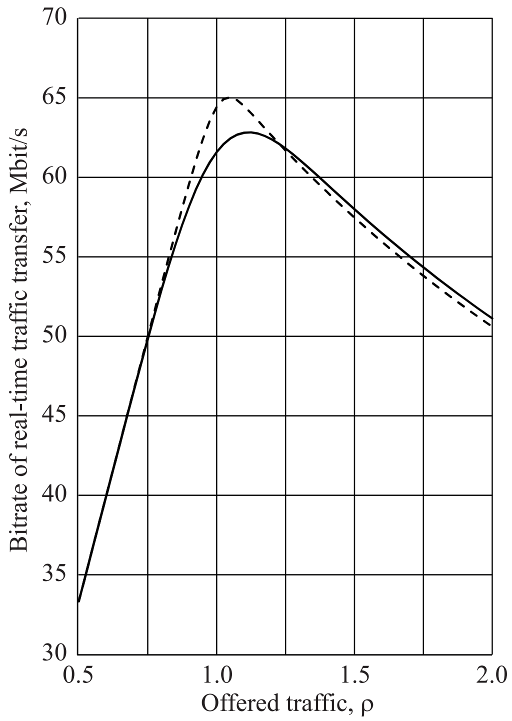

Fig. 5. Error in estimating the average bitrate used to service real-time traffic.

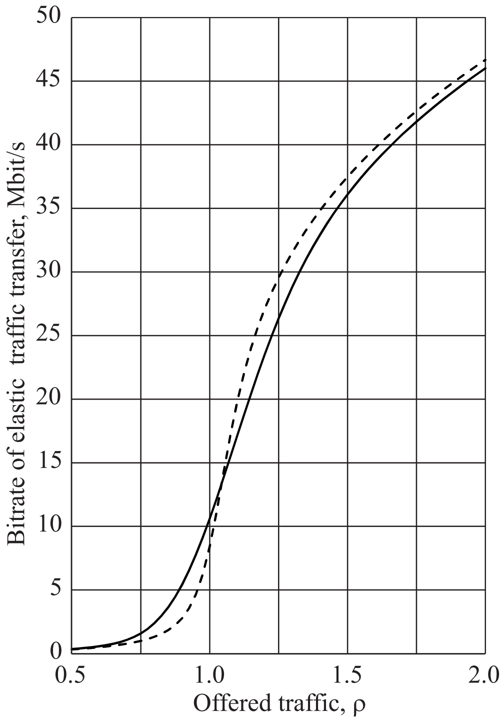

Fig. 6. Error in estimating the average bitrate used to service elastic data.

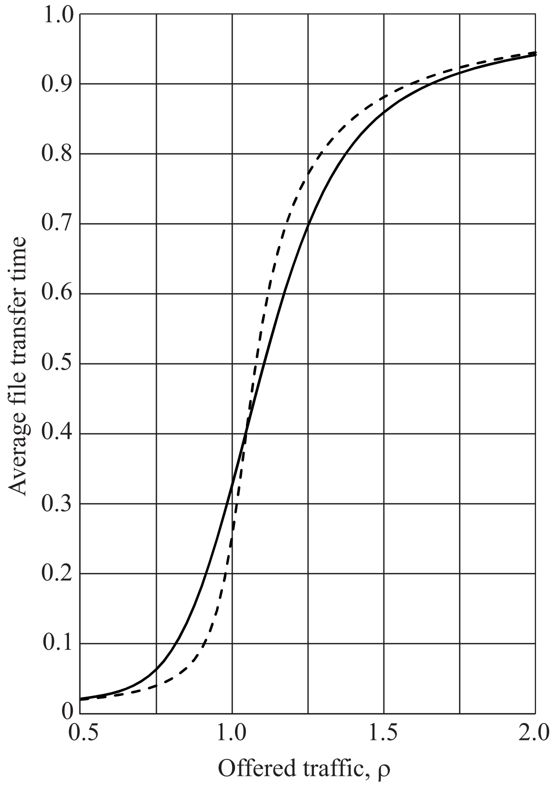

Fig. 7. Error in estimating the average file transfer time.

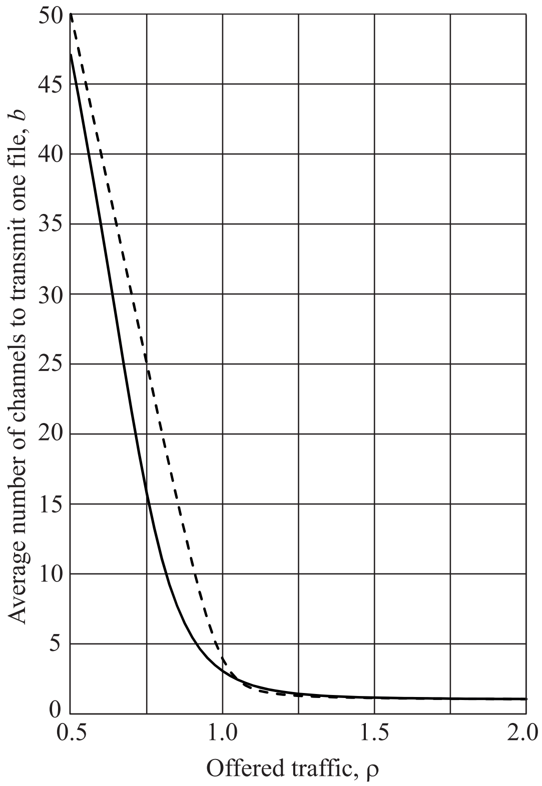

Fig. 8. Error in estimating the average number of channels used to transmit one file.

From the analysis of numerical data it follows that the obtained estimates have the following properties.

(1) They have good accuracy, which increases in the area with low and high loads. These results will be examined in more detail below.

(2) As the load on the channel increases, the values of the analyzed characteristics increase with the exception of the average value of the bitrate used to service the traffic of real-time services (see Fig. 5). This property is explained by the fact that as ρ increases, elastic data increasingly uses one channel to transfer a file, thereby displacing real-time traffic, which requires more channels for servicing.

(3) As a rule, the obtained approximate values of the characteristics at low loads give a lower estimate for the characteristics of the access node model under study, and at a heavy load they provide an upper estimate.

As noted earlier, the accuracy of estimates increases in the region of large and small losses. Let us present numerical data that confirm this conclusion. Tables 1 and 2 show the values of the quality of service characteristics of real-time traffic and elastic data for the access node model with the values of the input parameters used in calculating the curves shown in Figs. 3–8 and listed at the beginning of this Section. The presented numerical data confirm the asymptotic properties of the estimates. The error in estimates quickly decreases with an increase in ρ (Table 1) and with its decrease (Table 2). In the region of large losses, asymptotic properties manifest themselves for all characteristics listed in (2). In the low loss region—only for elastic data transmission characteristics.

The presented numerical data showed that the developed method has good accuracy for values of input parameters that correspond to practical applications, in particular for values of small losses,

Table 1. Accurate and approximate estimation of the characteristics of the access node model under heavy load conditions

ρ	π_2		π_d		h_d		z_r		z_d	
	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
1.00	0.1051	0.0496	0.0264	0.0085	0.3270	0.2546	61.58	64.42	10.61	8.41
1.25	0.3484	0.3624	0.0930	0.0806	0.6972	0.7713	61.81	61.68	26.34	29.54
1.50	0.5538	0.5716	0.1600	0.1493	0.8594	0.8811	57.99	57.46	36.09	37.47
1.75	0.6877	0.7021	0.2173	0.2081	0.9157	0.9233	54.33	53.80	41.81	42.65
2.00	0.7760	0.7873	0.2668	0.2591	0.9414	0.9449	51.12	50.60	46.01	46.67
2.50	0.8778	0.8848	0.3493	0.3437	0.9649	0.9661	45.77	45.30	52.32	52.84
3.00	0.9290	0.9334	0.4154	0.4112	0.9756	0.9762	41.53	41.12	57.03	57.47
4.00	0.9723	0.9742	0.5147	0.5122	0.9853	0.9855	35.28	34.96	63.76	64.09
5.00	0.9875	0.9884	0.5853	0.5838	0.9897	0.9898	30.88	30.63	68.39	68.66
7.50	0.9974	0.9975	0.6956	0.6951	0.9942	0.9943	23.90	23.76	75.65	75.80

Table 2. Accurate and approximate estimation of service characteristics of elastic traffic under light load conditions

ρ	h_d		z_r		z_d		b	
	Exact	Approx.	Exact	Approx.	Exact	Approx.	Exact	Approx.
0.500	0.02124	0.02000	33.33	33.33	0.35401	0.33333	47.078	50.000
0.400	0.01717	0.01666	26.66	26.66	0.22899	0.22222	58.225	60.000
0.300	0.01451	0.01428	20.00	20.00	0.14512	0.14285	68.908	70.000
0.200	0.01259	0.01250	13.33	13.33	0.08398	0.08333	79.378	80.000
0.150	0.01182	0.01176	10.00	10.00	0.05912	0.05882	84.565	85.000
0.100	0.01114	0.01111	6.666	6.666	0.03714	0.03703	89.728	90.000
0.075	0.01083	0.01081	5.000	5.000	0.02708	0.02702	92.302	92.500
0.050	0.01054	0.01052	3.333	3.333	0.01756	0.01754	94.871	95.000
0.025	0.01026	0.01025	1.666	1.666	0.00855	0.00854	97.437	97.500

where the problem of planning the volume of information transmission resource required by the load is solved, and for values of large losses, where the problem is solved estimating the volume of traffic offloaded to other access nodes or to other frequency bands in order to achieve specified QoS indicators. Let us give examples of solving the formulated problems.

6. THE USAGE OF THE OBTAINED RESULTS IN PRACTICAL APPLICATIONS

Estimating the required resource of an access node for a given load and the maximum permissible load for a given resource value are similar problems and are solved by brute force. In the first case, starting from a certain initial value, the throughput of the access node increases until the required result in terms of quality of service is achieved; in the second case, in a situation of overload, the input flow of requests is reduced for the same purpose. Let's start by solving the second problem.

Let us change the description of the procedure for generating an input stream in the model of a multiservice access node under study. Let us assume that the access node services represent a Poisson flow of requests of intensity λ, divided into $n+1$ service categories. The first n categories represent requests for the transmission of real-time traffic. The last $(n+1)$th category is requests for the transmission of elastic data traffic. Let us assume that $\lambda = \lambda_1 + \ldots + \lambda_n + \lambda_d$. With probability $p_k = \frac{\lambda_k}{\lambda}$, $k = 1, \ldots, n$ the request belongs to the kth category, requires c_k bit/s, $k = 1, \ldots, n$ and the resource occupies a random time, which has an exponential distribution with the parameter μ_k. With probability $p_d = \frac{\lambda_d}{\lambda}$ the request belongs to the category of elastic traffic and is served according to the corresponding rules introduced when describing the original node model (see Section 2). It is

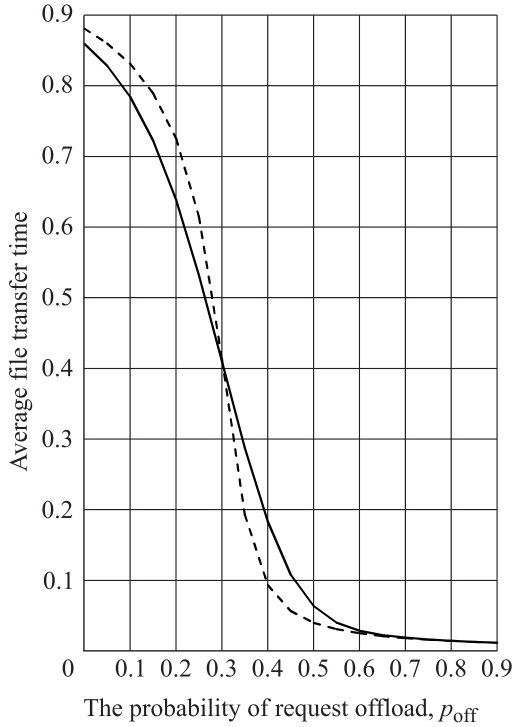

Fig. 9. Dependence of the average file transfer time estimate on p_{off}.

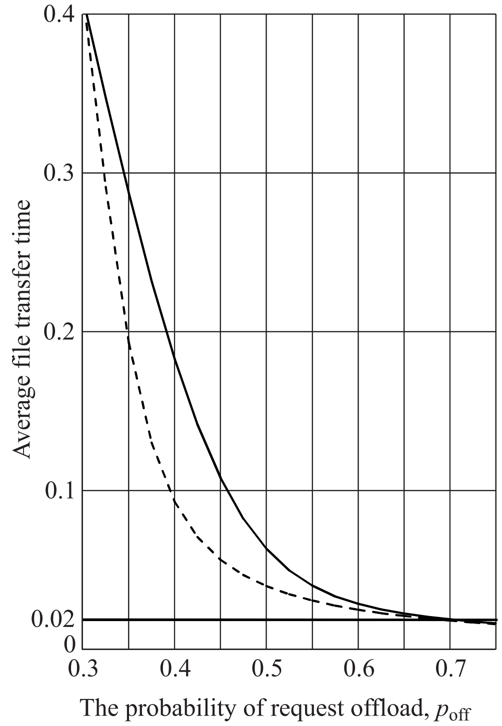

Fig. 10. Estimation of the share of offloaded traffic to ensure specified QoS indicators.

clear that the considered change in the procedure for generating the input stream did not change the mathematical description of the access node model under study.

Assume that existing or scheduled traffic is causing the access node's transmission capacity to become overloaded. This situation can be tracked with good accuracy using a method for assessing the characteristics of the quality of service for requests, based on the use of simplified equilibrium equations (see Section 4 and Table 1). Let's assume that in the current situation it is impossible to simply increase the throughput of the access node. The difficulties that arise can be dealt with by redirecting (they also say offloading) some part of the input flow to other access nodes so that the quality of service characteristics of the remaining part of the traffic on the available resource do not exceed standard values [24]. Let us denote by p_{off} the probability of offloading an incoming request.

The formal formulation of the problem is as follows. It is necessary to determine the value p_{off}, which provides a given level of quality of service for the remaining requests in the form of fulfilling the inequalities

$$\pi = \max(\pi_1, \ldots, \pi_n, \pi_d) < \pi^{\text{norm}}, \qquad h_d < h_d^{\text{norm}}. \tag{17}$$

Here π^{norm} and h_d^{norm}—are the values required under the terms of the service agreement, respectively, of the maximum loss of requests and the average time of elastic data transmission.

Let us consider the access node model for the values of the input parameters used in calculating the contents of Table 1. The values of the characteristics of servicing requests for $\rho = 1.5$ indicate that the node is in a state of overload and it is necessary to reduce the incoming flow of requests. Let's take the following values of normative indicators: $\pi^{\text{norm}} = 0.01$, $h_d^{\text{norm}} = 0.2$.

We solve the problem of estimating the probability of offloading in two stages. First, let's find the value p_{off}, which ensures the fulfillment of the inequality (17) for the average file transfer time.

Note that the choice of $h_d^{\text{norm}} = 0.2$ means that the average file transfer time is chosen 50 times less than the average file transfer time by one channel, and is 1.6 s. Figire 9 shows the change in h_d with increasing p_{off} in the range from 0 to 0.9. The exact value of the characteristic found as a result of solving the system of equilibrium equations (3) using the Gauss–Seidel iterative method, and its estimate found using simplified equilibrium equations (the curve is marked with a dotted line) are given. Note that the initial and final values of the range of changes p_{off} belong to the region where the characteristics estimates used are asymptotically accurate. This determines the high accuracy of calculating the characteristics of the original model, as evidenced by the curves shown in Fig. 9. A more detailed solution to the problem of determining p_{off} is shown in Fig. 10, where it is established that $p_{\text{off}} \approx 0.7$.

The second part of calculating the share of offloaded traffic is the need to satisfy the inequality $\pi \leqslant 0.01$. To establish this fact, we construct an upper bound for π, assuming that elastic files are transmitted using only one channel. This means that all requests arriving at the access node are serviced according to the traffic rules of real-time services. In this situation, the values of the characteristics can be calculated using recursion built for the multi-service Erlang model [7, 10, 22, 23]. The results obtained confirm the fulfillment of the inequality $\pi \leqslant 0.01$. Thus, the solution to the problem has been achieved for the offloading probability $p_{\text{off}} \approx 0.7$. The problem is solved in a similar way in a situation where only real-time service traffic or only elastic traffic is offloaded.

Estimation of the required resource of an access node for a given load is solved in the same sequence, so we will omit this part of the study.

7. CONCLUSION

A mathematical model of joint servicing of real-time traffic and elastic data traffic in a multi-service access node has been developed and studied. The arriving of requests for all service categories obeys the Poisson law; the duration of servicing requests for the transmission of real-time traffic has an exponential distribution, as do the volumes of transmitted elastic data. Elastic data is maintained in accordance with the provisions of the Processor Sharing discipline. An incoming request for real-time traffic transmission has priority in resource occupation, reducing, if necessary, the value of resource used by one file to a predetermined minimum value. A Markov process is constructed that describes the change in model states. Definitions of quality indicators for joint servicing of incoming requests for information services are provided. A system of statistical equilibrium equations is formed and its use for estimating the introduced characteristics is considered. A method for approximate calculation of characteristics is proposed, based on the construction of a system of simplified equilibrium equations. The use of the developed methods is shown to solve the problem of estimating the volume of traffic offloaded in an overload situation to other access nodes or to other frequency ranges in order to achieve specified QoS indicators and to solve the problem of planning the volume of required transmission resource of a multiservice access node. The results obtained are also applicable to the estimation of the required volume of resource of access nodes in satellite communication systems [25], data centers [26], and information centers [21].

Let us note the positive characteristics of the developed approximate method.

- The formation of simplified equilibrium equations is based on the use of local conservation laws that are fulfilled in individual macrostates of the model, selected in accordance with the structure of the state space used and the physical meaning of individual state components. Thus, it can be argued that this approach is based on the general principles of the functioning of communication system models described by Markov processes, which makes it possible to generalize the results obtained to other models of the formation of input flows of requests and resource allocation procedures, in particular to models with group arrival of requests and waiting.

- The obtained estimates of the quality of service characteristics of incoming requests have good accuracy for values of input parameters that correspond to practical applications, in particular, for values of small losses, where the problem of planning the volume of information transmission resource required by the load is solved, and for values of large losses, where the problem of estimating the volume of traffic offloaded to other access nodes or to other frequency ranges is solved in order to achieve specified QoS indicators.
- The computational algorithms used in the process of implementing the method are based on the use of simple recursive procedures.

REFERENCES

1. *Study on Scenarios and Requirements for Next Generation Access Technologies*, 3GPP Technical Report (TR) 138.913 version 15.0.0 Release 15. 2018.
2. *Network Slice Selection Services*, 3GPP Technical Specification (TS) 129.531 version 15.5.0. Release 15. 2019.
3. Evans, J. and Filsfils, C., *Deploying IP and MPLS QOS for Multiservice networks. Theory and Practice*, Morgan Kaufmann Publishers, Elsevier, 2007.
4. Ross, K.W., *Multiservice Loss Models for Broadband Telecommunication Networks*. London: Springer, 1995.
5. Iversen, V.B., *Teletraffic Engineering and Network Planning*, Technical University of Denmark, May 2010.
6. *Broadband Network Traffic. Performance Evaluation and Design of Broadband Multiservice Networks. Final Report of Action COST 242, James Roberts, Ed.,* Lecture notes in computer sciences, Springer, 1996.
7. Stepanov, S.N., *Osnovy Teletrafika Mul'tiservisnykh Setei* (The Foundation of Multiservice Network Teletraffic), Moscow: Eqo–Trends, 2010.
8. Bonald, T. and Feuillet, M., *Network Performance Analysis*, United Kingdom, London: Wiley, 2011.
9. Basharin, G.P., Gaidamaka, Yu.V., and Samouylov, K.E., Mathematical Theory of Teletraffic and Its Application to the Analysis of Multiservice Communication of Next Generation Networks, *Autom. Control Comput. Sci.*, 2013, vol. 47, no. 2, pp. 62–69.
10. Stepanov, S.N., *Teoriya teletrafika: kontseptsii, modeli, prilozheniya* (Teletraffic Theory: Concepts, Models, Applications), Moscow: Goryachaya Liniya–Telekom, 2015.
11. Stepanov, S.N. and Stepanov, M.S., Planning Transmission Resource at Joint Servicing of the Multiservice Real Time and Elastic Data Traffics, *Autom. Remote Control*, 2017, vol. 78, no. 11, pp. 2004–2015.
12. Shorgin, S., Samouylov, K., Gaidamaka, Y., Chukarin, A., Buturlin, I., and Begishev, V., Modeling Radio Resource Allocation Scheme with Fixed Transmission Zones for Multiservice M2M Communications in Wireless IoT Infrastructure, *Lecture Notes Comput. Sci.*, Springer, Cham., 2015, vol. 9012, pp. 473–483.
13. Begishev, V., Petrov, V., Samuylov, A., Moltchanov, D., Andreev, S., Koucheryavy, Y., and Samouylov, K., Resource Allocation and Sharing for Heterogeneous Data Collection over Conventional 3GPP LTE and Emerging NB-IoT Technologies, —it Comput. Commun., 2018, vol. 120, no. 2, pp.93–101.
14. Naumov, V., Gaidamaka, Yu., Yarkina, N., and Samouylov, K., *Matrix and Analytical Methods for Performance Analysis of Telecommunication Systems*, Springer Nature, Switzerland, 2021.
15. Kallos, G.A., Vassilakis, V.G., Moscholios, I.D., and Logothetis, M.D., Performance Modelling of W-CDMA Networks Supporting Elastic and Adaptive Traffic, *Proc. of 4 Int. Working Conference on Performance Modelling and Evaluation of Heterogeneous Networks (HET-NETs '06)*, Ilkley, 2006.
16. Gudkova, I.A. and Samouylov, K.E., Modelling a Radio Admission Control Scheme for Video Telephony Service in Wireless Networks, *Lecture Notes Comput. Sci.*, 2012, vol. 7469, pp. 208–215.
17. Bonald, T. and Virtamo, J., A Recursive Formula for Multirate Systems with Elastic Traffic, *IEEE Communicat. Lett.*, 2005, vol. 9, no. 8, pp. 753–755.

18. Stepanov, S.N. and Stepanov, M.S., Planning the Resource of Information Transmission for Connection Lines of Multiservice Hierarchical Access Networks, *Autom. Remote Control*, 2018, vol., 79. no. 8, pp. 1422–1433.

19. Stepanov, S.N. and Stepanov, M.S., Efficient Algorithm for Evaluating the Required Volume of Resource in Wireless Communication Systems under Joint Servicing of Heterogeneous Traffic for the Internet of Things, *Autom. Remote Control*, 2019, vol., 80, no. 11, pp. 2017–2032.

20. Stepanov, S.N. and Stepanov, M.S., Methods for Estimating the Required Volume of Resource for Multiservice Access Nodes, *Autom. Remote Control*, 2020, vol., 81, no. 12, pp. 2244–2261.

21. Stepanov, S.N., Markov Models with Retrials: The Calculation of Stationary Performance Measures Based on the Concept of Truncation, *Math. Comput. Model*, 1999, no. 30, pp. 207–228.

22. Stepanov, S. and Stepanov, M., Estimation of the Performance Measures of a Group of Servers Taking into Account Blocking and Call Repetition Before and After Server Occupation, *Mathematics*, 2021, vol. 9, no. 21.2811, pp. 1–24. https://doi.org/10.3390/math9212811

23. Fortet, R., and Grandjean, Ch., Congestion in a Loss System When Some Calls Want Several Devices Simultaneously, *Electr. Communicat.*, 1964, vol. 39, no. 4, pp. 513–526.

24. Kaufman, J.S., Blocking in a Shared Resource Environment, *IEEE Transact. Communicat.*, 1981, vol. 29, no. 1, pp. 1474–1481.

25. Chen, J., Chang, Z., Guo, X., Li, R., Han, Z., and Hamalainen, T., Resource Allocation and Computation Offloading for Multi-Access Edge Computing With Fronthaul and Backhaul Constraints, *IEEE Transactions on Vehicular Technology*, 2021, vol. 70, no. 8, pp. 8037–8049.

26. Stepanov, S.N., Andrabi, U.M., Stepanov, M.S. and Ndayikunda, J., Reservation Based Joint Servicing of Real Time and Batched Traffic in Inter Satellite Link, *Proc. of 2020 Systems of Signals Generating and Processing in the Field of on Board Communications*, Moscow, Russia, 2020, pp. 1–5. https://doi.org/10.1109/IEEECONF48371.2020.9078542

27. Volkov, A.O., Korobkina, A.V. and Stepanov, S.N., Development of Model and Algorithms for Servicing Real-Time and Data Traffic in a Cloud Computing System*Proc. of 2022 Systems of Signals Generating and Processing in the Field of on Board Communications*, Moscow, Russia, pp. 1–6. 2022. https://doi.org/10.1109/IEEECONF53456.2022.9744289

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ROBUST, ADAPTIVE, AND NETWORK CONTROL

Adaptive Observer of State and Disturbances for Linear Overparameterized Systems

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Abstract—The problem of state reconstruction is considered for a class of linear systems with time-invariant unknown parameters and overparameterization that are affected by external perturbations generated by a known exosystem with unknown initial conditions. An extended adaptive observer is proposed, which, in contrast to existing approaches, solves state and perturbation adaptive estimation problems for systems that are not represented in the observer canonical form. The obtained theoretical results are validated via mathematical modeling.

Keywords: estimation, identification, adaptive observer, persistent excitation, convergence, overparameterization

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1. INTRODUCTION

One of the problems of automatic control theory is the reconstruction of unmeasured state of completely observable linear systems:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= C^{\mathrm{T}}x(t) \end{aligned} \tag{1.1}$$

with unknown matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n}$, $C \in \mathbb{R}^{n}$.

To solve it, various observers based on the invariant ellipsoid technique [1], high-gain methods [2, 3], sliding mode approach [3, 4], and parametric identification theory [5, 6] have been proposed.

In contrast to other approaches, observers based on the methods of identification theory [5, 6] use parameter adaptation algorithms and, therefore, usually require less *a priori* information about the system parameters. However, since the baseline solutions [7–10], the class of systems, for which the adaptive observers can be designed, is traditionally restricted to models in the observer canonical form:

$$\begin{aligned} \dot{\xi}(t) &= A_0\xi(t) + \psi_a y(t) + \psi_b u(t) = A_a\xi(t) + \psi_b u(t), \\ y(t) &= C_0^{\mathrm{T}}\xi(t), \end{aligned} \tag{1.2}$$

$$A_0 = \begin{bmatrix} 0_n & \begin{matrix} I_{n-1} \\ 0_{n-1}^{\mathrm{T}} \end{matrix} \end{bmatrix}, \quad A_a = \begin{bmatrix} \psi_a & \begin{matrix} I_{n-1} \\ 0_{n-1}^{\mathrm{T}} \end{matrix} \end{bmatrix},$$

$$\psi_a = \begin{bmatrix} -a_{n-1} \\ -a_{n-2} \\ \vdots \\ -a_0 \end{bmatrix}, \quad \psi_b = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where ψ_a and ψ_b are parameters of the characteristic polynomials of the following linear operator

$$W_{uy}(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \ldots + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_0},$$

and they are related to the matrices of the model (1.1) via a transformation matrix T:

$$\begin{gathered}\psi_a = TAT^{-1}C_0,\ \ \psi_b = TB,\ \ C_0^{\mathrm{T}} = C^{\mathrm{T}}T^{-1},\\ \mathcal{O}_n = \mathcal{O}\begin{bmatrix}0_{1\times(n-1)} & 1\end{bmatrix}^{\mathrm{T}},\\ \mathcal{O}^{-1} = \begin{bmatrix}C & A^{\mathrm{T}}C & \cdots & (A^{n-1})^{\mathrm{T}}C\end{bmatrix}^{\mathrm{T}},\ \ T^{-1} = \begin{bmatrix}A^{n-1}\mathcal{O}_n & A^{n-2}\mathcal{O}_n & \cdots & \mathcal{O}_n\end{bmatrix}.\end{gathered} \tag{1.3}$$

The point is that the measurable control $u(t)$ and output $y(t)$ signals allow one to uniquely identify the parameters of such canonical state space form only [5, p. 269]. The states $\xi(t) \in \mathbb{R}^n$ of the model (1.2) are virtual and related to the plant physical states $x(t) \in \mathbb{R}^n$ via a non-singular transformation $\xi(t) = Tx(t)$.

Therefore, the estimates $\hat{\xi}(t)$ obtained by classical adaptive state observers [5, 6] of the following form ($\hat{\psi}_a(t), \hat{\psi}_b(t)$ are estimates of the parameters (1.3), L stands for the correction matrix, and the specific structures of functions $f_a(.), f_b(.), f_v(.)$ are defined in [5, 6]):

$$\begin{gathered}\dot{\hat{\xi}}(t) = A_0\hat{\xi}(t) + \hat{\psi}_a(t)y(t) + \hat{\psi}_b(t)u(t) + L\left(\hat{y}(t) - y(t)\right) + v(t),\\ \hat{y}(t) = C_0{}^{\mathrm{T}}\hat{\xi}(t),\\ \dot{\hat{\psi}}_a(t) = f_a\left(u, y, \hat{y}, \hat{\psi}_a\right),\\ \dot{\hat{\psi}}_b(t) = f_b\left(u, y, \hat{y}, \hat{\psi}_b\right),\\ \mathrm{Re}\left\{\lambda_i\left(A_0 + LC^{\mathrm{T}}\right)\right\} < 0,\\ v(t) = f_v\left(u, y, \dot{\hat{\psi}}_a, \dot{\hat{\psi}}_b\right)\ \text{or}\ v(t) = 0_n,\end{gathered} \tag{1.4}$$

not only do not coincide with $x(t)$, but also turn out to be useless, for example, to solve the problems of failure diagnostics, monitoring and storage of unmeasured variables of technological processes, design and online adjustment of digital twins and other practical scenarios.

The solution to this problem is to identify the linear transformation matrix T together with the parameters ψ_a and ψ_b. For one specific class of linear systems, an algorithm is proposed in [7] that forms an estimate of $\hat{T}(t)$ on the basis of the ones of $\hat{\psi}_a(t)$ and $\hat{\psi}_b(t)$. In the general case, the mapping $\hat{T}(t) = f_{\mathrm{T}}\left(\hat{\psi}_a(t),\ \hat{\psi}_b(t)\right)$ can be singular for certain values of the estimates $\hat{\psi}_a(t)$, $\hat{\psi}_b(t)$ (see Section VIII of [7]). In more recent papers [10–12] devoted to the development of methods to design adaptive observers (and even in the fundamental books on adaptive observers for linear systems [5, 6]), to the best of the authors' knowledge, the problem of the physical state $x(t)$ reconstruction and the estimation of the linear transformation matrix T with the help of adaptive observers was no longer touched upon.

In a recent paper [13] a new approach of adaptive reconstruction of the linear system physical state is presented instead of identification of the linear transformation matrix. It is proposed to overparameterize the matrices of the system (1.1) with respect to some physical parameters $\theta \in \mathbb{R}^{n_\theta}$ (such overparameterization is always possible if the model is obtained directly on the basis of the laws of mathematical physics—Kirchhoff, Euler–Lagrange, etc.):

$$\begin{gathered}\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) = \Phi^{\mathrm{T}}(x,\ u)\,\Theta_{AB}(\theta),\\ y(t) = C^{\mathrm{T}}x(t),\end{gathered} \tag{1.5}$$

and, using the following change of notation $\psi_a := \psi_a(\theta)$, $\psi_b := \psi_b(\theta)$, to take into account the dependence of the model (1.1) parameters from θ.

The fact that the overparameterization is considered allows one to link the matrices of the models (1.1) and (1.2) not via the above-mentioned transformation, but by means of some new functional transformations of the following form ($\theta = \mathcal{F}(\psi_{ab})$ is an inverse function, $\mathcal{L}_{ab} \in \mathbb{R}^{n_\theta \times 2n}$ stands for a matrix that defines some linear transformation, which ensures $\dim\{\psi_{ab}\} = \dim\{\theta\}$)

$$\Theta_{AB}(\theta) = (\Theta_{AB} \circ \mathcal{F})(\psi_{ab}),$$
$$\psi_{ab}(\theta) = \mathcal{L}_{ab}\begin{bmatrix}\psi_a(\theta)\\ \psi_b(\theta)\end{bmatrix}: \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta},$$

which provides much room to design the adaptive observers of physical states $x(t)$.

In [13] it is shown that, if the condition

$$\det{}^2\{\nabla_\theta \psi_{ab}(\theta)\} > 0, \quad \psi_{ab}(\theta) = \mathcal{L}_{ab}\begin{bmatrix}\psi_a(\theta)\\ \psi_b(\theta)\end{bmatrix}: \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta} \tag{1.6}$$

of existence of the inverse function $\mathcal{F}: \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta}$ holds and $\psi_{ab}(\theta)$ and $\Theta_{AB}(\theta)$ depend from θ in a polynomial fashion, then, using only the measurable signals $y(t), u(t)$ and known vector C, the following regression equations can be obtained without identification of the parameters $\psi_{ab}(\theta)$ and θ (where $\mathcal{Y}_{AB}(t)$, $\mathcal{Y}_L(t)$, $\mathcal{M}_{AB}(t)$, $\mathcal{M}_L(t)$ stand for the measurable signals):

$$\mathcal{Y}_{AB}(t) = \mathcal{M}_{AB}(t)\Theta_{AB}(\theta),$$
$$\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$$

and, as a consequence, an adaptive observer of the system (1.5) state can be implemented in the following form:

$$\begin{gathered}\dot{\hat{x}}(t) = \Phi^{\mathrm{T}}(\hat{x}, u)\hat{\Theta}_{AB}(t) - \hat{L}(t)(\hat{y}(t) - y(t)),\\ \dot{\hat{\Theta}}_{AB}(t) = f_{\Theta_{AB}}\left(\mathcal{Y}_{AB}, \mathcal{M}_{AB}, \hat{\Theta}_{AB}\right),\\ \dot{\hat{L}}(t) = f_L\left(\mathcal{Y}_L, \mathcal{M}_L, \hat{L}\right),\end{gathered} \tag{1.7}$$

where $\hat{L}(t)$ is an estimate of the matrix $L(\theta)$ such that $A(\theta) - L(\theta)C^{\mathrm{T}}$ is a Hurwitz one.

In other words, owing to the fact that $\Theta_{AB}(\theta)$ and $\psi_{ab}(\theta)$ are related to each other via the physical parameters θ, if the condition (1.6) is met, then, in accordance with [13], $\Theta_{AB}(\theta)$ and $L(\theta)$ can be identified without direct estimation of θ or $\psi_{ab}(\theta)$. In contrast to (1.4), the (1.7) observer allows one to obtain estimates of the physical state $x(t)$, and, contrary to [7], it is applicable to a wider class of systems and does not require direct identification of parameters $\psi_{ab}(\theta)$.

The aim of this paper is to extend the results of [13] to the class of linear systems with overparameterization that are affected by the external perturbations generated by a known exosystem with unknown initial conditions.

Main definitions

The definition of a heterogeneous mapping, the regressor persistent excitation condition, and the property of the Kreisselmeyer filtering [10] given below will be used throughout this paper[1].

[1] For the sake of brevity, hereafter the arguments θ and t will be omitted except the cases in which it is necessary for understanding.

Definition 1. A mapping $\mathcal{F} : \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\mathcal{F} \times m_\mathcal{F}}$ is heterogeneous of degree $\ell_\mathcal{F} \geqslant 1$, if there exists $\Pi_\mathcal{F}(\omega) \in \mathbb{R}^{n_\mathcal{F} \times n_\mathcal{F}}$, $\Xi_\mathcal{F}(\omega) = \overline{\Xi}_\mathcal{F}(\omega)\,\omega(t) \in \mathbb{R}^{\Delta_\mathcal{F} \times n_\theta}$, and mapping $\mathcal{T}_\mathcal{F} : \mathbb{R}^{\Delta_\mathcal{F}} \to \mathbb{R}^{n_\mathcal{F} \times m_\mathcal{F}}$ such that for all $\omega(t) \in \mathbb{R}$ and $\theta \in \mathbb{R}^{n_\theta}$ the following functional equation has a solution

$$\Pi_\mathcal{F}(\omega)\,\mathcal{F}(\theta) = \mathcal{T}_\mathcal{F}\left(\Xi_\mathcal{F}(\omega)\,\theta\right), \tag{1.8}$$

where

$$\begin{gathered}\det\left\{\Pi_\mathcal{F}(\omega)\right\} \geqslant \omega^{\ell_\mathcal{F}}(t),\\ \Xi_{\mathcal{F}ij}(\omega) = c_{ij}\omega^{\ell_{ij}}(t),\ \ \overline{\Xi}_{\mathcal{F}ij}(\omega) = c_{ij}\omega^{\ell_{ij}-1}(t),\\ c_{ij} \in \{0,1\},\ \ \ell_{ij} \geqslant 1.\end{gathered}$$

For example, $\mathcal{F}(\theta) = \mathrm{col}\{\theta_1\theta_2, \theta_1\}$ with $\Pi_\mathcal{F}(\omega) = \mathrm{diag}\{\omega^2, \omega\}$, $\Xi_\mathcal{F}(\omega) = \mathrm{diag}\{\omega, \omega\}$ is heterogeneous of degree $\ell_\mathcal{F} = 3$.

Using a known function $\mathcal{Y}_\theta(t) = \omega(t)\theta$, the main property $\Xi_\mathcal{F}(\omega)\,\theta = \overline{\Xi}_\mathcal{F}(\omega)\,\omega(t)\theta$ from Definition 1 allows one to obtain a linear regression equation with respect to $\mathcal{F}(\theta)$ in the following way:

$$\begin{gathered}\Pi_\mathcal{F}(\omega)\,\mathcal{F}(\theta) = \mathcal{T}_\mathcal{F}\left(\overline{\Xi}_\mathcal{F}(\omega)\,\mathcal{Y}_\theta\right),\\ \begin{bmatrix}\omega^2(t) & 0\\ 0 & \omega(t)\end{bmatrix}\mathcal{F}(\theta) = \begin{bmatrix}\mathcal{Y}_{1\theta}(t)\mathcal{Y}_{2\theta}(t)\\ \mathcal{Y}_{1\theta}(t)\end{bmatrix}.\end{gathered}$$

The elements of a mapping $\mathcal{F}(\theta)$ satisfy Definition 1 if they are polynomials or monomials of θ, as well as some of the irrational functions.

Definition 2. A regressor $\overline{\varphi}(t) \in \mathbb{R}^n$ is persistently exciting ($\overline{\varphi}(t) \in \mathrm{PE}$) if $\exists T > 0$ and $\alpha > 0$ such that $\forall t \geqslant t_0 \geqslant 0$ the following inequality holds

$$\int\limits_t^{t+T} \overline{\varphi}(\tau)\,\overline{\varphi}^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_n, \tag{1.9}$$

where $\alpha > 0$ is an excitation level, I_n stands for an identity matrix.

For a determinant of state of a stable ($l > 0$) dynamical filter

$$\dot{\varphi}(t) = -l\varphi(t) + \overline{\varphi}(t)\overline{\varphi}^{\mathrm{T}}(t), \quad \varphi(t_0) = 0_{n\times n}$$

it holds that

Proposition 1. (a) *If $\overline{\varphi}(t) \in \mathrm{PE}$, then for all $t \geqslant t_0 + T$ the following inequality holds*

$$\Delta(t) = \det\left\{\varphi(t)\right\} \geqslant \alpha^n e^{-nlT} = \Delta_{\min} > 0. \tag{1.10}$$

(b) *If there exists $t_e \in [t_0, \infty)$ such that for all $t \geqslant t_e$ equation* (1.10) *holds, then* $\overline{\varphi}(t) \in \mathrm{PE}$.

Proof of Proposition 1 is given in [14].

2. PROBLEM STATEMENT

We consider the following class of SISO-systems with overparameterization affected by a bounded external perturbation:

$$\begin{gathered}\dot{x}(t) = A(\theta)\,x(t) + B(\theta)\,u(t) + D(\theta)\,\delta(t) = \Phi^{\mathrm{T}}(x, u, \delta)\,\Theta_{AB}(\theta),\\ y(t) = C^{\mathrm{T}}x(t),\ \ x(t_0) = x_0,\end{gathered} \tag{2.1}$$

where

$$\Phi^{\mathrm{T}}(x,u,\delta)=\Big[I_n\otimes x^{\mathrm{T}}(t)\quad I_n\otimes u^{\mathrm{T}}(t)\quad I_n\otimes \delta^{\mathrm{T}}(t)\Big]\mathcal{D}_\Phi\in\mathbb{R}^{n\times n_\Theta},$$
$$\Theta_{AB}(\theta)=\mathcal{L}_\Phi\Big[vec^{\mathrm{T}}\Big(A^{\mathrm{T}}(\theta)\Big)\quad B^{\mathrm{T}}(\theta)\quad D^{\mathrm{T}}(\theta)\Big]^{\mathrm{T}}\in\mathbb{R}^{n_\Theta},$$

$x(t)\in\mathbb{R}^n$ are physical states of the system with unknown initial conditions x_0, $\delta(t)$ stands for a bounded external perturbation, $\Theta_{AB}\in\mathbb{R}^{n_\Theta}$, $\theta\in\mathbb{R}^{n_\theta}$ denote unknown vectors such that $n_\Theta\geqslant n_\theta$, $\mathcal{D}_\Phi\in\mathbb{R}^{(n^2+2n)\times n_\Theta}$, $\mathcal{L}_\Phi\in\mathbb{R}^{n_\Theta\times(n^2+2n)}$ are unknown matrices, the vector $C\in\mathbb{R}^n$ and mapping $\Theta_{AB}:\mathbb{R}^{n_\theta}\to\mathbb{R}^{n_\Theta}$ are known. Only the control $u(t)\in\mathbb{R}$ and output $y(t)\in\mathbb{R}$ signals are measurable.

The following assumptions are adopted for the control and disturbance signals.

Assumption 1. For all $t\geqslant t_0$ the control signal $u(t)$ ensures existence and boundedness of all trajectories of the system (2.1).

Assumption 2. The disturbance $\delta(t)$ is continuous and generated by a stable exosystem with time-invariant parameters:

$$\begin{gathered}\dot{x}_\delta(t)=\mathcal{A}_\delta x_\delta(t),\quad x_\delta(t_0)=x_{\delta 0},\\ \delta(t)=h_\delta^{\mathrm{T}}x_\delta(t),\end{gathered}\tag{2.2}$$

where $x_\delta(t)\in\mathbb{R}^{n_\delta}$ are states of the exosystem with unknown initial conditions $x_{\delta 0}$, $h_\delta\in\mathbb{R}^{n_\delta}$, $\mathcal{A}_\delta(t)\in\mathbb{R}^{n_\delta\times n_\delta}$ are known vector and matrix, which form an observable pair $\Big(h_\delta^{\mathrm{T}},\mathcal{A}_\delta\Big)$.

Taking into account the duality of the observation and control problems and following the results of the generalized pole placement theory [15, 16], we adopt an assumption that there exists a vector $L(\theta)\in\mathbb{R}^n$, which transforms an algebraic spectrum $\sigma\{.\}$ of the matrix $A^{\mathrm{T}}(\theta)-CL^{\mathrm{T}}(\theta)$ into a desired one.

Assumption 3. A pair $\Big(A^{\mathrm{T}}(\theta),\,C\Big)$ is controllable, there exists a known state matrix $\Gamma\in\mathbb{R}^{n\times n}$ of an exosystem

$$\begin{aligned}&\dot{\chi}(t)=\Gamma\chi(t),\\ &v(t)=B^{\mathrm{T}}(\theta)\,\chi(t)\end{aligned}\tag{2.3}$$

such that the pair $\Big(B^{\mathrm{T}}(\theta),\,\Gamma\Big)$ is observable and $\sigma\{A(\theta)\}\cap\sigma\{\Gamma\}=0$.

If Assumptions 1–3 are met, then the following observer of the state and perturbation can be introduced:

$$\begin{gathered}\dot{\hat{x}}(t)=\Phi^{\mathrm{T}}\Big(\hat{x},u,\hat{\delta}\Big)\hat{\Theta}_{AB}(t)-\hat{L}(t)\left(\hat{y}(t)-y(t)\right),\\ \hat{\delta}(t)=h_\delta^{\mathrm{T}}\Phi_\delta(t)\hat{x}_{\delta 0}(t),\\ \dot{\Phi}_\delta(t)=\mathcal{A}_\delta\Phi_\delta(t),\quad \Phi_\delta(t_0)=I_{n_\delta}.\end{gathered}\tag{2.4}$$

The aim is to augment the observer (2.4) with the estimation laws, which ensure that the following equalities hold

$$\begin{gathered}\lim_{t\to\infty}\|\tilde{x}(t)\|=0\ (\exp),\ \lim_{t\to\infty}\Big\|\tilde{\delta}(t)\Big\|=0\ (\exp),\ \lim_{t\to\infty}\|\tilde{\kappa}(t)\|=0\ (\exp),\\ \tilde{\kappa}(t)=\Big[\tilde{x}_{\delta 0}^{\mathrm{T}}(t)\quad \tilde{\Theta}_{AB}^{\mathrm{T}}(t)\quad \tilde{L}^{\mathrm{T}}(t)\Big]^{\mathrm{T}},\end{gathered}\tag{2.5}$$

where $\tilde{x}(t)=\hat{x}(t)-x(t)$ is the state (2.1) observation error, $\tilde{\delta}(t)=\hat{\delta}(t)-\delta(t)$ stands for the disturbance observation error, $\tilde{\Theta}_{AB}(t)=\hat{\Theta}_{AB}(t)-\Theta_{AB}(\theta)$ denotes the error of the system (2.1) parameters estimation, $\tilde{x}_{\delta 0}(t)=\hat{x}_{\delta 0}(t)-x_{\delta 0}$ is the observation error of the exosystem (2.2) initial conditions, $\tilde{L}(t)=\hat{L}(t)-L(\theta)$ stands for the error of $L(\theta)$ estimation.

Remark 1. Assumptions 1 and 3 are conventional for the adaptive observation [10–12] *and pole placement design* [15, 16] *problems, respectively. Assumption 2 restricts the class of permissible external perturbations.*

3. PREREQUISITES AND PRELIMINARY TRANSFORMATIONS

Before presenting the solution of the problem (2.5), the identifiability of the unknown parameters κ from the measurements of $y(t)$ and $u(t)$ is investigated. For this purpose, using the transformations (1.3), the system (2.1) is represented in the form (1.2):

$$\dot{\xi}(t) = A_0\xi(t) + \psi_a(\theta)\, y(t) + \psi_b(\theta)\, u(t) + \psi_d(\theta)\, \delta(t), \tag{3.1}$$

$$y(t) = C^{\mathrm{T}}x(t) = C_0^{\mathrm{T}}\xi(t), \quad \xi(t_0) = Tx_0 = \xi_0, \tag{3.2}$$

where $\psi_d(\theta) = TD(\theta)$, $\xi(t) \in \mathbb{R}^n$ are unmeasurable virtual state of the observer canonical form, the vector $C_0 \in \mathbb{R}^n$ and mappings ψ_a, ψ_b, ψ_d: $\mathbb{R}^{n_\theta} \to \mathbb{R}^n$ are known.

The following parametrization can be obtained for the unknown parameters $\eta(\theta) = \mathrm{col}\{\psi_a(\theta), \psi_b(\theta)\}$ of equation (3.1) in case Assumptions 1 and 2 are met.

Lemma 1. *The unknown parameters $\eta(\theta)$ satisfy the following linear regression model* [2]

$$\begin{gathered} \mathcal{Y}(t) = \Delta(t)\eta(\theta) + \epsilon(t), \\ \mathcal{Y}(t) = k(t) \times \mathrm{adj}\{\varphi(t)\}\, q(t), \;\; \Delta(t) = k(t) \times \det\{\varphi(t)\}, \end{gathered} \tag{3.3}$$

where

$$\begin{gathered} \dot{q}(t) = -k_2 q(t) + \overline{\varphi}_f(t)(\overline{q}(t) - k_1\overline{q}_f(t) - \beta^{\mathrm{T}}(F_f(t) + l y_f(t))),\; q(t_0) = 0_{2n}, \\ \dot{\varphi}(t) = -k_2\varphi(t) + \overline{\varphi}_f(t)\overline{\varphi}_f^{\mathrm{T}}(t),\; \varphi(t_0) = 0_{2n\times 2n}, \end{gathered} \tag{3.4}$$

$$\begin{gathered} \dot{\overline{q}}_f(t) = -k_1\overline{q}_f(t) + \overline{q}(t),\; \overline{q}_f(t_0) = 0, \\ \dot{\overline{\varphi}}_f(t) = -k_1\overline{\varphi}_f(t) + \overline{\varphi}(t),\; \overline{\varphi}_f(t_0) = 0_{2n}, \\ \dot{F}_f(t) = -k_1 F_f(t) + F(t),\; F_f(t_0) = 0_{n_\delta}, \\ \dot{y}_f(t) = -k_1 y_f(t) + y(t),\; y_f(t_0) = 0, \end{gathered} \tag{3.5}$$

$$\begin{gathered} \overline{q}(t) = y(t) - C_0^{\mathrm{T}}z(t),\; \overline{\varphi}(t) = \begin{bmatrix} \dot{\Omega}^{\mathrm{T}}(t)C_0 + N^{\mathrm{T}}(t)\beta \\ \dot{P}^{\mathrm{T}}(t)C_0 + H^{\mathrm{T}}(t)\beta \end{bmatrix}, \\ \dot{z}(t) = A_K z(t) + K y(t),\; z(t_0) = 0_n, \\ \dot{\Omega}(t) = A_K\Omega(t) + I_n y(t),\; \Omega(t_0) = 0_{n\times n}, \\ \dot{P}(t) = A_K P(t) + I_n u(t),\; P(t_0) = 0_{n\times n}, \\ \dot{F}(t) = GF(t) + Gly(t) - lC_0^{\mathrm{T}}\dot{z}(t),\; F(t_0) = 0_{n_\delta}, \\ \dot{H}(t) = GH(t) - lC_0^{\mathrm{T}}\dot{P}(t),\; H(t_0) = 0_{n_\delta\times n}, \\ \dot{N}(t) = GN(t) - lC_0^{\mathrm{T}}\dot{\Omega}(t),\; N(t_0) = 0_{n_\delta\times n}, \end{gathered} \tag{3.6}$$

and, if $\overline{\varphi}(t) \in \mathrm{PE}$, then for all $t \geqslant t_0 + T$ it holds that $\Delta_{\max} \geqslant \Delta(t) \geqslant \Delta_{\min} > 0$.

Here $\epsilon(t)$ is an exponentially decaying term, $k(t) \geqslant k_{\min} > 0$ stands for an amplitude modulator (can be time-varying), $k_1 > 0$, $k_2 > 0$ denote filters constants, $A_K = A_0 - KC_0^{\mathrm{T}}$, G are stable matrices of respective dimension, the vector $l \in \mathbb{R}^{n_\delta}$ is such that the pair $(G,\, l)$ is controllable, and

[2] Without loss of generality, further the exponentially decaying term $\epsilon(t)$ is omitted.

G is chosen considering $\sigma\left\{\mathcal{A}_\delta\right\} \cap \sigma\left\{G\right\} = 0$, the parameter $\beta \in \mathbb{R}^{n_\delta}$ is a solution of the following set of equations

$$\begin{gathered} M_\delta \mathcal{A}_\delta - G M_\delta = l \overline{h}_\delta^{\mathrm{T}},\ \overline{h}_\delta^{\mathrm{T}} = h_\delta^{\mathrm{T}} \mathcal{A}_\delta, \\ \beta = \overline{h}_\delta^{\mathrm{T}} M_\delta^{-1}. \end{gathered}$$

Proof of Lemma 1 is postponed to Appendix.

In the general case, the goal (2.5) cannot be achieved because only the parameters ψ_a, ψ_b of the characteristic polynomials of the transfer function $W_{uy}\left(s\right) = C^{\mathrm{T}}\left(sI_n - A\left(\theta\right)\right)^{-1} B\left(\theta\right)$ are identifiable on the basis of measurable signals $u(t)$, $y(t)$ via parameterization (3.3) if $\overline{\varphi}(t) \in \mathrm{PE}$. However, in the case that is important for practical scenarios, according to the problem statement, the parameters Θ_{AB}, ψ_d, L depend nonlinearly from the physical parameters θ in a known way. In their turn, the parameters ψ_a, ψ_b of the characteristic polynomials of the transfer function $W_{uy}\left(s\right)$ also depend nonlinearly from θ. Therefore, if the following condition is met

$$\det{}^2\left\{\nabla_\theta \psi_{ab}\left(\theta\right)\right\} > 0,\ \psi_{ab}\left(\theta\right) = \mathcal{L}_{ab}\eta\left(\theta\right) \in \mathbb{R}^{n_\theta}, \tag{3.7}$$

then, owing to the inverse function theorem [17], there exists an inverse mapping $\theta = \mathcal{F}\left(\psi_{ab}\right)$, and therefore, it becomes possible to: *i*) calculate the parameters of the system Θ_{AB} and observer L using ψ_{ab}, *ii*) obtain estimates $\hat{x}_{\delta 0}(t)$ of the initial conditions of the exosystem (2.2), *iii*) implement the adaptive observer (2.4), from which the estimates $\hat{x}(t)$ and $\hat{\delta}(t)$ are obtained.

In this paper, to solve the problem of reconstruction of unmeasurable state $x(t)$ and external perturbation $\delta(t)$ when the condition (3.7) is satisfied, the following hypotheses are additionally adopted with respect to $\psi_{ab}\left(\theta\right)$, $\Theta_{AB}\left(\theta\right)$, and $\psi_d\left(\theta\right)$.

Hypothesis 1. There exist the heterogeneous in the sense of (1.8) mappings $\mathcal{G}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta \times n_\theta}$, $\mathcal{S}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\theta}$ such that:

$$\begin{gathered} \mathcal{S}\left(\psi_{ab}\right) = \mathcal{G}\left(\psi_{ab}\right)\mathcal{F}\left(\psi_{ab}\right) = \mathcal{G}\left(\psi_{ab}\right)\theta, \\ \Pi_\theta\left(\omega\right)\mathcal{G}\left(\psi_{ab}\right) = \mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}\left(\omega\right)\psi_{ab}\right), \\ \Pi_\theta\left(\omega\right)\mathcal{S}\left(\psi_{ab}\right) = \mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}\left(\omega\right)\psi_{ab}\right), \end{gathered} \tag{3.8}$$

where $\Xi_{\mathcal{G}}\left(\omega\right) \in \mathbb{R}^{\Delta_{\mathcal{G}} \times n_\theta}$, $\Xi_{\mathcal{S}}\left(\omega\right) \in \mathbb{R}^{\Delta_{\mathcal{S}} \times n_\theta}$, $\det\left\{\Pi_\theta\left(\omega\right)\right\} \geqslant \omega^{\ell_\theta}(t)$, $\operatorname{rank}\left\{\mathcal{G}\left(\psi_{ab}\right)\right\} = n_\theta$, $\ell_\theta \geqslant 1$, $\mathcal{T}_{\mathcal{G}} : \mathbb{R}^{\Delta_{\mathcal{G}}} \to \mathbb{R}^{n_\theta \times n_\theta}$, $\mathcal{T}_{\mathcal{S}} : \mathbb{R}^{\Delta_{\mathcal{S}}} \to \mathbb{R}^{n_\theta}$ and all mappings are known.

Hypothesis 2. There exist the heterogeneous in the sense of (1.8) mappings $\mathcal{X}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\Theta \times n_\Theta}$, $\mathcal{Z}\colon \mathbb{R}^{n_\theta} \to \mathbb{R}^{n_\Theta}$ such that:

$$\begin{gathered} \mathcal{Z}\left(\theta\right) = \mathcal{X}\left(\theta\right)\Theta_{AB}\left(\theta\right), \\ \Pi_\Theta\left(\omega\right)\mathcal{X}\left(\theta\right) = \mathcal{T}_{\mathcal{X}}\left(\Xi_{\mathcal{X}}\left(\omega\right)\theta\right), \\ \Pi_\Theta\left(\omega\right)\mathcal{Z}\left(\theta\right) = \mathcal{T}_{\mathcal{Z}}\left(\Xi_{\mathcal{Z}}\left(\omega\right)\theta\right), \end{gathered} \tag{3.9}$$

where $\Xi_{\mathcal{X}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{X}} \times n_\theta}$, $\Xi_{\mathcal{Z}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{Z}} \times n_\theta}$, $\det\{\Pi_\Theta(\omega)\} \geqslant \omega^{\ell_\Theta}(t)$, $\operatorname{rank}\{\mathcal{X}(\theta)\} = n_\Theta$, $\ell_\Theta \geqslant 1$, $\mathcal{T}_{\mathcal{X}}\colon \mathbb{R}^{\Delta_{\mathcal{X}}} \to \mathbb{R}^{n_\Theta \times n_\Theta}$, $\mathcal{T}_{\mathcal{Z}}\colon \mathbb{R}^{\Delta_{\mathcal{Z}}} \to \mathbb{R}^{n_\Theta}$ and all mappings are known.

Hypothesis 3. There exist the heterogeneous in the sense of (1.8) mappings $\mathcal{W} :\ \mathbb{R}^{n_\theta} \to \mathbb{R}^{n}$, $\mathcal{R} :$ $\mathbb{R}^{n_\theta} \to \mathbb{R}^{n \times n}$ such that:

$$\begin{gathered} \mathcal{W}\left(\theta\right) = \mathcal{R}\left(\theta\right)\psi_d\left(\theta\right), \\ \Pi_{\psi_d}\left(\omega\right)\mathcal{R}\left(\theta\right) = \mathcal{T}_{\mathcal{R}}\left(\Xi_{\mathcal{R}}\left(\omega\right)\theta\right), \\ \Pi_{\psi_d}\left(\omega\right)\mathcal{W}\left(\theta\right) = \mathcal{T}_{\mathcal{W}}\left(\Xi_{\mathcal{W}}\left(\omega\right)\theta\right), \end{gathered} \tag{3.10}$$

where $\Xi_{\mathcal{W}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{W}} \times n_\theta}$, $\Xi_{\mathcal{R}}(\omega) \in \mathbb{R}^{\Delta_{\mathcal{R}} \times n_\theta}$, $\det\{\Pi_{\psi_d}(\omega)\} \geqslant \omega^{\ell_{\psi_d}}(t)$, $\operatorname{rank}\{\mathcal{R}(\theta)\} = n$, $\ell_{\psi_d} \geqslant 1$, $\mathcal{T}_{\mathcal{R}} :\ \mathbb{R}^{\Delta_{\mathcal{R}}} \to \mathbb{R}^{n \times n}$, $\mathcal{T}_{\mathcal{W}} :\ \mathbb{R}^{\Delta_{\mathcal{W}}} \to \mathbb{R}^{n}$ and all mappings are known.

Hypotheses 1–3 are met if the corresponding mappings are defined using elementary algebraic functions in a polynomial form. For example, for vectors $\Theta_{AB}(\theta) = col\left\{\theta_2^2\theta_1^2 + (\theta_2+\theta_1)^3,\ \theta_2\right\}$ and $\psi_{ab}(\theta) = col\left\{\theta_1\theta_2 + \theta_1^2,\ \theta_2+\theta_1\right\}$ the mappings from (3.9) and (3.8) are written as follows

$$\mathcal{T}_{\mathcal{X}}\left(\Xi_{\mathcal{X}}\left(\mathcal{M}_\theta\right)\theta\right) = \begin{bmatrix}\mathcal{M}_\theta^4 & 0\\ 0 & \mathcal{M}_\theta\end{bmatrix},\ \mathcal{T}_{\mathcal{Z}}\left(\Xi_{\mathcal{Z}}\left(\mathcal{M}_\theta\right)\theta\right) = \begin{bmatrix}\mathcal{M}_\theta^4\theta_2^2\theta_1^2 + \mathcal{M}_\theta^4(\theta_2+\theta_1)^3\\ \mathcal{M}_\theta\theta_2\end{bmatrix},$$

$$\mathcal{S}\left(\psi_{ab}\right) = \begin{bmatrix}\psi_{1ab}\\ \psi_{2ab}^2 - \psi_{1ab}\end{bmatrix},\ \mathcal{G}\left(\psi_{ab}\right) = \begin{bmatrix}\psi_{2ab} & 0\\ 0 & \psi_{2ab}\end{bmatrix}, \tag{3.11}$$

$$\mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}\left(\Delta\right)\psi_{ab}\right) = \begin{bmatrix}\Delta\psi_{2ab} & 0\\ 0 & \Delta^2\psi_{2ab}\end{bmatrix},\ \mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}\left(\Delta\right)\psi_{ab}\right) = \begin{bmatrix}\psi_{1ab}\Delta\\ \Delta^2\psi_{2ab}^2 - \Delta^2\psi_{1ab}\end{bmatrix}.$$

The essence of Hypotheses 1–3 is that, owing to the property $\Xi_{(.)}(\omega) = \overline{\Xi}_{(.)}(\omega)\,\omega(t)$, the linear regression equations with respect to the unknown parameters $\theta,\ \Theta_{AB}(\theta),\ \psi_d(\theta)$ can be parametrized on the basis of the measurable signals $\mathcal{Y}_{ab}(t) = \mathcal{L}_{ab}\mathcal{Y}(t) = \Delta(t)\psi_{ab}(\theta)$ and $\mathcal{Y}_\theta(t) = \mathcal{M}_\theta(t)\theta$, respectively.

For instance, equation (3.11) can be rewritten as:

$$\mathcal{T}_{\mathcal{Z}}\left(\overline{\Xi}_{\mathcal{Z}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right) = \begin{bmatrix}\mathcal{Y}_{2\theta}^2\mathcal{Y}_{1\theta}^2 + \mathcal{M}_\theta(\mathcal{Y}_{2\theta}+\mathcal{Y}_{1\theta})^3\\ \mathcal{Y}_{2\theta}\end{bmatrix},$$

$$\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \begin{bmatrix}\mathcal{Y}_{2ab} & 0\\ 0 & \Delta\mathcal{Y}_{2ab}\end{bmatrix},\ \mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \begin{bmatrix}\mathcal{Y}_{1ab}\\ \mathcal{Y}_{2ab}^2 - \Delta\mathcal{Y}_{1ab}\end{bmatrix},$$

and therefore, we directly have the following measurable linear regression equations[3]

$$\mathcal{T}_{\mathcal{Z}}\left(\overline{\Xi}_{\mathcal{Z}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right) = \mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\Theta_{AB}(\theta),$$
$$\mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\theta,$$

where the signals $\mathcal{Y}_\theta(t)$ and $\mathcal{M}_\theta(t)$ are calculated in the following way using the second equation:

$$\mathcal{Y}_\theta(t) = \operatorname{adj}\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}\mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right),$$
$$\mathcal{M}_\theta(t) = \det\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}.$$

The requirement (3.7) and Hypotheses 1–3, despite being seemed as mathematically restrictive, are practice-oriented and met for a large number of models of real technical systems.

[3] A measurable regression equation means that its regressor and regressand are measurable or can be computed, while its parameters are unknown.

4. MAIN RESULT

The solvability conditions (3.7)–(3.10) are assumed to be met and the error equations for the differences between (2.4) and (2.1), $\hat{\delta}(t)$ and $\delta(t)$ are written:

$$\begin{aligned}
\dot{\tilde{x}}(t) &= \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\hat{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) - \Phi^{\mathrm{T}}(x, u, \delta)\,\Theta_{AB} \\
&= \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\hat{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) - \Phi^{\mathrm{T}}(x, u, \delta)\Theta_{AB} \pm \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\Theta_{AB} \\
&= \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\tilde{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) - \Phi^{\mathrm{T}}(x, u, \delta)\Theta_{AB} + \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\Theta_{AB} \\
&= A(\theta)\,\tilde{x}(t) + D(\theta)\,\tilde{\delta}(t) + \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\tilde{\Theta}_{AB}(t) - \hat{L}(t)\tilde{y}(t) \pm L(\theta)\,\tilde{y}(t) \\
&= A_m\tilde{x}(t) + D(\theta)\,h_\delta^{\mathrm{T}}\Phi_\delta(t)\tilde{x}_{\delta 0}(t) + \Phi^{\mathrm{T}}\left(\hat{x}, u, \hat{\delta}\right)\tilde{\Theta}_{AB}(t) - \tilde{L}(t)\tilde{y}(t) \\
&= A_m\tilde{x}(t) + \phi^{\mathrm{T}}(t)\tilde{\kappa}(t), \\
\tilde{\delta}(t) &= h_\delta^{\mathrm{T}}\Phi_\delta(t)\hat{x}_{\delta 0} - h_\delta^{\mathrm{T}}\Phi_\delta(t)x_{\delta 0} = h_\delta^{\mathrm{T}}\Phi_\delta(t)\tilde{x}_{\delta 0}(t),
\end{aligned} \tag{4.1}$$

where

$$\phi^{\mathrm{T}}(t) = \begin{bmatrix} D(\theta)\,h_\delta^{\mathrm{T}}\Phi_\delta(t) & \Phi^{\mathrm{T}}\left(\hat{x},\ u,\ \hat{\delta}\right) & -\tilde{y}(t)I_n \end{bmatrix}$$

and $A_m = A(\theta) - L(\theta)\,C^{\mathrm{T}}$ is a Hurwitz matrix in accordance with Assumption 3.

In order to achieve the goal (2.5), using equations (4.1), an estimation law is required to be designed that ensures exponential convergence to zero of the error $\tilde{\kappa}(t)$ and exponential stability of the equilibrium point of the state observation error $\tilde{x}(t)$. Thus, the problem of reconstruction of the perturbation $\delta(t)$ and unmeasurable state $x(t)$ of the system (2.1) is reduced to the problem of parametric identification. Such problem, in its turn, can be solved if the assumptions (3.7)–(3.10) are met. To design an estimation law that is based on Hypotheses 1–3 and the results of Lemma 1, and ensures achievement of (2.5), we first parameterize the static regression equation with respect to κ.

Lemma 2. *The vector of unknown parameters κ satisfies the linear regression equation*

$$\begin{gathered}
\mathcal{Y}_\kappa(t) = \mathcal{M}_\kappa(t)\kappa, \\
\mathcal{Y}_\kappa(t) = \operatorname{adj}\left\{\operatorname{blkdiag}\left\{\mathcal{M}_{x_{\delta 0}}(t)I_{n_\delta},\ \mathcal{M}_{AB}(t)I_{n_\Theta},\ \mathcal{M}_L(t)I_n\right\}\right\}\begin{bmatrix}\mathcal{Y}_{x_{\delta 0}}(t) \\ \mathcal{Y}_{AB}(t) \\ \mathcal{Y}_L(t)\end{bmatrix}, \\
\mathcal{M}_\kappa(t) = \det\left\{\operatorname{blkdiag}\left\{\mathcal{M}_{x_{\delta 0}}(t)I_{n_\delta},\ \mathcal{M}_{AB}(t)I_{n_\Theta},\ \mathcal{M}_L(t)I_n\right\}\right\},
\end{gathered} \tag{4.2}$$

where:

1) *the regressand and regressor of the regression $\mathcal{Y}_{AB}(t) = \mathcal{M}_{AB}(t)\Theta(\theta)$, using the auxilary calculations*

$$\begin{gathered}
\mathcal{Y}_\theta(t) = \operatorname{adj}\left\{\mathcal{T}_\mathcal{G}\left(\Xi_\mathcal{G}(\Delta)\,\mathcal{Y}_{ab}\right)\right\}\mathcal{T}_\mathcal{S}\left(\Xi_\mathcal{S}(\Delta)\,\mathcal{Y}_{ab}\right), \\
\mathcal{M}_\theta(t) = \det\left\{\mathcal{T}_\mathcal{G}\left(\Xi_\mathcal{G}(\Delta)\,\mathcal{Y}_{ab}\right)\right\},
\end{gathered}$$

are defined as:

$$\begin{gathered}
\mathcal{Y}_{AB}(t) = \operatorname{adj}\left\{\mathcal{T}_\mathcal{X}\left(\Xi_\mathcal{X}(\mathcal{M}_\theta)\,\mathcal{Y}_\theta\right)\right\}\mathcal{T}_\mathcal{Z}\left(\Xi_\mathcal{Z}(\mathcal{M}_\theta)\,\mathcal{Y}_\theta\right), \\
\mathcal{M}_{AB}(t) = \det\left\{\mathcal{T}_\mathcal{X}\left(\Xi_\mathcal{X}(\mathcal{M}_\theta)\,\mathcal{Y}_\theta\right)\right\}.
\end{gathered}$$

2) *the regressand and regressor of the regression* $\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$ *are calculated as:*

$$\mathcal{Y}_L(t) = \mathrm{adj}\left\{\mathcal{T}_\mathcal{P}\left(\overline{\Xi}_\mathcal{P}(\mathcal{M}_{AB})\mathcal{Y}_{AB}\right)\right\}\mathcal{T}_\mathcal{Q}\left(\overline{\Xi}_\mathcal{Q}(\mathcal{M}_{AB})\mathcal{Y}_{AB}\right),$$
$$\mathcal{M}_L(t) = \det\left\{\mathcal{T}_\mathcal{P}\left(\overline{\Xi}_\mathcal{P}(\mathcal{M}_{AB})\mathcal{Y}_{AB}\right)\right\},$$
$$\mathcal{T}_\mathcal{P}\left(\Xi_\mathcal{P}(\mathcal{M}_{AB})\mathcal{Y}_{AB}\right) = vec^{-1}\Big\{\mathcal{M}_{AB}\,\mathrm{adj}\Big\{I_n \otimes vec^{-1}\left(\mathcal{L}_{A^\mathrm{T}}\mathcal{D}_\Phi\mathcal{Y}_{AB}\right)$$
$$- \mathcal{M}_{AB}\Gamma^\mathrm{T} \otimes I_n\Big\} vec\left(C(\mathcal{L}_B\mathcal{D}_\Phi\mathcal{Y}_{AB})^\mathrm{T}\right)\Big\}^\mathrm{T},$$
$$\mathcal{T}_\mathcal{Q}\left(\Xi_\mathcal{Q}(\mathcal{M}_{AB})\mathcal{Y}_{AB}\right) = \det\Big\{I_n \otimes vec^{-1}\left(\mathcal{L}_{A^\mathrm{T}}\mathcal{D}_\Phi\mathcal{Y}_{AB}\right) - \mathcal{M}_{AB}\Gamma^\mathrm{T} \otimes I_n\Big\}\mathcal{L}_B\mathcal{D}_\Phi\mathcal{Y}_{AB}.$$

3) *the regression* $\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}$*, considering the equations*

$$p(t) = \Delta(t)\overline{q}(t) - C_0^\mathrm{T}\Omega(t)\mathcal{L}_a\mathcal{Y}(t) - C_0^\mathrm{T}P(t)\mathcal{L}_b\mathcal{Y}(t),$$
$$\mathcal{Y}_{\psi_d}(t) = \mathrm{adj}\left\{\mathcal{T}_\mathcal{R}\left(\overline{\Xi}_\mathcal{R}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)\right\}\mathcal{T}_\mathcal{W}\left(\overline{\Xi}_\mathcal{W}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right),$$
$$\mathcal{M}_{\psi_d}(t) = \det\left\{\mathcal{T}_\mathcal{R}\left(\overline{\Xi}_\mathcal{R}(\mathcal{M}_\theta)\mathcal{Y}_\theta\right)\right\}$$

and filtering

$$\begin{aligned}
&\dot{V}(t) = A_K V(t) + \left(h_\delta^\mathrm{T}\Phi_\delta(t) \otimes I_n\right), \quad V(t_0) = 0_{n\times nn_\delta},\\
&\dot{p}_f(t) = -k_2\varphi(t) + \Delta(t)(I_{n_\delta} \otimes \mathcal{Y}_{\psi_d}(t))^\mathrm{T}V^\mathrm{T}(t)C_0\mathcal{M}_{\psi_d}(t)p(t),\ p_f(t_0) = 0_{n_\delta},\\
&\dot{V}_f(t) = -k_2V_f(t) + \Delta^2(t)(I_{n_\delta} \otimes \mathcal{Y}_{\psi_d}(t))^\mathrm{T}V^\mathrm{T}(t)C_0C_0^\mathrm{T}V(t)\left(I_{n_\delta} \otimes \mathcal{Y}_{\psi_d}(t)\right),\ V_f(t_0) = 0_{n_\delta\times n_\delta},
\end{aligned} \tag{4.3}$$

is defined as follows:

$$\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}, \quad \mathcal{Y}_{x_{\delta 0}}(t) = \mathrm{adj}\left\{V_f(t)\right\}p_f(t), \quad \mathcal{M}_{x_{\delta 0}}(t) = \det\left\{V_f(t)\right\},$$

and, if the conditions $\overline{\varphi}(t) \in \mathrm{PE}$, $\left(h_\delta^\mathrm{T}\Phi_\delta(t) \otimes I_n\right) \in \mathrm{PE}$ *are met, then for all* $t \geqslant t_0 + T$ *it holds that* $|\mathcal{M}_\kappa(t)| \geqslant \underline{\mathcal{M}_\kappa} > 0$.

Proof of Lemma 2 and definitions of the matrices $\mathcal{L}_{A^\mathrm{T}}, \mathcal{L}_B, \mathcal{L}_a, \mathcal{L}_b$ *are given in Appendix.*

Having at hand the regression equation (4.2) with a scalar regressor $\mathcal{M}_\kappa(t)$, which is bounded away from zero for all $t \geqslant t_0 + T$, and using the results from [13, 18], the estimation law is derived, which ensures that the goal (2.5) is achieved.

Theorem 1. *Let the vector* $D_{\max} \in \mathbb{R}^n$ *be known such that* $\|D(\theta)\| \leqslant \|D_{\max}\|$, *then, if* $\overline{\varphi}(t) \in \mathrm{PE}$, $\left(h_\delta^\mathrm{T}\Phi_\delta(t) \otimes I_n\right) \in \mathrm{PE}$ *and* $\gamma_0 > 0$, $\gamma_1 > 0$, *then the estimation law*

$$\begin{aligned}
\dot{\hat{\kappa}}(t) = \dot{\tilde{\kappa}}(t) &= -\gamma(t)\mathcal{M}_\kappa(t)\left(\mathcal{M}_\kappa(t)\hat{\kappa}(t) - \mathcal{Y}_\kappa(t)\right) = -\gamma(t)\mathcal{M}_\kappa^2(t)\tilde{\kappa}(t),\\
\gamma(t) &:= \begin{cases} 0, \text{ if } \Delta(t) < \rho \in [\Delta_{\min}; \Delta_{\max}), \\ \dfrac{\gamma_0\lambda_{\max}\left(\phi_{\max}(t)\phi_{\max}^\mathrm{T}(t)\right) + \gamma_1}{\mathcal{M}_\kappa^2(t)} \text{ otherwise}, \end{cases}\\
\phi_{\max}^\mathrm{T}(t) &= \begin{bmatrix} D_{\max}h_\delta^\mathrm{T}\Phi_\delta(t) & \Phi^\mathrm{T}\left(\hat{x},\ u,\ \hat{\delta}\right) & -\tilde{y}(t)I_n \end{bmatrix}
\end{aligned} \tag{4.4}$$

ensures the following properties:

1) $\forall t \geqslant t_0$ $\begin{bmatrix}\tilde{x}^\mathrm{T}(t) & \tilde{\kappa}^\mathrm{T}(t)\end{bmatrix}^\mathrm{T} \in L_\infty$;

2) $\forall t \geqslant t_0 + T$ *the error* $\begin{bmatrix}\tilde{x}^\mathrm{T}(t) & \tilde{\kappa}^\mathrm{T}(t)\end{bmatrix}^\mathrm{T}$ *converges exponentially to zero with the rate, which minimum value is directly proportional to* $\gamma_1 > 0$.

Proof of the first part of theorem is similar to the proof of the second part of Theorem 1 from [18], *proof of the second part of theorem coincides up to the notation with the proof of Theorem 1 from* [13].

Owing to the boundedness of $h_\delta^{\mathrm{T}}\Phi_\delta(t)$, the exponential convergence of the error $\tilde{\delta}(t)$ follows from the above-given theorem, which together with the exponential convergence of $\left[\tilde{x}^{\mathrm{T}}(t) \;\; \tilde{\kappa}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ means that the goal (2.5) is achieved.

Remark 2. The results of Lemma 2 describe the procedure to transform the regression equation (3.3) *with a scalar regressor with respect to the numerator and denominator parameters of the transfer function* $W_{uy}(s)$ *into a new equation* (4.2) *with respect to the observer parameters* (2.4). *Considering such recalculation, the division operations by time-dependent signals are not used, the parameters* $\eta(\theta)$, $\psi_{ab}(\theta)$ *or* θ *are not identified, and* $\mathcal{Y}_\kappa(t)$ *and* $\mathcal{M}_\kappa(t)$ *are calculated solely using the signals* $\mathcal{Y}(t)$ *and* $\Delta(t)$ *that are measurable according to the results of Lemma 1.*

Remark 3. The exponential stability conditions from the theorem are conservative. In practice, the knowledge of $D_{\max} \in \mathbb{R}^n$ *and* ρ, *as well as the implementation of the procedure to compute the eigenvalue* $\lambda_{\max}\left(\phi_{\max}(t)\phi_{\max}^{\mathrm{T}}(t)\right)$ *are not required, and the goal* (2.5) *can be achieved using any sufficiently large constant coefficient* $\gamma \geqslant \left(\gamma_{\min} \sim \frac{1}{\mathcal{M}_\kappa^2(t)}\right) > 0$, *which is a majorant for* $\lambda_{\max}\left(\phi_{\max}(t)\phi_{\max}^{\mathrm{T}}(t)\right)$.

5. DISCUSSION

In this section, four additional technical comments on the obtained results are given.

Comment 1. In accordance with the lower bound from (A.48), the regressor $\mathcal{M}_\kappa(t)$ is proportional to a power function $\Delta^{\ell_\theta\ell_\Theta n_\Theta+\ell_\theta\ell_\Theta n(n^3+n)+n_\delta^2\left(2\ell_\theta\ell_{\psi_d}+2\right)}(t)$. Therefore, if $\Delta(t) \ll 1$ or $\Delta(t) \gg 1$, then the computational elimination of the regressor excitation may occur inside a software implementation of the proposed approach:

$$\Delta(t) \ll 1 \Rightarrow \mathcal{M}_\kappa(t) \to 0 \text{ or } \Delta(t) \gg 1 \Rightarrow \mathcal{M}_\kappa(t) \to \infty,$$

i.e. $\mathcal{M}_\kappa(t)$ can become so small or so large that it can not be processed by a computer as its CPU has a limited registers length (for example, in Matlab/Simulink the numbers that are smaller than 10^{-309} or larger than 10^{309} are considered equal to zero and infinity, respectively).

This problem does not concern the theoretical results of the paper, but is related solely to the shortcomings of the existing computational devices. To prevent computational elimination of the regressor excitation, a time-varying amplitude modulator $k(t)$ should be used in accordance with the method of regressor excitation normalization:

$$\begin{gathered} k(t) \sim \frac{1}{\Delta(t)} \text{ or } k(t) := \begin{cases} 1, \text{ if } \Delta(t) < \rho \in [\Delta_{\min};\Delta_{\max}), \\ \dfrac{1}{\Delta(t)} \text{ otherwise,} \end{cases} \\ \text{or } k(t) := \begin{cases} 1, \text{ if } t < t_e \in [t_0;\infty), \\ \dfrac{1}{\Delta(t)} \text{ otherwise.} \end{cases} \end{gathered} \tag{5.1}$$

Moreover, implementing the parameterization (4.2) in practice, it is advisable to apply a multiplication by an amplitude modulator similar to (5.1) after each multiplication by the adjoint matrix adj$\{.\}$. The problem of computational elimination of the regressor excitation was discussed in more detail in Section 3.3 of [18].

Comment 2. The existing identification methods with the relaxed regressor excitation requirements do not allow one to ensure parametric convergence if the parameterized regression equation is affected even by an exponentially decaying perturbation [19].

To solve this problem, in [20] it is proposed to use integral filtering with periodic resetting after a given time interval. The method from [20] allows one to reduce the upper bound of the steady-state parametric error iteratively.

An alternative approach is to extend the identification problem via parameterization of the exponentially decaying perturbation as a linear regression with measurable regressor and unknown parameters—unmeasurable initial conditions [11–13]. This approach allows one to ensure the exponential convergence of the parametric error to zero when the relaxed regressor excitation requirements are met, but it is applicable only to perturbations that can be reduced to a linear regression model. The exponentially decaying perturbation $\epsilon(t)$ of (3.3) cannot be represented in such a way.

Therefore, in contrast to the results of [11–13], in this paper, to achieve the goal (2.5), instead of the relaxed conditions, a stricter one of the regressor persistent excitation (1.9) is required. If this condition is met, then the filters with memory from [11–13] are not required, and the exponential convergence of the parametric error is guaranteed even in case of existence of an exponentially decaying perturbation in the parameterization in use.

It is possible to relax the requirement of the regressor persistent excitation by application of the following filter instead of (3.4):

$$\begin{aligned}
\dot{q}(t) &= \int\limits_{t_\epsilon}^{t} e^{-k_2\tau}\overline{\varphi}_f(\tau)(\overline{q}(\tau) - k_1\overline{q}_f(\tau) - \beta^{\mathrm{T}}(F_f(\tau) + ly_f(\tau)))d\tau,\ q\left(t_\epsilon\right) = 0_{2n}, \\
\dot{\varphi}(t) &= \int\limits_{t_\epsilon}^{t} e^{-k_2\tau}\overline{\varphi}_f\left(\tau\right)\overline{\varphi}_f^{\mathrm{T}}\left(\tau\right)d\tau, \quad \varphi\left(t_\epsilon\right) = 0_{2n\times 2n},
\end{aligned} \tag{5.2}$$

where $t_\epsilon \gg t_0$ is a known time instant when the filtering is started.

If the time instant t_ϵ is chosen so that to satisfy the condition $\varepsilon(t) = o\left(\varphi(t)\eta(\theta)\right)$ from (A.26) for all $t \geqslant t_\epsilon$, and the regressor is finitely exciting over $[t_\epsilon,\, t_e]$, then the goal (2.5) is achieved under the relaxed regressor excitation requirement. More detailed properties of the extended observer on the basis of the parameterization with filtering (5.2) are studied in [21].

Comment 3. According to theorem 1, the proposed observer (2.4) + (4.4) ensures convergence of the state observation error to zero only if the persistent excitation requirements $\overline{\varphi}(t) \in \mathrm{PE}$ and $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right) \in \mathrm{PE}$ are met. As the signal $h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n$ is known for all $t \in [t_0,\, \in fty)$, then the condition $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right) \in \mathrm{PE}$ can be validated offline—before the observer implementation. The condition $\overline{\varphi}(t) \in \mathrm{PE}$ is, strictly speaking, unverifiable both offline and online, since it depends on all previous and future values of the regressor $\overline{\varphi}(t)$. Usually, to meet the regressor persistent excitation condition in linear systems parametrizations of the form (A.25), the control signal is formed so as to belong to a class of functions that are sufficiently rich of some order [5, 6], i.e., the functions that include a sufficient number of spectral lines (harmonics), as far as their Fourier expansion is considered. Considering the parametrization (A.25), (3.4) from this paper, unfortunately, at this stage, it is difficult to define the exact number of spectral lines that the control signal is to include in order to meet the condition $\overline{\varphi}(t) \in \mathrm{PE}$. This is one of the main disadvantages of the proposed solution, which significantly reduces its practical value.

However, owing to the implication from Proposition 1

$$\overline{\varphi}(t) \in \mathrm{PE} \Leftrightarrow \exists t \geqslant t_e \in [t_0, \infty)\ \ \Delta(t) \geqslant \Delta_{\mathrm{LB}} > 0\ ,$$

the proposed observer can be augmented with the following heuristic procedure to obtain the control signal that ensures $\overline{\varphi}(t) \in \mathrm{PE}$.

Initialization. *Set $k = 1$ and $m = 1$.*

Step 1. *Choose*

$$u(t) = u_b(t) + \sum_{i=1}^{m} a_i \sin(\omega_i t), \tag{5.3}$$

where $u_b(t)$ is a stabilizing component of the control signal, for example, a P-controller, a_i stands for an arbitrary amplitude of the i^{th} harmonic, and the frequencies $\omega_i(t)$ are such that $\omega_i \neq \omega_j$ for all $i \neq j$.

Step 2. *Apply the control signal $u(t)$ to the system and calculate the value of $\Delta(t)$ over $[t_{k-1}, t_k]$, where $t_k - t_{k-1}$ is a sufficiently large value.*

Step 3. *If there exists $\Delta_{LB} > 0$ such that $\Delta(t) \geqslant \Delta_{LB}$ for all $t \in [t_{k-1}, t_k]$, then, according to Proposition 1, it holds that*

$$\int\limits_{t}^{t+T} \overline{\varphi}(\tau)\, \overline{\varphi}^{\mathrm{T}}(\tau)\, d\tau \geqslant \alpha I_n \tag{5.4}$$

for all $t \in [t_{k-1}, t_k]$.

Assume that the result obtained at $[t_{k-1}, t_k]$ can be interpolated to the entire time axis $[t_0, \infty)$, then, based on Proposition 1, a control signal is found that satisfies the condition $\overline{\varphi}(t) \in \mathrm{PE}$.

If there is no $\Delta_{LB} > 0$ such that $\Delta(t) \geqslant \Delta_{LB}$ for all $t \in [t_{k-1}, t_k]$, then set $m = m + 1$ and $k = k + 1$, and go to Step 1.

The essence of the above-given algorithm is to increase iteratively the number of harmonics in the control signal until the scalar regressor $\Delta(t)$ becomes to be bounded away from zero over a sufficiently large time interval. Assuming that there exists $m_{\max}$ such that, when $m = m_{\max}$, then the control signal (5.3) ensures that the condition $\overline{\varphi}(t) \in \mathrm{PE}$ is met, then it can be claimed that this algorithm, in a finite number of iterations $m_{\max}$, allows one to generate a control signal that ensures that the condition $\overline{\varphi}(t) \in \mathrm{PE}$ is met. It should be noted that the above procedure is not mathematically rigorous because it is designed under the strict assumption that the fact that the condition (5.4) is met over a sufficiently large time interval means that the condition (5.4) is satisfied over the entire time axis. Generally speaking, such a conclusion cannot be made. However, as far as practical scenarios are concerned, the above-mentioned simplification is acceptable, and the described procedure can be efficient.

Comment 4. If, in addition to Hypotheses 1–3, in the extended system

$$\begin{gathered} \dot{x}_e(t) = \left(A_e(\theta) + A_\delta\right) x_e(t) + B_e u(t) = \Phi^{\mathrm{T}}(x_e,\ u)\, \Theta_{AB}(\theta) + A_\delta x_e(t), \\ y(t) = C_e^{\mathrm{T}} x_e(t), \quad x_e(t_0) = \begin{bmatrix} x_0 & x_{\delta 0} \end{bmatrix}^{\mathrm{T}}, \end{gathered} \tag{5.5}$$

$$A_e(\theta) = \begin{bmatrix} A(\theta) & D(\theta)\, h_\delta^{\mathrm{T}} \\ 0_{n_\delta \times n} & 0_{n_\delta \times n_\delta} \end{bmatrix}, \quad A_\delta = \begin{bmatrix} 0_{n \times n} & 0_{n \times n_\delta} \\ 0_{n_\delta \times n} & \mathcal{A}_\delta \end{bmatrix},$$

$$B_e(\theta) = \begin{bmatrix} B(\theta) \\ 0_{n_\delta} \end{bmatrix}, \quad x_e(t) = \begin{bmatrix} x(t) \\ x_\delta(t) \end{bmatrix}, \quad C_e = \begin{bmatrix} C \\ 0_{n_\delta} \end{bmatrix},$$

$$\Phi^{\mathrm{T}}(x_e,\ u) = \begin{bmatrix} I_{n+n_\delta} \otimes x_e^{\mathrm{T}}(t) & I_{n+n_\delta} \otimes u^{\mathrm{T}}(t) \end{bmatrix} \mathcal{D}_\Phi \in \mathbb{R}^{(n+n_\delta) \times n_\Theta},$$

$$\Theta_{AB}(\theta) = \mathcal{L}_\Phi \begin{bmatrix} vec^{\mathrm{T}}\left(A_e^{\mathrm{T}}(\theta)\right) & B_e^{\mathrm{T}}(\theta) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_\Theta}$$

the pair $\left(C_e^{\mathrm{T}},\ A_e\right)$ is observable, then the extended observer

$$\dot{\hat{x}}_e(t) = \Phi^{\mathrm{T}}\left(\hat{x}_e,\ u\right)\hat{\Theta}_{AB}(t) + A_\delta \hat{x}_\delta(t) - \hat{L}_e(t)\left(\hat{y}(t) - y(t)\right), \tag{5.6}$$

which is augmented only with the estimation laws for $\Theta_{AB}(\theta)$ and $L_e(\theta)$, (i) does not require to parametrize (4.3) the regression equation $\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}$, (ii) does not require to identify the initial conditions $x_{\delta 0}$ of the exosystem (2.2), and, (iii) if the condition (1.9) is met in case (3.4) is used or the regressor finite excitation condition is met in case (5.2) is used, ensures the exponential convergence to zero of the errors $\tilde{x}(t)$, $\tilde{\delta}(t)$. In (5.5) $\hat{L}_e(t)$ is an estimate of the vector $L_e(\theta) \in \mathbb{R}^{n+n_\delta}$, which ensures that the matrix $A_e(\theta) + A_\delta$ has the desired algebraic spectrum. The linear regression equation with respect to $L_e(\theta)$ is parameterised in the same way as $\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$, but in the space $n + n_\delta$. More detailed properties of the alternative version of the extended observer are given in [21].

6. MATHEMATICAL MODELLING

In Matlab/Simulink the numerical experiments with the proposed adaptive observer have been conducted. The simulation was done using numerical integration by the explicit Euler method with a constant discretization step of $\tau_s = 10^{-4}$ s.

A two-mass elastic mechanical system shown in Fig. 1 was chosen as a plant.

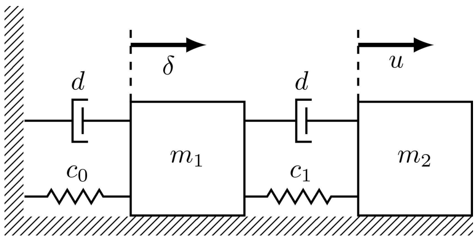

Fig. 1. Two-mass elastic mechanical system.

Here $c_0 > 0,\ c_1 > 0$ denote spring stiffness, $d > 0$ is a damping coefficient, $m_1 > 0,\ m_2 > 0$ are reduced masses of the bodies.

The mathematical model of the system under consideration was written as the following system of differential equations:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\theta_1\left(\theta_2+\theta_3\right) & -2\theta_1\theta_4 & \theta_1\theta_3 & \theta_1\theta_4 \\ 0 & 0 & 0 & 1 \\ \theta_5\theta_3 & \theta_5\theta_4 & -\theta_3\theta_5 & -2\theta_4\theta_5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \theta_5 \end{bmatrix}u + \begin{bmatrix} 0 \\ \theta_1 \\ 0 \\ 0 \end{bmatrix}\delta, \\ y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}x, \end{aligned} \tag{6.1}$$

where

$$\theta = col\left\{m_1^{-1},\ c_0,\ c_1,\ d,\ m_2^{-1}\right\},$$
$$\Theta(\theta) = col\left\{1,\ \theta_1\left(\theta_2+\theta_3\right),\ \theta_1\theta_4,\ \theta_1\theta_3,\ \theta_3\theta_5,\ \theta_4\theta_5,\ \theta_1,\ \theta_5\right\}.$$

In the observer canonical form (3.1) the parameters of the system (6.1) were defined as follows:

$$\psi_a(\theta) = \begin{bmatrix} -2\theta_4(\theta_1+\theta_5) \\ -\theta_1\left(3\theta_5\theta_4^2+\theta_2+\theta_3\right)-\theta_3\theta_5 \\ -2\theta_1\theta_4\theta_5(\theta_2+\theta_3) \\ -\theta_1\theta_2\theta_3\theta_5 \end{bmatrix},\quad \psi_b(\theta) = \begin{bmatrix} 0 \\ \theta_5 \\ 2\theta_1\theta_4\theta_5 \\ \theta_1\theta_5(\theta_2+\theta_3) \end{bmatrix},$$
$$\psi_d(\theta) = \begin{bmatrix} 0 & 0 & \theta_1\theta_4\theta_5 & \theta_1\theta_3\theta_5 \end{bmatrix}^{\mathrm{T}}, \tag{6.2}$$

from which it followed that the condition (3.7) was met for

$$\psi_{ab}(\theta) = \begin{bmatrix} -2\theta_4(\theta_1+\theta_5) \\ -\theta_1\left(3\theta_5\theta_4^2+\theta_2+\theta_3\right)-\theta_3\theta_5 \\ \theta_1\theta_5(\theta_2+\theta_3) \\ 2\theta_1\theta_4\theta_5 \\ \theta_5 \end{bmatrix}.$$

The regression equation (4.2) was parameterised using the transformations introduced in Hypotheses 1–3 and Lemma 2.

Step 1. The derivation of the parametrization $\mathcal{Y}_\theta(t) = \mathcal{M}_\theta(t)\theta$. The following set of the nonlinear algebraic equations were solved with respect to θ

$$\psi_{ab}(\theta) = \begin{bmatrix} -2\theta_4(\theta_1+\theta_5) \\ -\theta_1\left(3\theta_5\theta_4^2+\theta_2+\theta_3\right)-\theta_3\theta_5 \\ \theta_1\theta_5(\theta_2+\theta_3) \\ 2\theta_1\theta_4\theta_5 \\ \theta_5 \end{bmatrix} = \begin{bmatrix} \psi_{1ab} \\ \psi_{2ab} \\ \psi_{3ab} \\ \psi_{4ab} \\ \psi_{5ab} \end{bmatrix},$$

and, using such solution, the mappings $\mathcal{S}(\psi_{ab})$ and $\mathcal{G}(\psi_{ab})$ from (3.8) were obtained:

$$\mathcal{S}(\psi_{ab}) = \begin{bmatrix} \psi_{4ab}\psi_{5ab} \\ \psi_{3ab}\psi_{5ab}(-\psi_{4ab}-\psi_{1ab}\psi_{5ab}) \\ +\,\psi_{4ab}\left((\psi_{2ab}\psi_{5ab}+\psi_{3ab})\psi_{5ab}+\frac{3}{4}\psi_{4ab}(-\psi_{4ab}-\psi_{1ab}\psi_{5ab})\right) \\ (\psi_{2ab}\psi_{5ab}+\psi_{3ab})\psi_{5ab}+\frac{3}{4}\psi_{4ab}(-\psi_{4ab}-\psi_{1ab}\psi_{5ab}) \\ -\psi_{4ab}-\psi_{1ab}\psi_{5ab} \\ \psi_{5ab}^2 \end{bmatrix},$$

$$\mathcal{G}(\psi_{ab}) = \operatorname{diag}\begin{bmatrix} -\psi_{4ab}-\psi_{1ab}\psi_{5ab} \\ \psi_{4ab}\psi_{5ab}^3 \\ -\psi_{5ab}^3 \\ 2\psi_{5ab}^2 \\ \psi_{5ab} \end{bmatrix}.$$

Then the mappings $\mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}(\Delta)\mathcal{Y}_{ab}\right)$, $\mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}(\Delta)\mathcal{Y}_{ab}\right)$ were defined as follows:

$$\mathcal{T}_{\mathcal{S}}\left(\Xi_{\mathcal{S}}(\Delta)\mathcal{Y}_{ab}\right)=\begin{bmatrix}\mathcal{Y}_{4ab}\mathcal{Y}_{5ab}\\ \mathcal{Y}_{3ab}\mathcal{Y}_{5ab}\left(-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\right)\\ +\mathcal{Y}_{4ab}\left(\left(\mathcal{Y}_{2ab}\mathcal{Y}_{5ab}+\Delta\mathcal{Y}_{3ab}\right)\mathcal{Y}_{5ab}+\frac{3}{4}\Delta\mathcal{Y}_{4ab}\left(-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\right)\right)\\ \left(\mathcal{Y}_{2ab}\mathcal{Y}_{5ab}+\Delta\mathcal{Y}_{3ab}\right)\mathcal{Y}_{5ab}+\frac{3}{4}\mathcal{Y}_{4ab}\left(-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\right)\\ -\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\\ \mathcal{Y}_{5ab}^{2}\end{bmatrix},$$

$$\mathcal{T}_{\mathcal{G}}\left(\Xi_{\mathcal{G}}(\Delta)\mathcal{Y}_{ab}\right)=\operatorname{diag}\begin{bmatrix}-\Delta\mathcal{Y}_{4ab}-\mathcal{Y}_{1ab}\mathcal{Y}_{5ab}\\ \mathcal{Y}_{4ab}\mathcal{Y}_{5ab}^{3}\\ -\mathcal{Y}_{5ab}^{3}\\ 2\mathcal{Y}_{5ab}^{2}\\ \Delta\mathcal{Y}_{5ab}\end{bmatrix},$$

which allowed one to compute $\mathcal{Y}_{\theta}(t)$ and $\mathcal{M}_{\theta}(t)$.

Step 2. Using the above-obtained equation $\mathcal{Y}_{\theta}(t)=\mathcal{M}_{\theta}(t)\theta$, the following mappings were obtained

$$\mathcal{T}_{\mathcal{Z}}\left(\Xi_{\mathcal{Z}}(\mathcal{M}_{\theta})\mathcal{Y}_{\theta}\right)=\operatorname{col}\left\{\mathcal{M}_{\theta},\ \mathcal{Y}_{1\theta}\left(\mathcal{Y}_{2\theta}+\mathcal{Y}_{3\theta}\right),\ \mathcal{Y}_{1\theta}\mathcal{Y}_{4\theta},\ \mathcal{Y}_{1\theta}\mathcal{Y}_{3\theta},\ \mathcal{Y}_{3\theta}\mathcal{Y}_{5\theta},\ \mathcal{Y}_{4\theta}\mathcal{Y}_{5\theta},\ \mathcal{Y}_{1\theta},\ \mathcal{Y}_{5\theta}\right\},$$
$$\mathcal{T}_{\mathcal{X}}\left(\Xi_{\mathcal{X}}(\mathcal{M}_{\theta})\mathcal{Y}_{\theta}\right)=\operatorname{blkdiag}\left\{\mathcal{M}_{\theta},\ \mathcal{M}_{\theta}^{2}I_{5},\ \mathcal{M}_{\theta},\ \mathcal{M}_{\theta}\right\},$$

therefore, we could calculate $\mathcal{Y}_{AB}(t)$ and $\mathcal{M}_{AB}(t)$.

Step 3. Having equation $\mathcal{Y}_{AB}(t)=\mathcal{M}_{AB}(t)\Theta_{AB}(\theta)$ at hand and using equations from the second statement of lemma 2, the values of $\mathcal{Y}_{L}(t)$ and $\mathcal{M}_{L}(t)$ were computed.

Step 4. Applying equation $\mathcal{Y}_{\theta}(t)=\mathcal{M}_{\theta}(t)\theta$ from the first step, the following mappings were obtained:

$$\mathcal{T}_{\mathcal{W}}\left(\Xi_{\mathcal{W}}(\mathcal{M}_{\theta})\mathcal{Y}_{\theta}\right)=\operatorname{col}\left\{0,\ 0,\ \mathcal{Y}_{1\theta}\mathcal{Y}_{4\theta}\mathcal{Y}_{5\theta},\ \mathcal{Y}_{1\theta}\mathcal{Y}_{3\theta}\mathcal{Y}_{5\theta}\right\},$$
$$\mathcal{T}_{\mathcal{R}}\left(\Xi_{\mathcal{R}}(\mathcal{M}_{\theta})\mathcal{Y}_{\theta}\right)=\operatorname{diag}\left\{\mathcal{M}_{\theta},\ \mathcal{M}_{\theta},\ \mathcal{M}_{\theta}^{3},\ \mathcal{M}_{\theta}^{3}\right\},$$

which allowed one to calculate $\mathcal{Y}_{\psi_d}(t)$, $\mathcal{M}_{\psi_d}(t)$ and $\mathcal{Y}_{x_{\delta 0}}(t)$, $\mathcal{M}_{x_{\delta 0}}(t)$ on the basis of equations from the third statement of Lemma 2.

At this point, having $\mathcal{Y}_{AB}(t),\mathcal{Y}_{L}(t),\mathcal{Y}_{x_{\delta 0}}(t)$ at hand, equation (4.2) with measurable regressand $\mathcal{Y}_{\kappa}(t)$ and regressor $\mathcal{M}_{\kappa}(t)$ could be obtained, and the observer (2.4) with estimation law (4.4) was going to be implemented.

The unknown parameters of the system (6.1), the disturbance exosystem (2.2) and exosystem (2.3) parameters were picked as:

$$\begin{gathered}\theta=[1\ \ 0.5\ \ 0.75\ \ 0.25\ \ 0.5]^{\mathrm{T}},\ x_0=\begin{bmatrix}0 & 0 & -1 & 0\end{bmatrix}^{\mathrm{T}},\ x_{\delta 0}=\begin{bmatrix}-4 & 1\end{bmatrix}^{\mathrm{T}},\\ \mathcal{A}_{\delta}=\begin{bmatrix}0 & 1\\ -5 & 0\end{bmatrix},\ h_{\delta}^{\mathrm{T}}=\begin{bmatrix}1 & 0\end{bmatrix},\ \sigma\{\Gamma\}=\begin{bmatrix}-1 & -1 & -1 & -1\end{bmatrix}^{\mathrm{T}}.\end{gathered}\tag{6.3}$$

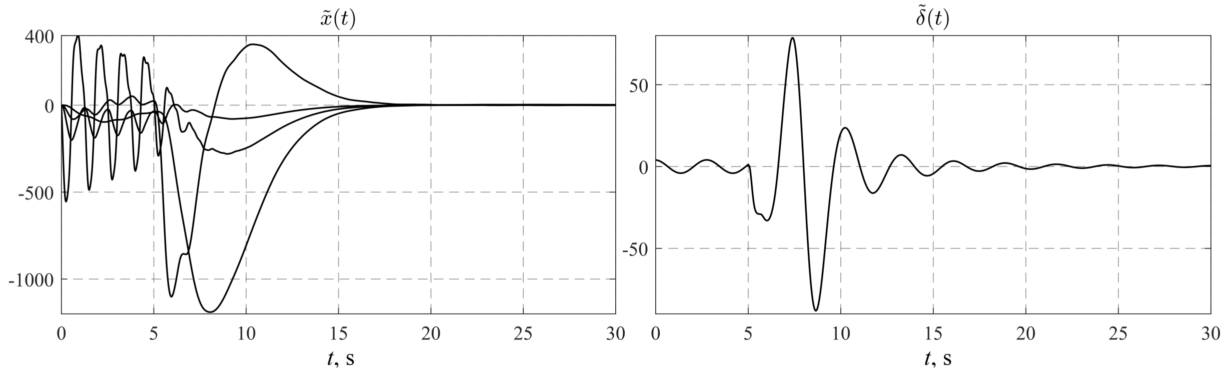

Fig. 2. Behavior of $\tilde{x}(t)$ and $\tilde{\delta}(t)$.

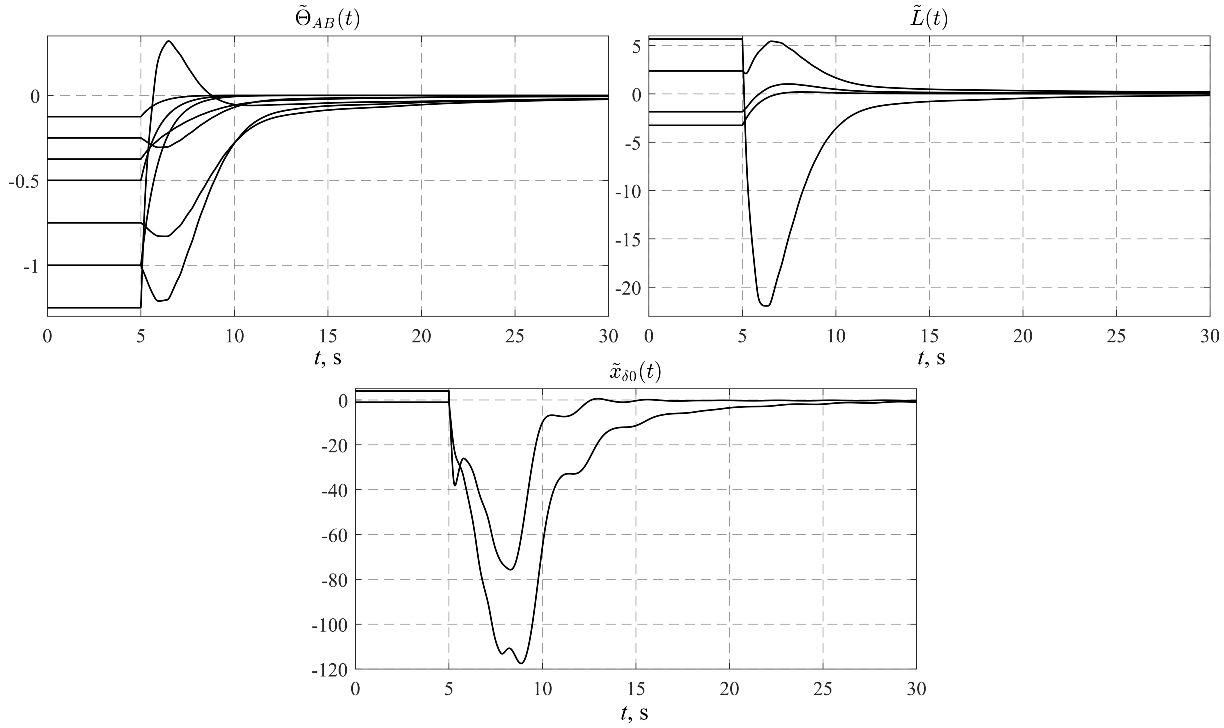

Fig. 3. Behaviour of $\tilde{\Theta}_{AB}(t)$, $\tilde{L}(t)$, $\tilde{x}_{\delta 0}(t)$.

The control signal $u(t)$ was obtained from a P-controller with the reference signal that was chosen by trial and error so as to ensure $\overline{\varphi}(t) \in \mathrm{PE}$:

$$\begin{aligned} &u(t) = 50\,(r(t) - y(t))\,, \\ &r(t) = 25\sin(10t) + 25\sin(20t) + 100\cos(0.1t)\,. \end{aligned} \tag{6.4}$$

The parameters of the filters (3.4)–(3.6), (4.3) and estimation law (4.4) were set as follows

$$\begin{gathered} \det\{sI_4 - A_K\} = (s+1)^4, \;\; G = \begin{bmatrix} -4 & 1 \\ -2 & 0 \end{bmatrix}, \;\; l = \begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathrm{T}}, \\ \beta = \begin{bmatrix} 15 & -5.5 \end{bmatrix}^{\mathrm{T}}, \;\; k(t) = \begin{cases} 1, & \text{if } \; t < 5 \\ \Delta^{-1}(t), & \text{if } \; t \geqslant 5, \end{cases} \;\; k_1 = 25, \;\; k_2 = 0.1, \\ \rho = 10^{-4}, \;\; \gamma_0 = 10^{-9}, \;\; \gamma_1 = 1. \end{gathered} \tag{6.5}$$

Figure 2 shows the behavior of the state $\tilde{x}(t)$ and the external perturbation $\tilde{\delta}(t)$ observation errors.

The peaking in $\tilde{x}(t)$ over [5, 15] was caused primarily the fact that the error equation (4.1) with non-zero initial conditions [22] was included in the closed loop with $\hat{L}(t)\left(\hat{y}(t) - y(t)\right)$. The peaking in $\tilde{\delta}(t)$ could be explained by the fact that the behavior of $\tilde{x}_{\delta 0}(t)$ was affected by the perturbation $\epsilon(t)$.

Figure 3 depicts the behavior of the parametric errors $\tilde{\Theta}_{AB}(t)$, $\tilde{L}(t)$, and $\tilde{x}_{\delta 0}(t)$.

The oscillations of the obtained transients of the parametric errors $\tilde{\Theta}_{AB}(t)$, $\tilde{L}(t)$, $\tilde{x}_{\delta 0}(t)$ were caused by the influence of the exponentially decaying perturbation $\epsilon(t)$ from the parameterization (3.3). In general, the simulation results validated that the goal (2.5) was achieved.

7. CONCLUSION

An adaptive observer of state and perturbations for linear systems with overparameterization is developed. If the condition of the regressor persistent excitation (sufficient richness of the control/reference signal) is met, the solution provides exponential convergence to zero of the observation errors of the system state and the external perturbation generated by a known exosystem with unknown initial conditions. Unlike the closest analogues [10–12], the proposed observer allows one to reconstruct not virtual but physical state of the system represented in an arbitrary state space form.

The scopes of the further research can be:

— an application of the proposed observer to solve control problems with dynamic feedback;

— the relaxation of (1.9) by substituting (3.3) with a parametrization, which does not include $\epsilon(t)$ (a preliminary result for this problem has been obtained in comment 2 and [21]);

— the extension of the obtained results to systems with new, possibly nonlinear, models of the external perturbations;

— taking into consideration the additive disturbances, which affect the measurable output signal $y(t)$ directly;

— following [12], to reduce the transients peaking amplitude by estimation of the state $x(t)$ with the help of an algebraic equation instead of the differential one (a preliminary result for this problem has been obtained in [23]).

FUNDING

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APPENDIX

Proof of Lemma 1. The parameterization (3.3) is obtained as a combination of the results from [12, 24] with the dynamic regressor extension and mixing procedure from [14, 19]. The proof of lemma 1 is derived on the basis of Lemma 1 and Theorem 2 from [24]. To make it easier to understand the adopted notation and ensure that the results of the paper are self-contained, we next present the proof of this lemma in accordance with the one in [24]. In contrast to the results [24], in this paper, owing to Assumption 2, β is known, which allows one not to avoid overparameterization in (3.3) (see (A.23)).

Step 1. The following error is considered:

$$\tilde{\xi}(t) = \xi(t) - z(t) - \Omega(t)\psi_a(\theta) - P(t)\psi_b(\theta). \tag{A.1}$$

The time derivative of (A.1) is written:

$$\begin{gathered}\dot{\tilde{\xi}}(t)=A_0\xi(t)+\psi_a(\theta)y(t)+\psi_b(\theta)u(t)+\psi_d(\theta)\delta(t)-A_Kz(t)\\ -Ky(t)-(A_K\Omega(t)+I_ny(t))\psi_a(\theta)-(A_KP(t)+I_nu(t))\psi_b(\theta)\\ =A_0\xi(t)-A_Kz(t)-Ky(t)-A_K\Omega(t)\psi_a(\theta)-A_KP(t)\psi_b(\theta)+\psi_d(\theta)\delta(t)\\ =A_K\tilde{\xi}(t)+\psi_d(\theta)\delta(t).\end{gathered}\tag{A.2}$$

The solution of equation (A.2) is obtained as

$$\tilde{\xi}(t)=e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right)+\overline{\delta}(t),\tag{A.3}$$

where the external perturbation $\overline{\delta}(t)$ is described as a set of equations

$$\begin{cases}\dot{\overline{\delta}}(t)=A_K\overline{\delta}(t)+\psi_d\left(\theta\right)\delta(t),\\ v_f(t)=C_0^{\mathrm{T}}\overline{\delta}(t).\end{cases}\tag{A.4}$$

Having substituted (A.3) into (A.1), it is written:

$$\begin{gathered}e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right)+\overline{\delta}(t)=\xi(t)-z(t)-\Omega(t)\psi_a\left(\theta\right)-P(t)\psi_b\left(\theta\right),\\ \Updownarrow\\ \xi(t)=e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right)+\overline{\delta}(t)+z(t)+\Omega(t)\psi_a\left(\theta\right)+P(t)\psi_b\left(\theta\right).\end{gathered}\tag{A.5}$$

Equation (A.5) is multiplied by C_0^{T} to obtain:

$$y(t)=C_0^{\mathrm{T}}\xi(t)=C_0^{\mathrm{T}}z(t)+C_0^{\mathrm{T}}\Omega(t)\psi_a\left(\theta\right)+C_0^{\mathrm{T}}P(t)\psi_b\left(\theta\right)+v_f(t)+C_0^{\mathrm{T}}e^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right).\tag{A.6}$$

Considering (A.6), the function $\overline{q}=y(t)-C_0^{\mathrm{T}}z(t)$ is differentiated:

$$\dot{\overline{q}}(t)=C_0^{\mathrm{T}}\dot{\Omega}(t)\psi_a\left(\theta\right)+C_0^{\mathrm{T}}\dot{P}(t)\psi_b\left(\theta\right)+\dot{v}_f(t)+C_0^{\mathrm{T}}A_Ke^{A_K(t-t_0)}\tilde{\xi}\left(t_0\right).\tag{A.7}$$

Step 2. The next aim is to parametrize the term $\dot{v}_f(t)$ of equation (A.7) as a linear regression equation with a measurable regressor. For this purpose, the system (A.4) is rewritten as a transfer function:

$$v_f(t)=C_0^{\mathrm{T}}(sI_n-A_K)^{-1}\psi_d\left(\theta\right)\delta(t)=W_f\left[\delta(t)\right].\tag{A.8}$$

The derivative of the perturbation $\delta(t)$ is represented as:

$$\dot{\delta}(t)=h_\delta^{\mathrm{T}}\mathcal{A}_\delta x_\delta(t)+\delta\left(t_0\right)D_\delta(t),\tag{A.9}$$

where $D_\delta(t)$ is a Dirac delta function.

A virtual signal $\delta_d(t)=h_\delta^{\mathrm{T}}\mathcal{A}_\delta x_\delta(t)$ is introduced into consideration. Then the following equalities hold

$$\begin{gathered}\dot{x}_\delta(t)=\mathcal{A}_\delta x_\delta(t),\\ \delta_d(t)=\overline{h}_\delta^{\mathrm{T}}x_\delta(t),\ \overline{h}_\delta^{\mathrm{T}}=h_\delta^{\mathrm{T}}\mathcal{A}_\delta.\end{gathered}\tag{A.10}$$

Equation (A.8) is differentiated, and then (A.9), (A.10) are substituted into the obtained result to write:

$$\begin{aligned}\dot{v}_f&=sW_f\left[\delta(t)\right]=W_f\left[\dot{\delta}(t)\right]\\ &=W_f\left[h_\delta^{\mathrm{T}}A_\delta x_\delta(t)+\delta\left(t_0\right)D_\delta(t)\right]=\underbrace{W_f\left[\delta_d(t)\right]}_{v_f(t)}+W_f\left[\delta\left(t_0\right)D_\delta(t)\right].\end{aligned}\tag{A.11}$$

Thus, owing to the fact that the matrix A_K is a Hurwitz one, it is sufficient to parametrize $\upsilon_f(t)$ to parameterize $\dot{\upsilon}_f(t)$. For this purpose, an auxiliary signal $\zeta(t) = M_\delta x_\delta(t)$ is considered, where the transformation matrix M_δ is a solution of the Sylvester equation

$$M_\delta \mathcal{A}_\delta - G M_\delta = l \overline{h}_\delta^{\mathrm{T}}, \tag{A.12}$$

which has a unique solution [15, 16, 24] as, owing to Assumption 2, the pair $\left(h_\delta^{\mathrm{T}},\ \mathcal{A}_\delta\right)$ is observable and, following the premises of this lemma, $(G,\ l)$ is controllable and $\sigma\left\{\mathcal{A}_\delta\right\} \cap \sigma\left\{G\right\} = 0$.

We differentiate $\zeta(t)$ to obtain:

$$\dot{\zeta}(t) = M_\delta \mathcal{A}_\delta x_\delta(t) = G M_\delta x_\delta(t) + l \overline{h}_\delta^{\mathrm{T}} x_\delta(t) = G\zeta(t) + l\delta_d(t), \tag{A.13}$$

form which, considering $x_\delta(t) = M_\delta^{-1}\zeta(t)$, it follows that

$$\delta_d(t) = \overline{h}_\delta^{\mathrm{T}} M_\delta^{-1}\zeta = \beta^{\mathrm{T}}\zeta,\ \beta = \overline{h}_\delta^{\mathrm{T}} M_\delta^{-1}. \tag{A.14}$$

Taking into account (A.14), equation (A.11) is rewritten as:

$$\begin{aligned} \dot{\upsilon}_f(t) &= W_f\left[\beta^{\mathrm{T}}\zeta(t)\right] + W_f\left[\delta\left(t_0\right) D_\delta(t)\right] \\ &= \beta^{\mathrm{T}} W_f\left[\zeta(t)\right] + W_f\left[\delta\left(t_0\right) D_\delta(t)\right] = \beta^{\mathrm{T}}\zeta_w(t) + W_f\left[\delta\left(t_0\right) D_\delta(t)\right]. \end{aligned} \tag{A.15}$$

The signal $\upsilon_f(t)$ is filtered via (A.13) instead of $\delta_d(t)$:

$$\zeta_f(t) = (sI - G)^{-1} l\left[\upsilon_f(t)\right] + e^{G(t-t_0)}\zeta_f\left(t_0\right), \tag{A.16}$$

then, owing to $\zeta(t) = (sI - G)^{-1} l\left[\delta_d(t)\right] + e^{G(t-t_0)}\zeta\left(t_0\right)$, the following equality holds:

$$\begin{aligned} \zeta_w(t) = W_f\left[\zeta(t)\right] &= W_f\left[(sI - G)^{-1} l\left[\delta_d(t)\right] + e^{G(t-t_0)}\zeta\left(t_0\right)\right] \\ &= (sI - G)^{-1} l W_f\left[\delta_d(t)\right] + W_f\left[e^{G(t-t_0)}\zeta\left(t_0\right)\right] \\ &= (sI - G)^{-1} l\upsilon_f + W_f\left[e^{G(t-t_0)}\zeta\left(t_0\right)\right] \\ &= \zeta_f(t) - e^{G(t-t_0)}\zeta_f\left(t_0\right) + W_f\left[e^{G(t-t_0)}\zeta\left(t_0\right)\right]. \end{aligned} \tag{A.17}$$

Having substituted (A.17) into (A.15), it is written:

$$\dot{\upsilon}_f(t) = \beta^{\mathrm{T}}\zeta_f(t) - \beta^{\mathrm{T}} e^{G(t-t_0)}\zeta_f\left(t_0\right) + \beta^{\mathrm{T}} W_f\left[e^{G(t-t_0)}\xi\left(t_0\right)\right] + W_f\left[\delta\left(t_0\right) D_\delta(t)\right]. \tag{A.18}$$

The following observer of state $\zeta_f(t)$ is introduced:

$$\hat{\zeta}_f(t) = F(t) + H(t)\psi_b\left(\theta\right) + N(t)\psi_a\left(\theta\right) + ly(t). \tag{A.19}$$

Considering equations (A.7), (A.11), (A.16), (A.19), the error is differentiated $\tilde{\zeta}_f(t) = \zeta_f(t) - \hat{\zeta}_f(t)$ to obtain:

$$\begin{aligned}
\dot{\tilde{\zeta}}_f &= G\zeta_f(t) + l\upsilon_f(t) - GF(t) - Gly(t) + lC_0^{\mathrm{T}}\dot{z}(t) \\
&- \left(GH(t) - lC_0^{\mathrm{T}}\dot{P}(t)\right)\psi_b(\theta) - \left(GN(t) - lC_0^{\mathrm{T}}\dot{\Omega}(t)\right)\psi_a(\theta) \\
&- lC_0^{\mathrm{T}}\dot{z}(t) - lC_0^{\mathrm{T}}\dot{\Omega}(t)\psi_a(\theta) - lC_0^{\mathrm{T}}\dot{P}(t)\psi_b(\theta) \\
&- l\left(\upsilon_f(t) + W_f\left[\delta(t_0)D_\delta(t)\right]\right) - lC_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) \\
&= G\zeta_f(t) - \underbrace{GF(t) - Gly(t) - GH(t)\psi_b(\theta) - GN(t)\psi_a(\theta)}_{G\hat{\zeta}_f(t)} \\
&- lW_f\left[\delta(t_0)D_\delta(t)\right] - lC_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) \\
&= G\tilde{\zeta}_f - lC_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) - lW_f\left[\delta(t_0)D_\delta(t)\right].
\end{aligned} \tag{A.20}$$

The set of equations (A.20) is solved:

$$\tilde{\zeta}_f(t) = \zeta_f(t) - \hat{\zeta}_f(t) = e^{G(t-t_0)}\tilde{\zeta}_f(t_0) - \mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right], \tag{A.21}$$

which allows one to rewrite (A.18) as follows:

$$\begin{aligned}
\dot{\upsilon}_f(t) &= \beta^{\mathrm{T}}\hat{\zeta}_f(t) + \beta^{\mathrm{T}}e^{G(t-t_0)}\tilde{\zeta}_f(t_0) \\
&-\beta^{\mathrm{T}}\mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right] \\
&-\beta^{\mathrm{T}}e^{G(t-t_0)}\zeta_f(t_0) + \beta^{\mathrm{T}}W_f\left[e^{G(t-t_0)}\xi(t_0)\right] + W_f\left[\delta(t_0)D_\delta(t)\right] \\
&= \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \beta^{\mathrm{T}}H(t)\psi_b(\theta) + \beta^{\mathrm{T}}N(t)\psi_a(\theta) \\
&+\beta^{\mathrm{T}}e^{G(t-t_0)}\tilde{\zeta}_f(t_0) - \beta^{\mathrm{T}}\mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right] \\
&-\beta^{\mathrm{T}}e^{G(t-t_0)}\zeta_f(t_0) + \beta^{\mathrm{T}}W_f\left[e^{G(t-t_0)}\xi(t_0)\right] + W_f\left[\delta(t_0)D_\delta(t)\right],
\end{aligned} \tag{A.22}$$

where $\mathfrak{H}[.] = (sI_{n_\delta} - G)^{-1}l[.]$.

Equation (A.22) is substituted into (A.7) to obtain:

$$\begin{aligned}
\dot{\overline{q}}(t) &= C_0^{\mathrm{T}}\dot{\Omega}(t)\psi_a(\theta) + C_0^{\mathrm{T}}\dot{P}(t)\psi_b(\theta) \\
&+ \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \beta^{\mathrm{T}}H(t)\psi_b(\theta) + \beta^{\mathrm{T}}N(t)\psi_a(\theta) \\
&+ \beta^{\mathrm{T}}e^{G(t-t_0)}\tilde{\zeta}_f(t_0) - \beta^{\mathrm{T}}\mathfrak{H}\left[C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) + W_f\left[\delta(t_0)D_\delta(t)\right]\right] \\
&- \beta^{\mathrm{T}}e^{G(t-t_0)}\zeta_f(t_0) + \beta^{\mathrm{T}}W_f\left[e^{G(t-t_0)}\xi(t_0)\right] \\
&+ W_f\left[\delta(t_0)D_\delta(t)\right] + C_0^{\mathrm{T}}A_K e^{A_K(t-t_0)}\tilde{\xi}(t_0) \\
&= \overline{\varphi}^{\mathrm{T}}(t)\eta(\theta) + \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \overline{\varepsilon}(t),
\end{aligned} \tag{A.23}$$

where $\overline{\varepsilon}(t)$ are aggregated exponentially vanishing functions.

Step 3. The next aim is to transform the regression equation (A.23) into the form of (3.3) via application of the dynamic regressor extension and mixing procedure. For this purpose, considering (A.23), (3.5), the signal $\chi(t) = \overline{q}(t) - k_1\overline{q}_f(t)$ is differentiated to obtain:

$$\begin{aligned}
\dot{\chi}(t) &= \overline{\varphi}^{\mathrm{T}}(t)\eta(\theta) + \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \overline{\varepsilon}(t) - k_1\left(-k_1\overline{q}_f(t) + \overline{q}(t)\right) \\
&= -k_1\chi(t) + \overline{\varphi}^{\mathrm{T}}(t)\eta(\theta) + \beta^{\mathrm{T}}\left(F(t) + ly(t)\right) + \overline{\varepsilon}(t).
\end{aligned} \tag{A.24}$$

The solution of the differential equation (A.24) allows one to write:

$$\overline{q}(t) - k_1\overline{q}_f(t) - \beta^{\mathrm{T}}\left(F_f(t) + ly_f(t)\right) = e^{-k_1(t-t_0)}\overline{q}\left(t_0\right) + \overline{\varphi}_f^{\mathrm{T}}(t)\eta\left(\theta\right) + \overline{\varepsilon}_f(t), \tag{A.25}$$

where $\dot{\overline{\varepsilon}}_f(t) = -k_1\overline{\varepsilon}_f(t) + k_1\overline{\varepsilon}(t),\ \overline{\varepsilon}_f\left(t_0\right) = 0.$

Owing to (A.25), the solution of the first differential equation from (3.4) satisfies the following equation

$$q(t) = \varphi(t)\eta\left(\theta\right) + \varepsilon(t), \tag{A.26}$$

where $\dot{\varepsilon}(t) = -k_2\varepsilon(t) + \overline{\varphi}_f(t)\left(\overline{\varepsilon}_f(t) + e^{-k_1(t-t_0)}\overline{q}\left(t_0\right)\right),\ \varepsilon\left(t_0\right) = 0_{2n}.$

Having multiplied equation (A.26) by $k(t)\mathrm{adj}\left\{\varphi(t)\right\}$ and applied the property

$$\mathrm{adj}\left\{\varphi(t)\right\}\varphi(t) = \det\left\{\varphi(t)\right\}I_{2n},$$

equation (3.3) is obtained with $\epsilon(t) = k(t)\mathrm{adj}\left\{\varphi(t)\right\}\varepsilon(t)$.

In accordance with Lemma 6.8 from [6], when $\overline{\varphi}(t)\in\mathrm{PE}$, it also holds that $\overline{\varphi}_f(t)\in\mathrm{PE}$. Following Proposition 1, when $\overline{\varphi}_f(t)\in\mathrm{PE}$, then it holds that $\Delta(t) \geqslant \Delta_{\min} > 0$. Since the signals $y(t),\ u(t)$ are bounded by Assumption 1, owing to the stability of the filters (3.4)–(3.6), the inequality $\Delta_{\max} \geqslant \Delta(t)$ holds for all $t \geqslant t_0$. Then for all $t \geqslant t_0 + T$ it holds that $\Delta_{\max} \geqslant \Delta(t) \geqslant \Delta_{\min} > 0$, which completes the proof of lemma.

Proof of Lemma 2. According to definition 1 and hypothesis 1 and owing to

$$\begin{gathered}
\Xi_{\mathcal{S}}\left(\Delta\right) = \overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\Delta(t),\ \Xi_{\mathcal{G}}\left(\Delta\right) = \overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\Delta(t),\\
\mathcal{Y}_{ab}(t) = \mathcal{L}_{ab}\mathcal{Y}(t) = \Delta(t)\mathcal{L}_{ab}\eta\left(\theta\right) = \Delta(t)\psi_{ab}\left(\theta\right),\\
\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\Delta(t)\psi_{ab}\left(\theta\right) = \overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}(t),\\
\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\Delta(t)\psi_{ab}\left(\theta\right) = \overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}(t)
\end{gathered}$$

it follows from (3.9) that

$$\mathcal{T}_{\mathcal{S}}\left(\overline{\Xi}_{\mathcal{S}}\left(\Delta\right)\mathcal{Y}_{ab}\right) = \mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\theta. \tag{A.27}$$

Then, having multiplied (A.27) by $\mathrm{adj}\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}$, the following regression equation is obtained

$$\mathcal{Y}_{\theta}(t) = \mathcal{M}_{\theta}(t)\theta, \tag{A.28}$$

which is used together with (3.8) to write:

$$\mathcal{T}_{\mathcal{Z}}\left(\overline{\Xi}_{\mathcal{Z}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right) = \mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\Theta_{AB}\left(\theta\right), \tag{A.29a}$$

$$\mathcal{T}_{\mathcal{W}}\left(\overline{\Xi}_{\mathcal{W}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right) = \mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\psi_d\left(\theta\right). \tag{A.29b}$$

Having multiplied (A.29a) by $\mathrm{adj}\left\{\mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_{\theta}\right)\mathcal{Y}_{\theta}\right)\right\}$, the regression equation $\mathcal{Y}_{AB}(t) = \mathcal{M}_{AB}(t)\Theta_{AB}\left(\theta\right)$ is obtained.

The next aim it to parametrize equation with respect to $L\left(\theta\right)$. If Assumption 2 is met, then, following the generalized pole placement theory [15, 16], the vector $L\left(\theta\right)$ can be obtained as a solution of the following set of equations

$$\begin{cases} A^{\mathrm{T}}(\theta)M - M\Gamma = CB^{\mathrm{T}}(\theta),\\ B^{\mathrm{T}}(\theta) = L^{\mathrm{T}}(\theta)M, \end{cases} \tag{A.30}$$

which has a unique solution [15, 16], as, following Assumption 3, the pair $\left(A^{\mathrm{T}}(\theta),\ C\right)$ is controllable, the pair $\left(B^{\mathrm{T}}(\theta),\ \Gamma\right)$ is observable and $\sigma\left\{A^{\mathrm{T}}(\theta)\right\} \cap \sigma\{\Gamma\} = 0$.

Having vectorized the first equation from (A.30) and considered the property $vec\,(AB) = (I \otimes A)\,vec\,(B) = \left(B^{\mathrm{T}} \otimes I\right) vec\,(A)$, it is obtained that:

$$\left(I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right) vec\,(M) = vec\left(CB^{\mathrm{T}}(\theta)\right). \tag{A.31}$$

As equations (A.30), (A.31) have unique solutions, then

$$\det\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} \neq 0,$$

and therefore, having multiplied (A.31) by an adjoint matrix $\operatorname{adj}\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\}$, it is written:

$$\begin{gathered}\det\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} vec\,(M) \\ = \operatorname{adj}\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} vec\left(CB^{\mathrm{T}}(\theta)\right).\end{gathered} \tag{A.32}$$

The obtained result is devectorized $\left(vec^{-1}\{.\}\right)$ and substituted into the second equation of (A.30):

$$\begin{gathered}\underbrace{\det\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} B(\theta)}_{\mathcal{Q}(\Theta_{AB})} \\ = \underbrace{vec^{-1}\left\{\operatorname{adj}\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} vec\left(CB^{\mathrm{T}}(\theta)\right)\right\}^{\mathrm{T}}}_{\mathcal{P}(\Theta_{AB})} L(\theta),\end{gathered} \tag{A.33}$$

where $\det\{\mathcal{P}(\Theta_{AB})\} \neq 0$.

The following equalities are introduced:

$$\begin{gathered}\mathcal{M}_{AB}(t)A^{\mathrm{T}}(\theta) = vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right), \\ \mathcal{M}_{AB}(t)B^{\mathrm{T}}(\theta) = \left[\mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right]^{\mathrm{T}}, \\ \mathcal{M}_{AB}(t)B(\theta) = \mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t).\end{gathered} \tag{A.34}$$

Having multiplied (A.33) by $\Pi_L\left(\mathcal{M}_{AB}\right) = \mathcal{M}_{AB}^{n^2+1}I_n$, used the properties $c^n\det\{A\} = \det\{cA\}$, $c^{n-1}\operatorname{adj}\{A\} = \operatorname{adj}\{cA\}$, $A \in \mathbb{R}^{n\times n}$ and substituted (A.34), it is obtained:

$$\begin{gathered}\mathcal{T}_{\mathcal{P}}\left(\Xi_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\Theta_{AB}\right) = \Pi_L\left(\mathcal{M}_{AB}\right)\mathcal{P}\left(\Theta_{AB}\right) = \mathcal{M}_{AB}^{n^2+1}\mathcal{P}\left(\Theta_{AB}\right) \\ = \mathcal{M}_{AB}^{n^2+1}vec^{-1}\left\{\operatorname{adj}\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} vec\left(CB^{\mathrm{T}}(\theta)\right)\right\}^{\mathrm{T}} \\ = vec^{-1}\left\{\mathcal{M}_{AB}\operatorname{adj}\{I_n \otimes vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}\right) - \mathcal{M}_{AB}\Gamma^{\mathrm{T}} \otimes I_n\}\,vec\left(C(\mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB})^{\mathrm{T}}\right)\right\}^{\mathrm{T}}, \\ \mathcal{T}_{\mathcal{Q}}\left(\Xi_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)\Theta_{AB}\right) = \Pi_L\left(\mathcal{M}_{AB}\right)\mathcal{Q}\left(\Theta_{AB}\right) = \mathcal{M}_{AB}^{n^2+1}\mathcal{Q}\left(\Theta_{AB}\right) \\ = \mathcal{M}_{AB}^{n^2+1}\det\left\{I_n \otimes A^{\mathrm{T}}(\theta) - \Gamma^{\mathrm{T}} \otimes I_n\right\} B(\theta) \\ = \det\left\{I_n \otimes vec^{-1}\left(\mathcal{L}_{A^{\mathrm{T}}}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right) - \mathcal{M}_{AB}(t)\Gamma^{\mathrm{T}} \otimes I_n\right\}\left(\mathcal{L}_{B}\mathcal{D}_{\Phi}\mathcal{Y}_{AB}(t)\right),\end{gathered} \tag{A.35}$$

where $\Xi_{\mathcal{P}}\left(\mathcal{M}_{AB}\right) = \Xi_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right) = \mathcal{M}_{AB}(t)$.

The following regression equation is written on the basis of equations (A.33) and (A.35):

$$\mathcal{T}_{\mathcal{Q}}\left(\overline{\Xi}_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right) = \mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)L(\theta), \tag{A.36}$$

where $\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right) = \overline{\Xi}_{\mathcal{Q}}\left(\mathcal{M}_{AB}\right) = 1$.

Having multiplied (A.36) by $\operatorname{adj}\left\{\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)\right\}$, the regression equation $\mathcal{Y}_L(t) = \mathcal{M}_L(t)L(\theta)$ is obtained.

The next aim is to derive the regression equation with respect to $x_{\delta 0}$. Using the properties of the vectorization operation

$$vec\left(\psi_d(\theta)h_\delta^{\mathrm{T}}\Phi_\delta(t)x_{\delta 0}\right) = \underbrace{\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right)}_{n\times n_\delta}\underbrace{vec\left(h_\delta^{\mathrm{T}}\Phi_\delta\right)}_{n_\delta},$$
$$vec\left(\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right)vec\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\right)\right) = \underbrace{\left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right)}_{n\times nn_\delta}\underbrace{vec\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right)}_{nn_\delta},$$

equation (3.1) is rewritten as follows:

$$\begin{aligned}\dot{\xi}(t) &= A_0\xi(t) + \psi_a(\theta)y(t) + \psi_b(\theta)u(t)\\ &+ \left(h_\delta^{\mathrm{T}}\Phi_\delta(t) \otimes I_n\right)vec\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right).\end{aligned} \tag{A.37}$$

The following error is introduced:

$$e(t) = \xi(t) - z(t) - \Omega(t)\psi_a(\theta) - P(t)\psi_b(\theta) - V(t)vec\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right). \tag{A.38}$$

Having differentiated (A.38), equation $\dot{e}(t) = A_K e(t)$ is obtained in a similar way as (A.2). Then, having multiplied (A.38) by C_0^{T}, it is written:

$$\begin{aligned}\overline{q}(t) &= C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right) + C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta)\\ &+ C_0^{\mathrm{T}}P(t)\psi_b(\theta) + C_0^{\mathrm{T}}V(t)vec\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right).\end{aligned} \tag{A.39}$$

Using the properties

$$x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta) = \psi_d(\theta)x_{\delta 0},$$
$$vec\left(\psi_d(\theta)x_{\delta 0}\right) = \underbrace{\left(I_{n_\delta} \otimes \psi_d(\theta)\right)}_{nn_\delta\times n_\delta}x_{\delta 0},$$

equation (A.39) is transformed into

$$\begin{aligned}\overline{q}(t) &= C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right) + C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta)\\ &+ C_0^{\mathrm{T}}P(t)\psi_b(\theta) + C_0^{\mathrm{T}}V(t)vec\left(x_{\delta 0}^{\mathrm{T}} \otimes \psi_d(\theta)\right).\end{aligned} \tag{A.40}$$

To compensate for the unknown terms $C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta) + C_0^{\mathrm{T}}P(t)\psi_b(\theta)$, the following auxiliary signal is introduced

$$\begin{aligned}\overline{p}_e(t) &= \Delta(t)C_0^{\mathrm{T}}\Omega(t)\psi_a(\theta) + \Delta(t)C_0^{\mathrm{T}}P(t)\psi_b(\theta)\\ &= C_0^{\mathrm{T}}\Omega(t)\mathcal{L}_a\mathcal{Y}(t) + C_0^{\mathrm{T}}P(t)\mathcal{L}_b\mathcal{Y}(t),\end{aligned} \tag{A.41}$$

where

$$\mathcal{L}_a\mathcal{Y}(t) = \mathcal{L}_a\Delta(t)\eta(\theta) = \Delta(t)\mathcal{L}_a\eta(\theta) = \Delta(t)\psi_a(\theta),$$
$$\mathcal{L}_b\mathcal{Y}(t) = \Delta(t)\mathcal{L}_b\eta(\theta) = \Delta(t)\psi_b(\theta).$$

Having multiplied (A.40) by $\Delta(t)$ and subtracted (A.41) from the obtained result, it is written:

$$\begin{aligned} p(t) &= \Delta(t)\overline{q}(t) - \overline{p}_e(t) \\ &= \Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\psi_d(\theta)\right)x_{\delta 0} + \Delta(t)C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right) \\ &= \Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\psi_d(\theta)\right)x_{\delta 0} + \Delta(t)C_0^{\mathrm{T}}e^{A_K(t-t_0)}e\left(t_0\right). \end{aligned} \tag{A.42}$$

To implement the multiplier $\psi_d(\theta)$ indirectly, equation (A.29b) is multiplied by $\operatorname{adj}\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}$:

$$\begin{aligned} \mathcal{Y}_{\psi_d}(t) &= \mathcal{M}_{\psi_d}(t)\psi_d(\theta), \\ \mathcal{Y}_{\psi_d}(t) &= \operatorname{adj}\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}\mathcal{T}_{\mathcal{W}}\left(\overline{\Xi}_{\mathcal{W}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right), \\ \mathcal{M}_{\psi_d}(t) &= \det\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}. \end{aligned} \tag{A.43}$$

The multiplication of (A.42) by $\mathcal{M}_{\psi_d}(t)$ and substitution of (A.43) into the obtained result allow one to write:

$$\begin{aligned} \mathcal{M}_{\psi_d}(t)p(t) &= \mathcal{M}_{\psi_d}(t)\Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\psi_d(\theta)\right)x_{\delta 0} \\ &= \Delta(t)C_0^{\mathrm{T}}V(t)\left(I_{n_\delta}\otimes\mathcal{Y}_{\psi_d}(t)\right)x_{\delta 0}. \end{aligned} \tag{A.44}$$

Having filtered (A.44) via (4.3) and multiplied the obtained result by $\operatorname{adj}\left\{V_f(t)\right\}$, the regression equation $\mathcal{Y}_{x_{\delta 0}}(t) = \mathcal{M}_{x_{\delta 0}}(t)x_{\delta 0}$ is obtained, which completes proof of statement that equations (4.2) can be formed on the basis of the measurable signals.

Following Lemma 1, if $\overline{\varphi}(t)\in\mathrm{PE}$, then for all $t\geqslant t_0+T$ it holds that $\Delta(t)\geqslant\Delta_{\min}>0$, and, owing to Hypotheses 1–3 and proved inequalities:

$$\begin{gathered} \det{}^2\left\{\mathcal{X}(\theta)\right\}>0,\quad \det{}^2\left\{\mathcal{R}(\theta)\right\}>0, \\ \det{}^2\left\{\mathcal{G}\left(\psi_{ab}\right)\right\}>0,\quad \det{}^2\left\{\mathcal{P}\left(\Theta_{AB}\right)\right\}>0, \\ \det\left\{\Pi_\theta\left(\Delta\right)\right\}\geqslant\Delta^{\ell_\theta}(t),\quad \det\left\{\Pi_\Theta\left(\mathcal{M}_\theta\right)\right\}\geqslant\mathcal{M}_\theta^{\ell_\Theta}(t), \\ \det\left\{\Pi_{\psi_d}\left(\mathcal{M}_\theta\right)\right\}\geqslant\mathcal{M}_\theta^{\ell_{\psi_d}}(t),\quad \det\left\{\Pi_L\left(\mathcal{M}_{AB}\right)\right\}\geqslant\mathcal{M}_{AB}^{n^3+n}(t), \end{gathered}$$

we have that, if $\overline{\varphi}(t)\in\mathrm{PE}$, then for all $t\geqslant t_0+T$ the following holds:

$$\begin{aligned} \left|\mathcal{M}_\theta(t)\right| &= \left|\det\left\{\mathcal{T}_{\mathcal{G}}\left(\overline{\Xi}_{\mathcal{G}}\left(\Delta\right)\mathcal{Y}_{ab}\right)\right\}\right| = \left|\det\left\{\Pi_\theta\left(\Delta\right)\right\}\det\left\{\mathcal{G}\left(\psi_{ab}\right)\right\}\right| \\ &\geqslant \left|\det\left\{\mathcal{G}\left(\psi_{ab}\right)\right\}\right|\Delta_{\min}^{\ell_\theta} = \underline{\mathcal{M}_\theta} > 0, \\ \left|\mathcal{M}_{AB}(t)\right| &= \left|\det\left\{\mathcal{T}_{\mathcal{X}}\left(\overline{\Xi}_{\mathcal{X}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\}\right| = \left|\det\left\{\Pi_\Theta\left(\mathcal{M}_\theta\right)\right\}\det\left\{\mathcal{X}(\theta)\right\}\right| \\ &\geqslant \left|\det{}^{\ell_\Theta}\left\{\mathcal{G}\left(\psi_{ab}\right)\right\}\right|\left|\det\left\{\mathcal{X}(\theta)\right\}\right|\Delta_{\min}^{\ell_\theta\ell_\Theta} = \underline{\mathcal{M}_{AB}} > 0, \\ \left|\mathcal{M}_{\psi_d}(t)\right| &= \det\left\{\mathcal{T}_{\mathcal{R}}\left(\overline{\Xi}_{\mathcal{R}}\left(\mathcal{M}_\theta\right)\mathcal{Y}_\theta\right)\right\} = \left|\det\left\{\Pi_{\psi_d}\left(\mathcal{M}_\theta\right)\right\}\det\left\{\mathcal{R}(\theta)\right\}\right| \\ &\geqslant \left|\det{}^{\ell_{\psi_d}}\left\{\mathcal{G}\left(\psi_{ab}\right)\right\}\right|\left|\det\left\{\mathcal{R}(\theta)\right\}\right|\Delta_{\min}^{\ell_\theta\ell_{\psi_d}} = \underline{\mathcal{M}_{\psi_d}} > 0, \\ \left|\mathcal{M}_L(t)\right| &= \left|\det\left\{\mathcal{T}_{\mathcal{P}}\left(\overline{\Xi}_{\mathcal{P}}\left(\mathcal{M}_{AB}\right)\mathcal{Y}_{AB}\right)\right\}\right| = \left|\det\left\{\Pi_L\left(\mathcal{M}_{AB}\right)\right\}\det\left\{\mathcal{P}\left(\Theta_{AB}\right)\right\}\right| \\ &\geqslant \left|\det\left\{\mathcal{P}\left(\Theta_{AB}\right)\right\}\right|\mathcal{M}_{AB}^{n^3+n} \geqslant \left|\det\left\{\mathcal{P}\left(\Theta_{AB}\right)\right\}\right|\underline{\mathcal{M}_{AB}^{n^3+n}} = \underline{\mathcal{M}_L} > 0. \end{aligned}$$

To obtain the lower bound for the regressor $\mathcal{M}_{x\delta 0}(t)$, first of all, such bound needs to be derived for the solution of the differential equation for $V_f(t)$ in case $\overline{\varphi}(t) \in \mathrm{PE}$ and $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right) \in \mathrm{PE}$:

$$
\begin{aligned}
V_f(t) &= \int\limits_{t_0}^{t} e^{-k_2(t-\tau)}\Delta^2(\tau)(I_{n_\delta}\otimes\mathcal{Y}_{\psi_d}(\tau))^{\mathrm{T}}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)\left(I_{n_\delta}\otimes\mathcal{Y}_{\psi_d}(\tau)\right)d\tau \\
&= (I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\int\limits_{t_0}^{t} e^{-k_2(t-\tau)}\mathcal{M}_{\psi_d}^2(\tau)\Delta^2(\tau)V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\left(I_{n_\delta}\otimes\psi_d(\theta)\right) \\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\int\limits_{t_0}^{t} e^{-k_2(t-\tau)}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\left(I_{n_\delta}\otimes\psi_d(\theta)\right) \\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\left[\int\limits_{t_0}^{t-\overline{k}T} e^{-k_2(t-\tau)}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\right. \\
&\qquad\left. + \sum_{k=1}^{\overline{k}}\int\limits_{t-kT}^{t-kT+T} e^{-k_2(t-\tau)}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\right]\left(I_{n_\delta}\otimes\psi_d(\theta)\right) \\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2 e^{-k_2t}(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\sum_{k=1}^{\overline{k}}\int\limits_{t-kT}^{t-kT+T} e^{k_2\tau}V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau(I_{n_\delta}\otimes\psi_d(\theta)) \\
&\geqslant \underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\sum_{k=1}^{\overline{k}} e^{-k_2kT}\int\limits_{t-kT}^{t-kT+T} V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau\left(I_{n_\delta}\otimes\psi_d(\theta)\right),
\end{aligned}
$$

where $\overline{k} \geqslant k \geqslant 1$ are integers.

In accordance with Lemma 6.8 from [6], if $\left(h_\delta^{\mathrm{T}}\Phi_\delta(t)\otimes I_n\right) \in \mathrm{PE}$, then the following inequality holds

$$
\int\limits_{t}^{t+T} V^{\mathrm{T}}(\tau)C_0C_0^{\mathrm{T}}V(\tau)d\tau \geqslant \alpha I_{nn_\delta}, \tag{A.45}
$$

and, using the properties of the Kronecker product, it is obtained that:

$$
\begin{aligned}
(I_{n_\delta}\otimes\psi_d(\theta))^{\mathrm{T}}\underbrace{(I_{n_\delta}\otimes\psi_d(\theta))}_{nn_\delta\times n_\delta} &= \left(I_{n_\delta}^{\mathrm{T}}\otimes\psi_d^{\mathrm{T}}(\theta)\right)\left(I_{n_\delta}\otimes\psi_d(\theta)\right) \\
&= I_{n_\delta}\otimes\psi_d^{\mathrm{T}}(\theta)\psi_d(\theta) = \underbrace{\psi_d^{\mathrm{T}}(\theta)\psi_d(\theta)}_{>0} I_{n_\delta}.
\end{aligned} \tag{A.46}
$$

Then for all $t \geqslant t_0 + T$ it holds that:

$$
V_f(t) \geqslant \underbrace{\underline{\mathcal{M}_{\psi_d}^2}\Delta_{\min}^2\alpha\sum_{k=1}^{\overline{k}} e^{-k_2kT}\psi_d^{\mathrm{T}}(\theta)\psi_d(\theta)}_{>0} I_{n_\delta} \geqslant \sqrt[n_\delta]{\underline{\mathcal{M}_{x\delta 0}}}I_{n_\delta}, \tag{A.47}
$$

from which for all $t \geqslant t_0 + T$ we have $\mathcal{M}_{x\delta 0} \geqslant \underline{\mathcal{M}_{x\delta 0}} > 0$, which allows one to obtain:

$$
\forall t \geqslant t_0 + T \;\; |\mathcal{M}_\kappa(t)| = \left|\mathcal{M}_{AB}^{n_\Theta}(t)\mathcal{M}_L^n(t)\mathcal{M}_{x\delta 0}^{n_\delta}(t)\right| \geqslant \underline{\mathcal{M}_{AB}^{n_\Theta}}\underline{\mathcal{M}_L^n}\underline{\mathcal{M}_{x\delta 0}^{n_\delta}} = \underline{\mathcal{M}_\kappa} > 0. \tag{A.48}
$$

This completes proof of Lemma 2.

REFERENCES

1. Khlebnikov, M.V., Polyak, B.T., and Kuntsevich, V.M., Optimization of linear systems subject to bounded exogenous disturbances: The invariant ellipsoid technique, *Autom. Remote Control*, 2011, vol. 72, no. 11, pp. 2227–2275.

2. Khalil, H.K. and Praly, L., High-gain observers in nonlinear feedback control, *Int. J. Robust Nonlinear Control*, 2014, vol. 24, no. 6, pp. 993–1015.

3. Krasnova, S.A. and Utkin, V.A., *Cascade Design of State Observers for Dynamical Systems*, Moscow: Nauka, 2006. In Russian.

4. Shtessel, Y., Edwards, C., Fridman, L., and Levant, A., *Sliding Mode Control and Observation*, New York: Springer New York, 2014.

5. Ioannou, P. and Sun, J., *Robust Adaptive Control*, New York: Dover, 2013.

6. Narendra, K.S. and Annaswamy, A.M., *Stable Adaptive Systems*, Courier Corporation, 2012.

7. Carroll, R. and Lindorff, D., An adaptive observer for single-input single-output linear systems, *IEEE Trans. Automat. Control*, 1973, vol. 18, no. 5, pp. 428–435.

8. Kudva, P. and Narendra, K.S., Synthesis of an adaptive observer using Lyapunov's direct method, *Int. J. Control*, 1973, vol. 18, no. 6, pp. 1201–1210.

9. Luders, G. and Narendra, K.S., Stable adaptive schemes for state estimation and identification of linear systems, *IEEE Trans. Automat. Control*, 1974, vol. 19, no. 6, pp. 841–847.

10. Kreisselmeier, G., Adaptive observers with exponential rate of convergence, *IEEE Trans. Automat. Control*, 1977, vol. 22, no. 1, pp. 2–8.

11. Katiyar, A., Roy, S.B., and Bhasin, S., Initial Excitation Based Robust Adaptive Observer for MIMO LTI Systems, *IEEE Trans. Automat. Control*, 2022.

12. Bobtsov, A., Pyrkin, A., Vedyakov, A., Vediakova, A., and Aranovskiy, S., A Modification of Generalized Parameter-Based Adaptive Observer for Linear Systems with Relaxed Excitation Conditions, *IFACPapersOnLine*, 2022, vol. 55, no. 12, pp. 324–329.

13. Glushchenko, A. and Lastochkin, K., Exponentially Stable Adaptive Observation for Systems Parameterized by Unknown Physical Parameters, *arXiv preprint arXiv:2212.08405*, 2022, pp. 1–6.

14. Aranovskiy, S., Ushirobira, R., Korotina, M., and Vedyakov, A., On preserving-excitation properties of Kreisselmeiers regressor extension scheme, *IEEE Trans. Automat. Control*, 2022, pp. 1–6.

15. Bhattacharyya, S.P. and De Souza, E., Pole assignment via Sylvester's equation, *Syst. Control Lett.*, 1982, vol. 1, no. 4, pp. 261–263.

16. Dudarenko, N.A., Slita, O.V., and Ushakov, A.V., Algebraic conditions of generalized modal control, *IFAC Proc. Volumes*, 2012, vol. 45, no. 13, pp. 150–155.

17. Poznyak, A.S., *Advanced Mathematical Tools for Automatic Control Engineers*, Elsevier Science, 2009.

18. Glushchenko, A.I., Lastochkin, K.A., and Petrov, V.A., Exponentially stable adaptive control. Part I. Time-invariant plants, *Autom. Remote Control*, 2022, vol. 83, no. 4, pp. 548–578.

19. Ortega, R., Nikiforov, V., and Gerasimov, D., On modified parameter estimators for identification and adaptive control. A unified framework and some new schemes, *Ann. Rev. Control*, 2020, vol. 50, pp. 278–293.

20. Katiyar, A., Basu Roy, S., and Bhasin, S., Finite excitation based robust adaptive observer for MIMO LTI systems, *Int. J. Adaptive Control Signal Proc.*, 2022, vol. 36, no. 2, pp. 180–197.

21. Glushchenko, A. and Lastochkin, K., Extended Adaptive Observer for Linear Systems with Overparametrization, *Proceedings of 2023 31st Mediterranean Conference on Control and Automation (MED)*, Limassol: IEEE, 2023, pp. 789–794.

22. Polyak, B.T. and Smirnov, G., Large deviations for non-zero initial conditions in linear systems, *Automatica*, 2016, vol. 74, pp. 297–307.

23. Glushchenko, A. and Lastochkin, K., Parameter Estimation-Based Observer for Linear Systems with Polynomial Overparametrization, *Proceedings of 2023 31st Mediterranean Conference on Control and Automation (MED)*, Limassol: IEEE, 2023, pp. 795–799.

24. Nikiforov, V.O., Observers of external deterministic disturbances. II. Objects with unknown parameters, *Autom. Remote Control*, 2004, vol. 65, no. 11, pp. 1724–1732.

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Exponentially Stable Adaptive Control. Part III. Time-Varying Plants

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Abstract—A state-feedback adaptive control system is proposed for a class of linear systems in the controllable canonical form with time-varying unknown parameters described by known nonstationary exosystems with unknown initial conditions. The solution ensures global exponential stability of the closed-loop system in case a regressor is finitely exciting, and also does not require *a priori* information about the sign of the high-frequency gain (control direction). The obtained theoretical results are validated via mathematical modeling.

Keywords: adaptive control, time-varying parameters, parametric error, finite excitation, identification, exponential stability

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1. INTRODUCTION

Considering plants with time-invariant parameters, classical algorithms of the model reference adaptive control guarantee asymptotic stability of the tracking error (the difference between the plant and reference model state vectors) [1]. However, as far as practical scenarios are concerned, real physical systems are often described by models with time-varying parameters. Under these conditions, conventional solutions face difficulties caused by the need to compensate for a term in the derivative of the Lyapunov function that is proportional to the unknown parameters change rate [1, p. 552].

If the unknown time-varying parameters converge exponentially to a constant value, then asymptotic stability of the tracking error is still ensured [2, p. 339]. In case the unknown parameters change their values arbitrarily, and the restrictive condition of the regressor persistent excitation is met, then the baseline solutions guarantee the boundedness of all signals and the convergence of the tracking error to a compact set. If the above-mentioned condition is not satisfied, then the application of robust modifications of the basic adaptive laws allows one to ensure similar properties for the closed-loop system [1, 2].

Further the existing methods to improve the properties of the baseline solutions are reviewed.

In [3, 4] the method of congelation of variables is proposed that is capable of reduction of the time-varying parameters estimation problem to the one of their mean identification. The upper bound of the parameters variance is assumed to be known, and the control law is designed using static nonlinear damping. In this scheme, the adaptive law compensates for the unknown parameters mean, whereas the nonlinear damping is responsible for their variance rejection. Such a solution guarantees asymptotic stability of the tracking error and boundedness of all signals of the closed-loop system. The disadvantage of the above-described method is that the power functions from the plant state (regressor) are used to form the control signal [5, pp. 222–223]. As it is highlighted in [6], another disadvantage of such method is that the parametric uncertainty represented as $\theta(t)\varphi(x(t))$ with $\varphi(0) \neq 0$ can not be compensated.

In [7–9] an approach based on the application of the composite adaptive laws is proposed, in which the parameters are adjusted using both tracking and prediction errors. Compared to the basic solutions [1, 2], under the same assumptions (the regressor is persistently exciting and the robust modifications are used) the composite laws ensure convergence of the tracking error to a compact set of a smaller size. In contrast to the method of congelation of variables [3, 4], the asymptotic stability is not guaranteed. A detailed review of some composite laws for the case of the time-varying parameters is given in the introduction of [10].

In [11] a simple scheme to adjust the geometric mean pole of a closed-loop system is proposed, which guarantees asymptotic stability of the tracking error. The disadvantage of the solution is the need to know the plant input matrix. In [12] the adaptive law with astatism of the first order is developed, which extends the applicability of the basic adaptive laws to a class of systems with linearly varying unknown parameters. In [13–15] a scheme of robust control of nonstationary linear systems represented as transfer functions is proposed. A disadvantage of the approach is the need to know the plant control direction and upper bounds for all unknown time-varying parameters of the system.

The above discussed approaches [3–15] do not consider *a priori* information about the structure of a function according to which the system changes its parameters. However, as is known [16], taking such information into account can significantly improve the properties of a closed-loop system. Recently, in [17, 18] a state observer of time-varying systems based on a parametric identification has been proposed. The system parameters are described by known nonstationary exosystems with unknown initial conditions. The problem of the system state reconstruction is reduced to the identification of the initial conditions of both plant and exosystems. If the regressor finite excitation condition (observability of the system over a finite time interval) is satisfied, then exponential or finite-time convergence of the parametric and state observation errors is ensured.

Based on the results of [17, 18], in this study it is proposed to reduce the control problem to the estimation of the initial conditions of the exosystems used to generate the system parameters.

A class of linear completely controllable systems with time-varying parameters described by known nonstationary exosystems with unknown initial conditions is considered. The difference between ideal and actual control signals is represented as a linear regression equation with an unknown regressor with respect to the unknown initial conditions of the exosystems. Then, in accordance with the results from [18] and using the measurable state and control signal, a measurable regression equation,[1] which scalar regressor is bounded away from zero, is derived with respect to the initial conditions of the exosystems. Using the results of the first part [19] of this paper series, an adaptive law is derived on the basis of such regression, which, in contrast to [3–15], guarantees exponential convergence of the tracking error to zero if the regressor finite excitation (FE) requirement is met.

In addition to FE, the proposed extension of the results from [19] to a class of systems with time-varying parameters also requires:

— the lower and upper bounds of the absolute value of the high-frequency gain to be known;

— the sign of the high-frequency gain to be constant;

— application of the projection operator, which prevents division by zero in the control law.

Compared to [3–11, 13–15], the proposed approach requires knowledge of the state and output matrices of the nonstationary exosystems, and hence, of the physical nature of the processes that cause the system parameters variation.

[1] A measurable regression equation means that its regressor and regressand are measurable or can be computed, while its parameters are unknown.

Main definitions

To prove theorem and propositions, the regressor finite excitation definition and corollary of the Kalman–Yakubovich–Popov (KYP) lemma [1, 2] are used further .

Definition 1. A regressor $\omega(t)$ is finitely exciting $\omega(t) \in \mathrm{FE}$ over $[t_r^+; t_e]$ if there exists $t_r^+ \geqslant 0$, $t_e > t_r^+$ and α such that the following inequality holds

$$\int\limits_{t_r^+}^{t_e} \omega(\tau)\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_{n\times n}, \tag{1.1}$$

where $\alpha > 0$ is an excitation level, $I_{n\times n}$ stands for an identity matrix.

Corollary 1. *For any matrix $D > 0$, a controllable pair (A, B) with $B \in \mathbb{R}^{n\times m}$, a Hurwitz matrix $A \in \mathbb{R}^{n\times n}$ there exist matrices $P = P^{\mathrm{T}} > 0$, $Q \in \mathbb{R}^{n\times m}$, $K \in \mathbb{R}^{m\times m}$ and a number $\mu > 0$ such that*

$$\begin{gathered} A^{\mathrm{T}}P + PA = -QQ^{\mathrm{T}} - \mu P, \quad PB = QK, \\ K^{\mathrm{T}}K = D + D^{\mathrm{T}}. \end{gathered} \tag{1.2}$$

2. PROBLEM STATEMENT

A class of continuous linear time-varying systems is considered[2]:

$$\begin{gathered} \forall t \geqslant t_0^+ \; \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x\left(t_0^+\right) = x_0, \\ A(t) = A_0 + e_1\vartheta^{\mathrm{T}}(t), \quad B(t) = e_1\beta(t), \\ A_0 = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \multicolumn{2}{c}{0_{1\times n}} \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \end{gathered} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is a system state with unknown initial conditions x_0, $u(t) \in \mathbb{R}$ stands for a control signal, $A_0 \in \mathbb{R}^{n\times n}$ denotes a known state matrix, $B(t) \in \mathbb{R}^n$, $\vartheta(t) \in \mathbb{R}^n$ are unknown vectors, t_0^+ stands for a known initial time instant. The pair $(A(t), B(t))$ is completely controllable for all $t \geqslant t_0^+$ in a sense of criterion from [20].

The following assumptions are adopted for the unknown parameters of the system (2.1).

Assumption 1. The vectors $\vartheta(t)$, $B(t)$ are bounded, continuous and formed by time-varying exosystems[3]:

$$\begin{gathered} \begin{cases} \dot{x}_\vartheta(t) = \mathcal{A}_\vartheta(t)x_\vartheta(t), \quad x_\vartheta\left(t_0^+\right) = x_{\vartheta_0}, \\ \vartheta(t) = h_\vartheta x_\vartheta(t), \end{cases} \\ \begin{cases} \dot{x}_B(t) = \mathcal{A}_B(t)x_B(t), \quad x_B\left(t_0^+\right) = x_{B_0}, \\ B(t) = h_B x_B(t), \end{cases} \end{gathered} \tag{2.2}$$

where $x_\vartheta(t) \in \mathbb{R}^{n_\vartheta}$, $x_B(t) \in \mathbb{R}^{n_B}$ are exosystems state with unknown initial conditions $x_\vartheta\left(t_0^+\right)$, $x_B\left(t_0^+\right)$, $h_\vartheta \in \mathbb{R}^{n\times n_\vartheta}$, $h_B \in \mathbb{R}^{n\times n_B}$; $\mathcal{A}_\vartheta(t) \in \mathbb{R}^{n_\vartheta\times n_\vartheta}$, $\mathcal{A}_B(t) \in \mathbb{R}^{n_B\times n_B}$ denote known vectors and matrices.

Assumption 2. Lower $\beta_{\min} > 0$ and upper $\beta_{\max} > \beta_{\min}$ bounds are known for $|\beta(t)|$.

[2] The obtained results can be generalized to MIMO systems in case if the structures of matrices $A(t) \in \mathbb{R}^{n\times n}$ and $B(t) \in \mathbb{R}^{n\times m}$ are known.

[3] In general case, the matrices $\mathcal{A}_\vartheta(t)$, $\mathcal{A}_B(t)$ can depend on the system state $x(t)$ in a nonlinear fashion.

Assumption 3. The sign of the high-frequency gain $\beta(t)$ is constant but unknown ($\mathrm{sgn}(\beta(t)) = const$).

The required control quality for a closed-loop system with the control signal $u(t)$ and plant (2.1) is defined using the time-invariant reference model:

$$\forall t \geqslant t_0^+ \ \dot{x}_{ref}(t) = A_{ref}x_{ref}(t) + B_{ref}r(t), \quad x_{ref}\left(t_0^+\right) = x_{0ref}, \tag{2.3}$$

where $x_{ref}(t) \in \mathbb{R}^n$ is a reference model state with known initial conditions x_{0ref}, $r(t) \in \mathbb{R}$ stands for a reference signal, $A_{ref} \in \mathbb{R}^{n\times n}$ denotes a Hurwitz state matrix of the reference model, $B_{ref} \in \mathbb{R}^n$ is a reference model control input vector.

Having the plant (2.1), reference model (2.3) and a controllable pair $(A(t),\, B(t))$, if Assumption 3 is met, then it is assumed that the matching conditions are satisfied.

Assumption 4. There exists a matrix $K_x(t) = A_{ref} - A(t) \in \mathbb{R}^{n\times n}$ and a vector $K_r(t) = \left[B^{\mathrm{T}}(t)B(t)\right]^{-1}B^{\mathrm{T}}(t) \in \mathbb{R}^{1\times n}$ such that the following holds

$$A(t) + B(t)K_r(t)K_x(t) = A_{ref}, \quad B(t)K_r(t)B_{ref} = B_{ref}. \tag{2.4}$$

Considering Assumption 2, the error equation between the plant (2.1) and the reference model (2.3) is written as

$$\begin{aligned}\dot{e}_{ref}(t) &= A_{ref}e_{ref}(t) + B(t)u(t) - (A_{ref} - A(t))\,x(t) - B_{ref}r(t)\\ &= A_{ref}e_{ref}(t) + B(t)\,(u(t) - u^*(t))\,,\end{aligned} \tag{2.5}$$

where $e_{ref}(t) = x(t) - x_{ref}(t), \quad u^*(t) = K_r(t)\left(K_x(t)x(t) + B_{ref}r(t)\right).$

The aim is to derive a control law $u(t)$ that ensures achievement of the following goal:

$$\Phi(t) \in \mathrm{FE} \ \Rightarrow \lim_{t\to\infty}\|e_{ref}(t)\| = 0 \ (\exp)\,, \tag{2.6}$$

where $\Phi(t)$ is some generalized vector of measurable signals.

Remark 1. Assumption 1 is to single out a group of systems, for which the problem of exponentially stable control (2.6) *is stated and solved in this paper, from the general class of linear systems with time-varying parameters. The proposed solution uses information about $\beta_{\max} > \beta_{\min} > 0$, which is required by Assumption 2. Mathematical modeling indicates that one can choose $\beta_{\max} \to \infty$, $\beta_{\min} \to 0$, which somewhat relaxes the strictness of this condition. Assumption 3 guarantees the continuity of the coefficients $K_x(t)$, $K_r(t)$ of the control law $u^*(t)$. Assumption 4 implies that the respective matrices of the reference model* (2.3) *and the plant* (2.1) *have the same structure.*

Remark 2. Systems (2.1) *with matched parametric uncertainty* (2.4) *are quite widespread, as far as practical scenarios are concerned. For example, the Euler angles dynamics equations of a solid body under the assumption of its symmetry are represented as a second-order system with matched uncertainty. Another good example of a control problem with matched uncertainty is the control of manipulator coordinates using the Euler–Lagrange formalism.*

3. MAIN RESULT

In Subsection 3.1 the stated problem of exponentially stable control (2.6) is reduced to the one of identification of the initial conditions x_{B_0}, x_{ϑ_0}. In subsection 3.2 a regression equation with respect to x_{B_0}, x_{ϑ_0} is derived on the basis of the measurable signals, and an adaptive law is introduced that allows one to achieve the goal (2.6).

3.1. Control Law Parametrization

The control law $u^*(t)$ is to be written via the plant measurable state (2.1), (2.2) and unknown parameters x_{B_0}, x_{ϑ_0}. For this equation (2.2) is solved:

$$\begin{gathered}\vartheta(t) = h_\vartheta \Phi_\vartheta(t) x_{\vartheta_0},\\ B(t) = h_B \Phi_B(t) x_{B_0},\end{gathered} \tag{3.1.1}$$

where, if Assumptions 2 and 3 are met, the following inequality holds

$$\begin{gathered}0 < \beta_{\min}^2 \leqslant x_{B_0}^{\mathrm{T}} G(t) x_{B_0} \leqslant \lambda_{\max}\left(G(t)\right) \|x_{B_0}\|^2,\\ G(t) = \Phi_B^{\mathrm{T}}(t) h_B^{\mathrm{T}} h_B \Phi_B(t),\end{gathered} \tag{3.1.2}$$

and the fundamental matrices $\Phi_\vartheta(t)$ and $\Phi_B(t)$ are measurable and defined as

$$\begin{gathered}\dot{\Phi}_\vartheta(t) = \mathcal{A}_\vartheta(t)\Phi_\vartheta(t), \quad \Phi_\vartheta\left(t_0^+\right) = I_{n_\vartheta},\\ \dot{\Phi}_B(t) = \mathcal{A}_B(t)\Phi_B(t), \quad \Phi_B\left(t_0^+\right) = I_{n_B}.\end{gathered}$$

Considering (2.4) and (3.1.1), the ideal control law $u^*(t)$ is rewritten in the required form

$$\begin{aligned}u^*(t) &= K_r(t)\left(K_x(t)x(t) + B_{ref}r(t)\right)\\ &= \frac{x_{B_0}^{\mathrm{T}}\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{x_{B_0}^{\mathrm{T}}\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}h_B\Phi_B(t)x_{B_0}}\Big((A_{ref} - A_0)x(t) - e_1\vartheta^{\mathrm{T}}(t)x(t) + B_{ref}r(t)\Big)\\ &= \frac{x_{B_0}^{\mathrm{T}}\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{F(t)}\, e_1\Big(e_1^{\mathrm{T}}(A_{ref} - A_0)x(t) - x_{\vartheta_0}^{\mathrm{T}}\Phi_\vartheta^{\mathrm{T}}(t)h_\vartheta^{\mathrm{T}}x(t) + e_1^{\mathrm{T}}B_{ref}r(t)\Big),\end{aligned} \tag{3.1.3}$$

where $F(t) = x_{B_0}^{\mathrm{T}} G(t) x_{B_0} > 0$.

As according to (2.5) the parameters x_{B_0} and x_{ϑ_0} are unknown, then equation (3.1.3) motivates to introduce the control law with the adjustable parameters:

$$u(t) = \frac{\hat{x}_{B_0}^{\mathrm{T}}(t)\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{\hat{F}(t)} e_1\Big(e_1^{\mathrm{T}}\left(A_{ref} - A_0\right)x(t) - \hat{x}_{\vartheta_0}^{\mathrm{T}}(t)\Phi_\vartheta^{\mathrm{T}}(t)h_\vartheta^{\mathrm{T}}x(t) + e_1^{\mathrm{T}}B_{ref}r(t)\Big), \tag{3.1.4}$$

where $\hat{F}(t) = \hat{x}_{B_0}^{\mathrm{T}}(t) G(t) \hat{x}_{B_0}(t)$.

Proposition 1. *The error $u(t) - u^*(t)$ between the actual* (3.1.4) *and ideal* (3.1.3) *control signals is written as*

$$u(t) - u^*(t) = \tilde{\theta}^{\mathrm{T}}(t)\omega(t), \tag{3.1.5}$$

where $\tilde{\theta}(t) = \begin{bmatrix}\tilde{x}_{\vartheta_0}^{\mathrm{T}}(t) & \tilde{x}_{B_0}^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n_\vartheta + n_B}$ is a parametric error, $\omega(t) \in \mathbb{R}^{n_\vartheta + n_B}$ stands for an unmeasurable regressor.

Proof of Proposition 1 and functional definition of the regressor $\omega(t)$ are given in Appendix.

Having substituted (3.1.5) into (2.5), it is obtained that:

$$\dot{e}_{ref}(t) = A_{ref}e_{ref}(t) + B(t)\tilde{\theta}^{\mathrm{T}}(t)\omega(t). \tag{3.1.6}$$

Then, according to the first paper [19] in this series, the following transformations are to be defined

$$\begin{gathered}\Phi(t) = \mathcal{F}_1\left(t, x(t), u(t), \Phi_\vartheta(t), \Phi_B(t)\right), \quad z(t) = \mathcal{F}_2\left(x(t)\right),\\ \dot{\hat{\theta}}(t) = \mathcal{G}\left(\Phi(t), z(t)\right),\end{gathered}$$

which together ensure that the goal of exponentially stable control in the augmented tracking error space is achieved:

$$\Phi(t) \in \mathrm{FE} \;\Rightarrow \lim_{t\to\infty} \|\xi(t)\| = 0 \;\text{(exp)}\,, \tag{3.1.7}$$

where $\xi(t) = \left[e_{ref}^{\mathrm{T}}(t) \;\; \tilde{\theta}^{\mathrm{T}}(t)\right]^{\mathrm{T}}$ is the augmented tracking error.

3.2. Adaptive Law Design

The next aim is to derive the regression equation with respect to the unknown time-invariant parameters x_{B_0} and x_{ϑ_0} of the ideal control law (3.1.3). The result of such parametrization is represented as the following proposition.

Proposition 2. *Using (i) the states of the set of filters ($A_K \in \mathbb{R}^{n\times n}$ is a Hurwitz matrix)*

$$\begin{aligned} \dot{\bar{x}}(t) &= A_K\bar{x}(t) - A_K x(t),\; \bar{x}\left(t_0^+\right) = 0_n, \\ \dot{\varphi}(t) &= A_K\varphi(t) + e_1 x^{\mathrm{T}}(t) h_\vartheta \Phi_\vartheta(t),\; \varphi(t_0^+) = 0_{n\times n_\vartheta}, \\ \dot{\psi}(t) &= A_K\psi(t) + h_B\Phi_B(t)u(t),\; \psi\left(t_0^+\right) = 0_{n\times n_B}, \\ \dot{\upsilon}(t) &= A_K\upsilon(t) + A_0 x(t),\; \upsilon\left(t_0^+\right) = 0_n, \end{aligned} \qquad A_K = \begin{bmatrix} K \in \mathbb{R}^n & \begin{matrix} I_{(n-1)\times(n-1)} \\ 0_{1\times(n-1)} \end{matrix} \end{bmatrix}, \tag{3.2.1}$$

(ii) normalization procedures

$$\begin{gathered} z(t) := \frac{e_1^{\mathrm{T}}\left(x(t) - \bar{x}(t) - \upsilon(t)\right)}{1 + \Phi^{\mathrm{T}}(t)\Phi(t)} \quad \Psi^{\mathrm{T}}(t) := \frac{\Phi^{\mathrm{T}}(t)}{1 + \Phi^{\mathrm{T}}(t)\Phi(t)}, \\ \Phi^{\mathrm{T}}(t) := \left[e_1^{\mathrm{T}} e^{A_K\left(t - t_0^+\right)} \;\; e_1^{\mathrm{T}}\varphi(t) \;\; e_1^{\mathrm{T}}\psi(t)\right], \end{gathered} \tag{3.2.2}$$

(iii) dynamic extension

$$\dot{\Delta}(t) = e^{-\sigma\left(t - t_0^+\right)}\Psi(t)\Psi^{\mathrm{T}}(t), \quad \Delta\left(t_0^+\right) = 0_{(n_\vartheta + n_B + n)\times(n_\vartheta + n_B + n)}, \tag{3.2.3a}$$

$$\dot{y}(t) = e^{-\sigma\left(t - t_0^+\right)}\Psi(t)z(t), \quad y\left(t_0^+\right) = 0_{(n_\vartheta + n_B + n)} \tag{3.2.3b}$$

(iv) and mixing

$$Y(t) := \operatorname{adj}\{\Delta(t)\}\, y(t), \quad \Omega(t) := \det\{\Delta(t)\}\,, \tag{3.2.3c}$$

the following regression equation with respect to the parameters x_{B_0} and x_{ϑ_0} is obtained:

$$\Upsilon(t) := \mathfrak{L}Y(t) = \Omega\left(\Phi(t)\right)\theta, \quad \mathfrak{L} = \left[0_{(n_\vartheta + n_B)\times n} \; I_{(n_\vartheta + n_B)\times(n_\vartheta + n_B)}\right], \tag{3.2.4}$$

where, if $\Phi(t) \in \mathrm{FE}$, then $\forall t \geqslant t_e$ it holds that $\Omega_{UB}(t) \geqslant \Omega(t) \geqslant \Omega_{LB} > 0$.

Proof of Proposition 2 is postponed to Appendix.

If $\omega(t)$ is measurable, then, using equation (3.2.4) and the results of Theorem 1 from [19], the adaptive law can be derived, which ensures that the goal (3.1.7) is achieved. The following theorem is to obtain the adaptive law, which ensures (3.1.7) in case $\omega(t)$ is unmeasurable.

Theorem 1. *Let Assumptions 1–4 be met and $\Phi(t) \in \mathrm{FE}$, then the adaptive law*

$$\begin{gathered} \dot{\hat{\theta}}(t) = -\gamma(t)\Omega(t)\left(\Omega(t)\hat{\theta}(t) - \Upsilon(t)\right) = -\gamma(t)\Omega^2(t)\tilde{\theta}(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0, \\ \gamma(t) = \begin{cases} 0, \;\; \text{if} \;\; \Omega(t) < \rho \in \left(0; \Omega_{LB}\right], \\ \dfrac{\gamma_0\lambda_{\max}\left(\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t)\right) + \gamma_1}{\Omega^2(t)} \quad \text{otherwise} \end{cases} \end{gathered} \tag{3.2.5}$$

in case $\gamma_0 > 0,\ \gamma_1 \geqslant 0$ *ensures the following:*

1) $\left|\tilde{\theta}_i\left(t_a\right)\right| \leqslant \left|\tilde{\theta}_i\left(t_b\right)\right| \ \ \forall t_a \geqslant t_b;$

2) $\left\{\begin{array}{c} \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \\ \left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t)x_{B_0}\right| \end{array}\right\} \Rightarrow \hat{F}(t) > 0;$

3) $\forall t \geqslant t_0^+$ *boundedness of* $\xi(t) \in L_\infty$;

4) *exponential convergence of* $\xi(t)$ *to zero for all* $t \geqslant t_e$.

Proof of theorem and definitions of $V_1(t)$, $\hat{\omega}(t)$ are presented in Appendix.

In case the conditions of the second statement of theorem are not met, then a division by zero may occur in the control law (3.1.4). That is why, considering a practical scenario, the law (3.1.4) should be augmented with the projection operator:

$$\begin{gathered} u(t) = \frac{\hat{x}_{B_0}^{\mathrm{T}}(t)\Phi_B^{\mathrm{T}}(t)h_B^{\mathrm{T}}}{\hat{F}_{\mathrm{prj}}(t)} e_1\left(e_1^{\mathrm{T}}\left(A_{ref} - A_0\right)x(t) - \hat{x}_{\vartheta_0}^{\mathrm{T}}(t)\Phi_\vartheta^{\mathrm{T}}(t)h_\vartheta^{\mathrm{T}}x(t) + e_1^{\mathrm{T}}B_{ref}r(t)\right), \\ \hat{F}_{\mathrm{prj}}(t) := \begin{cases} \hat{x}_{B_0}^{\mathrm{T}}(t)G(t)\hat{x}_{B_0}(t), \text{ if } \hat{x}_{B_0}^{\mathrm{T}}(t)G(t)\hat{x}_{B_0}(t) > \beta_{\min}^2 > 0, \\ \beta_{\min}^2 \text{ otherwise.} \end{cases} \end{gathered} \tag{3.2.6}$$

The proposed transformations $\mathcal{F}_1(.)$ and $\mathcal{F}_2(.)$ are described by (3.2.1), (3.2.2), and $\mathcal{G}(.)$ — by (3.2.3)–(3.2.5), respectively. In general, the designed adaptive control system includes the control law (3.1.4), procedures of the measurable signals processing (3.2.1)–(3.2.4) and the adaptive law (3.2.5). The filtering (3.2.1) allows one, using the measurable signals $x(t)$, $u(t)$, $\Phi_\vartheta(t)$, $\Phi_B(t)$, to obtain the static regression equation with respect to the unknown parameters x_{B_0}, x_{ϑ_0}, x_0. The normalization (3.2.2) guarantees that the regressor $\Psi(t)$ is bounded, which, owing to Proposition 1 from [19], is sufficient to state that $\Omega(t)$ is upper bounded. The procedures of extension and mixing (3.2.3) are used to transform the vector regressor $\Psi(t)$, first, into a matrix one $\Delta(t)$, and then — into a scalar one $\Omega(t)$. The division by $\Omega^2(t)$ used in the adaptive law (3.2.5) is a safe operation as $\Omega(t) \geqslant \Omega_{\mathrm{LB}} > 0$, and in case of proper choice of ρ it allows one to ensure the convergence of $\tilde{\theta}(t)$ to zero with the rate defined as $\gamma_0\lambda_{\max}\left(\hat{\omega}(t)\hat{\omega}^{\mathrm{T}}(t)\right) + \gamma_1$.

According to the results of theorem, unlike most known approaches [3–15] to control linear systems with time-varying parameters, the proposed system ensures exponentially stable control (2.6).

Remark 3. The application of the projection operator (3.2.6) *is a classical and well-known tool to avoid singularity in adaptive control schemes* (*see, for example,* [1, p. 400]). *In case conditions from the second statement of theorem are met, the choice* $\beta_{\min} \to 0$ *guarantees that there are no switches in* (3.2.6). *Otherwise, the choice* $\beta_{\min} \to 0$ *ensures that the number of switches is finite.*

Remark 4. Over $\left[t_0^+; t_e\right]$ *or when* $\Phi(t) \notin \mathrm{FE}$, *the loop of adjustment* (3.2.5) *of the control law* (3.1.4) *parameters is open, and in case of arbitrary choice of the initial conditions* $\hat{\theta}\left(t_0^+\right)$ *the control quality can be arbitrarily poor up to loss of stability. Therefore, in practice, for the proposed adaptive system:*

i) *the choice of the initial conditions* $\hat{\theta}\left(t_0^+\right)$ *should be made using robust control techniques to ensure that the following system is asymptotically stable*

$$\dot{x}(t) = \left(A(t) + B(t)\hat{K}_r(t)\hat{K}_x(t)\right)x(t) \ \textit{when} \ \ \dot{\hat{\theta}}(t) \equiv 0 \ \textit{for all} \ \ t \geqslant t_0^+,$$

ii) *the control law* (3.1.4) *should be augmented with some robust term, which guarantees the boundedness of the error* $\xi(t)$ *and acceptable control quality.*

For example, a) *in case* $\operatorname{sgn}(\beta(t))$ *and bounds* $\beta_{\min}$, $\beta_{\max}$ *are known, the following control law can be applied*

$$u(t) = \{(3.1.4)\,,\ (3.2.6)\} - \gamma_3 \operatorname{sgn}(\beta(t))\, e_{ref}(t) P e_1 \hat{\omega}(t) \hat{\omega}^{\mathrm{T}}(t), \quad \gamma_3 > 0$$

(*see Lemma 2.2 from* [5]), b) *if* $\operatorname{sgn}(\beta(t))$ *is unknown, but* $\beta_{\min}$, $\beta_{\max}$ *are known, then the law with the damping and Nussbaum function* [21] can be used:

$$\begin{gathered}
u(t) = \{(3.1.4)\,, (3.2.6)\} - \gamma_3 N\left(w(t)\right) e_{ref}^{\mathrm{T}}(t) P e_1 \hat{\omega}(t) \hat{\omega}^{\mathrm{T}}(t), \quad \gamma_3 > 0, \\
N\left(w(t)\right) = w^2(t) \cos\left(w(t)\right), \\
\dot{w}(t) = \gamma_3 \gamma_4 e_{ref}^{\mathrm{T}}(t) P e_1 e_1^{\mathrm{T}} P e_{ref}(t) \hat{\omega}(t) \hat{\omega}^{\mathrm{T}}(t), \quad \gamma_4 > 0.
\end{gathered}$$

4. NUMERICAL EXPERIMENTS

In Matlab/Simulink the numerical experiments with the proposed adaptive control system have been conducted for cases when the conditions of the second statement of theorem are met and violated. The simulation was done using numerical integration by the explicit Euler method with a constant discretization step of $\tau_s = 10^{-4}$ s.

4.1. $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \operatorname{sgn}\left(V_1^{\mathrm{T}}(t) x_{B_0}\right)$ *and* $\left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t) x_{B_0}\right|$

The matrices of the plant (2.1) were defined as follows for all $t \geqslant 0$:

$$\begin{gathered}
A(t) = \begin{bmatrix} 0 & 1 \\ a_1 \sin(a_2 t) & a_4 e^{a_3 t} + a_5\left(1 - e^{a_3 t}\right) \end{bmatrix}, \\
B(t) = \begin{bmatrix} 0 \\ b_1 \cos(b_2 t) + b_4 e^{b_3 t} + b_5 \end{bmatrix}, \quad x\left(t_0^+\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix},
\end{gathered} \tag{4.1.1}$$

where $a_2 =1$, $a_3 = b_3 = -0.25$, $b_2 = \sqrt{12}$ are known constants, $a_1 = -10$, $a_4 = 1$, $a_5 = 7$, $b_1 = 0.25$, $b_4 = -2$, $b_5 = -4$ denotes unknown constants.

Then the matrices and exosystem (2.2) initial conditions took the form:

$$\begin{gathered}
\mathcal{A}_\vartheta(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a_2^2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad x_{\vartheta_0} = \begin{bmatrix} -a_1 a_2 \\ 0 \\ a_4 - a_5 \\ a_5 \end{bmatrix}, \quad h_\vartheta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \\
\mathcal{A}_B(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -b_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_3 \end{bmatrix}, \quad x_{B_0} = \begin{bmatrix} 0 \\ b_1 \\ b_5 \\ b_4 \end{bmatrix}, \quad h_B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.
\end{gathered} \tag{4.1.2}$$

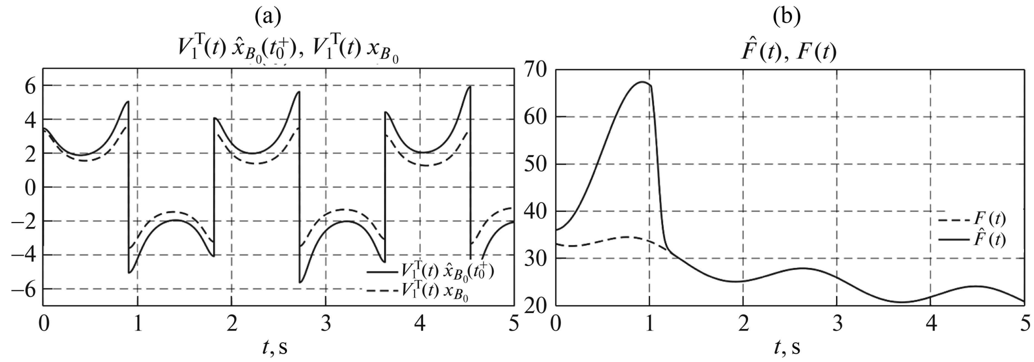

Fig. 1. Behavior of (a) $V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)$ and $V_1^{\mathrm{T}}(t)x_{B_0}$, (b) $\hat{F}(t)$ and $F(t)$.

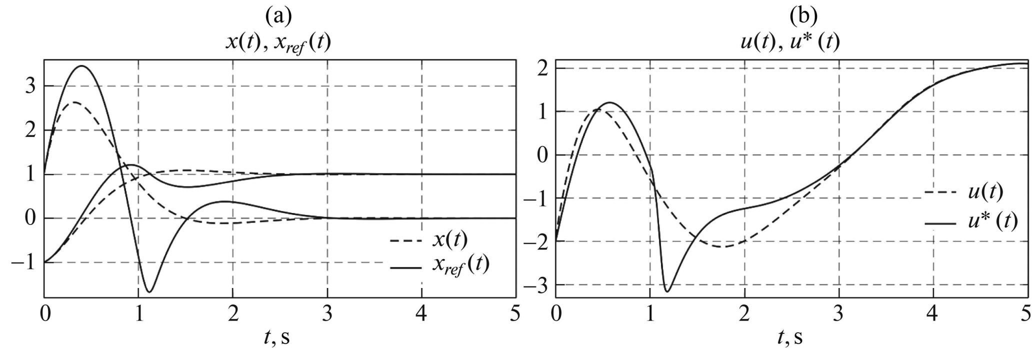

Fig. 2. Behavior of (a) $x(t)$ and $x_{ref}(t)$, (b) $u^*(t)$ and $u(t)$.

The matrices of the reference model (2.3), filters parameters (3.2.1), (3.2.3) and some parameters of the adaptive law (3.2.5) were picked as:

$$\begin{gathered}A_{ref} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix}, \quad B_{ref} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}, \quad A_K = \begin{bmatrix} -20 & 1 \\ -100 & 0 \end{bmatrix}, \quad \sigma = 5, \\ \rho = 10^{-62}, \quad \beta_{\min} = 0.1, \quad \beta_{\max} = 10, \quad \gamma_0 = 10^{-8}, \quad \gamma_1 = 0, \\ \hat{\theta}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -8 & 1 \end{bmatrix}^{\mathrm{T}}. \end{gathered} \tag{4.1.3}$$

First of all, it was checked whether the requirements of the second statement of theorem were met for such an experiment. Figure 1 depicts comparison of the functions $V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)$ with $V_1^{\mathrm{T}}(t)x_{B_0}$ and $\hat{F}(t)$ with $F(t)$.

The discontinuities in Fig. 1 were caused by the change of the direction of the eigenvector $V_1(t)$ (the elements of the matrix $G(t)$ crossed zero). It follows from Fig. 1a that the chosen initial conditions (4.1.3) guaranteed that the conditions of the second statement of theorem were met. Together Figs. 1a and 1b confirm the implication

$$\left\{ \begin{gathered} \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \\ \left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t)x_{B_0}\right| \end{gathered} \right\} \Rightarrow \hat{F}(t) > 0.$$

Having validated that $\hat{F}(t) > 0$, the modelling was continued. Figure 2a presents comparison of states of the reference model $x_{ref}(t)$ (when $x_{0ref} = x_0$) and the plant $x(t)$, and in Fig. 2b the ideal $u^*(t)$ and actual $u(t)$ control signals are compared.

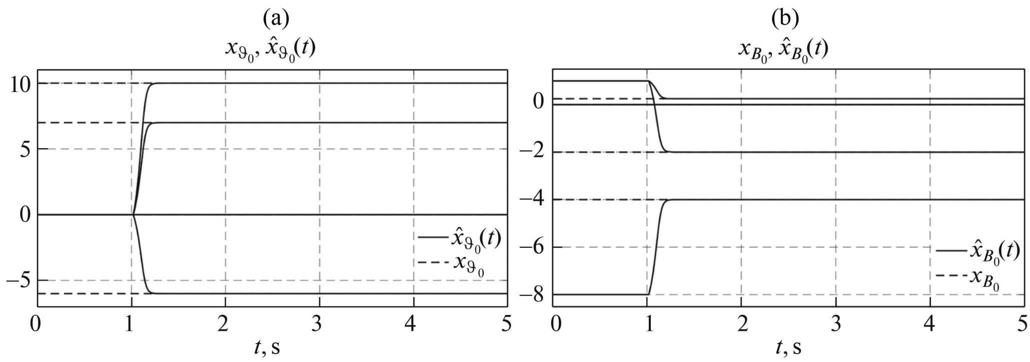

Fig. 3. Behaviour of (a) x_{ϑ_0} and $\hat{x}_{\vartheta_0}(t)$, (b) x_{B_0} and $\hat{x}_{B_0}(t)$.

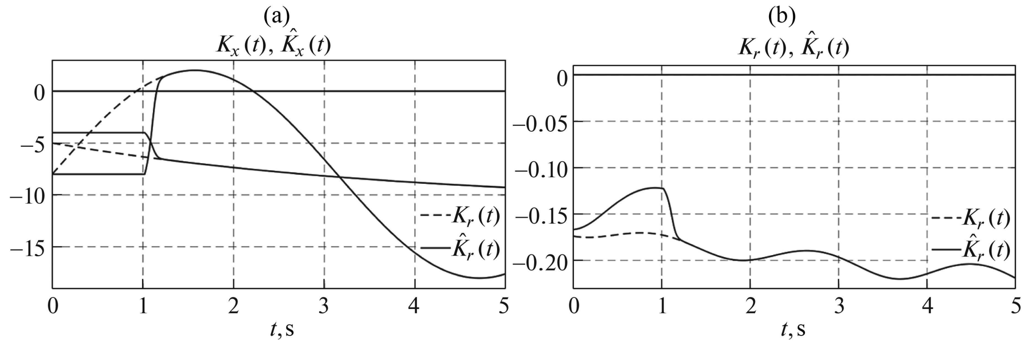

Fig. 4. Behavior of (a) $\hat{K}_x(t)$ and $K_x(t)$, (b) $\hat{K}_r(t)$ and $K_r(t)$.

In Fig. 3a the parameters x_{ϑ_0} and $\hat{x}_{\vartheta_0}(t)$ are compared, while in Fig. 3b — x_{B_0} and $\hat{x}_{B_0}(t)$.

Figure 4 demonstrates the comparison of the parameters $K_x(t)$, $K_r(t)$ and their estimates $\hat{K}_x(t)$, $\hat{K}_r(t)$ calculated with the help of $\hat{x}_{\vartheta_0}(t)$, $\hat{x}_{B_0}(t)$.

The simulation results confirm the theoretical conclusions of Theorem 1. Indeed, when $\gamma_0 > 0$, $\gamma_1 \geqslant 0$ the proposed adaptive system guaranteed that the goal (2.6) was achieved.

The transients shown in Figs. 2–4 confirm the shortcoming of the proposed system noted in Remark 3. Over the time interval [0; 1] the control system functioned with an open adaptive loop (3.2.5), which resulted in oscillations of $x(t)$.

4.2. $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$

The same plant was considered (4.1.1), (4.1.2) and the same parameters (4.1.3) of the reference model (2.3), filters (3.2.1), (3.2.3), adaptive law (3.2.5) were used, but under more realistic scenario for practice $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$. The modified control law (3.2.6) was chosen and, according to the first set of experiments, we picked $\gamma_0 = 10^{-10}$, $\gamma_1 = 10$ and $\beta_{\min} = 1$, $\rho = 10^{-81}$, $\hat{\theta}_0 = \begin{bmatrix} 0 & -8 & -2 & -2 & 0 & 1 & -8 & 1 \end{bmatrix}^{\mathrm{T}}$.

Figure 5 depicts the comparison of the function $V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)$ with $V_1^{\mathrm{T}}(t)x_{B_0}$, and $\hat{F}(t)$, $\hat{F}_{\mathrm{prj}}(t)$ with $F(t)$.

The discontinuities in Fig. 5 were caused by the change of the direction of the eigenvector $V_1(t)$ (the elements of the matrix $G(t)$ crossed zero). It follows from Fig. 5a that the chosen initial conditions guaranteed that the condition $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$ was met. Figure 5b demonstrates the effect of the projection operator (3.2.6) application. Together Figs. 5a and 5b

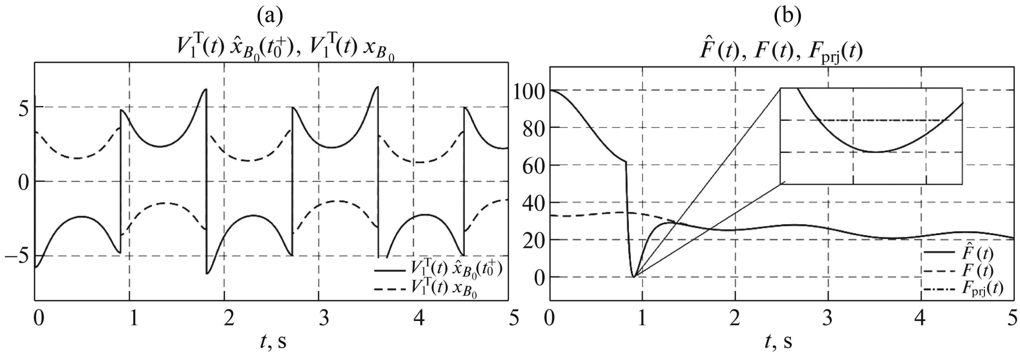

Fig. 5. Behavior of (a) $\hat{\Theta}_0^i(t)$ and $\Theta_0^i(t)$, (b) $\hat{F}(t)$, $\hat{F}_{\mathrm{prj}}(t)$ and $F(t)$.

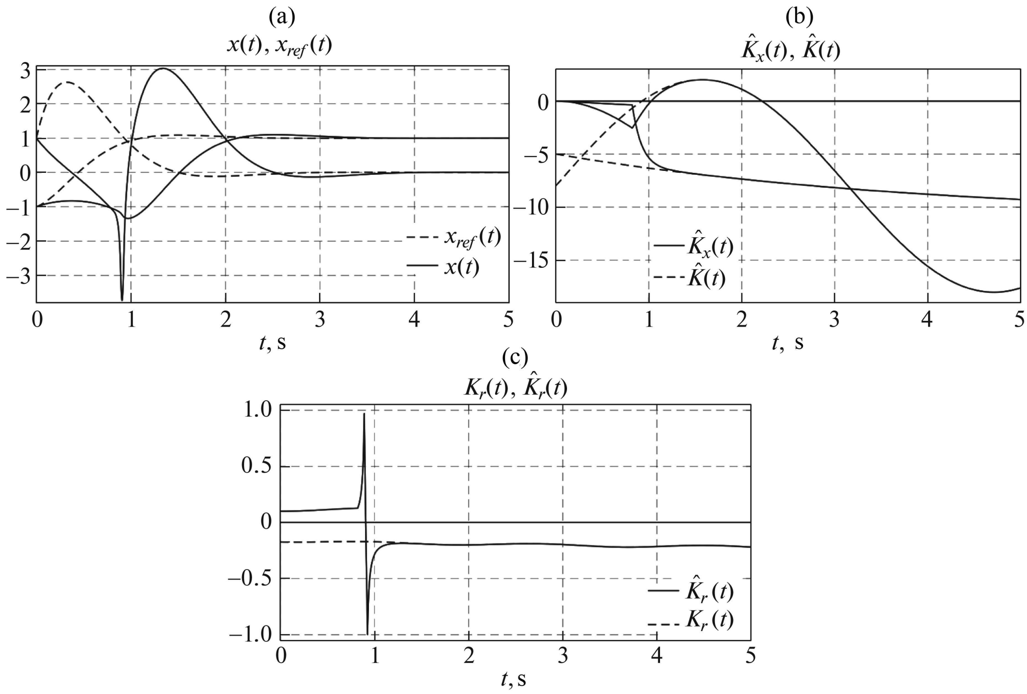

Fig. 6. Comparison of: (a) $x(t)$ and $x_{ref}(t)$, (b) $K_x(t)$ and $\hat{K}_x(t)$, (c) $K_r(t)$ and $\hat{K}_r(t)$.

confirm the implication

$$\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \Rightarrow \hat{F}(t) \geqslant 0.$$

Figure 6 depicts the comparison of the states of the reference model $x_{ref}(t)$ (when $x_{0ref} = x_0$) and the plant $x(t)$, as well as the parameters $K_x(t)$, $K_r(t)$ and their estimates $\hat{K}_x(t)$, $\hat{K}_r(t)$.

The experimental results confirm the capability of the adjustable control law (3.2.6) to effectively avoid possible division by zero when $\operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) \neq \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right)$.

Conducted experiments fully confirmed all theoretical conclusions of Theorem 1, Remarks 3 and 4.

5. CONCLUSION

The results of the first paper of this series were extended to the class of linear systems with time-varying unknown parameters described by known nonstationary exosystems with unknown initial conditions.

For this class of systems, the control system was proposed that solved the problem of tracking the trajectories of a time-invariant reference model by a time-varying plant. The control signal was computed using measurable signals and unknown initial conditions of the exosystems that generated the system parameters. To identify such initial conditions, an adaptive law was proposed that ensured exponential stability of the tracking error $e_{ref}(t)$ if the regressor was finitely exciting. The solution did not require to know the sign of the high-frequency gain, but requires known bounds of its absolute value.

The result had a drawback in common with [19, 22], namely it required the condition of the regressor finite excitation to be met to ensure boundedness of the tracking error. In Remark 4, some ways of dealing with this problem for single-input systems were given. For systems with multiple inputs, the problem to ensure tracking error boundedness without knowing the sign of the control allocation matrix is an open one.

The scope of further research could be to extend the results to (a) output-feedback control problems for systems with time-varying parameters, (b) control problems when matching conditions are violated, and (c) systems with multiple inputs.

FUNDING

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APPENDIX

Proof of Proposition 1. The following estimates are defined:

$$\begin{aligned} \hat{\vartheta}(t) &= h_{\vartheta}\Phi_{\vartheta}(t)\hat{x}_{\vartheta_0}(t), \\ \hat{B}(t) &= h_B\Phi_B(t)\hat{x}_{B_0}(t), \\ v(t) &= e_1^{\mathrm{T}}\left(A_{ref} - A_0\right)x(t) + e_1^{\mathrm{T}}B_{ref}r(t). \end{aligned} \tag{A.1}$$

Considering (A.1), the subtraction $u(t) - u^*(t)$ is written as follows (for the sake of brevity, time dependencies are omitted for a while):

$$\begin{aligned} u - u^* = \frac{\hat{B}^{\mathrm{T}}}{\hat{F}}e_1\left(v - \hat{\vartheta}^{\mathrm{T}}x\right) - \frac{B^{\mathrm{T}}}{F}e_1\left(v - \vartheta^{\mathrm{T}}x\right) \\ \pm\frac{B^{\mathrm{T}}}{F}e_1\left(v - \hat{\vartheta}^{\mathrm{T}}x\right) = -\frac{B^{\mathrm{T}}e_1}{F}\tilde{\vartheta}^{\mathrm{T}}x + \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}} - \frac{B^{\mathrm{T}}e_1}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right). \end{aligned} \tag{A.2}$$

The subtraction $\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}} - \frac{B^{\mathrm{T}}e_1}{F}$ is transformed into a linear regression with respect to $\tilde{B}$ and $\tilde{F}$:

$$\begin{aligned} \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}} - \frac{B^{\mathrm{T}}e_1}{F} &= \frac{\hat{B}^{\mathrm{T}}e_1F \pm \hat{B}^{\mathrm{T}}e_1\hat{F} - B^{\mathrm{T}}e_1\hat{F}}{\hat{F}F} \\ &= \frac{-\hat{B}^{\mathrm{T}}e_1\left(\hat{F} - F\right) + \left(\hat{B}^{\mathrm{T}} - B^{\mathrm{T}}\right)e_1\hat{F}}{\hat{F}F} \\ &= \frac{-\hat{B}^{\mathrm{T}}e_1\tilde{F} + \tilde{B}^{\mathrm{T}}e_1\hat{F}}{\hat{F}F} = \frac{-\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\tilde{F} + \tilde{B}^{\mathrm{T}}\frac{e_1}{F}. \end{aligned} \tag{A.3}$$

The error $\tilde{F}$ is considered separately:

$$\begin{aligned}\tilde{F} &= \hat{B}^{\mathrm{T}}\hat{B} - B^{\mathrm{T}}B + B^{\mathrm{T}}\hat{B} - B^{\mathrm{T}}\hat{B} \\ &= \left(\hat{B}^{\mathrm{T}} - B^{\mathrm{T}}\right)\hat{B} + B^{\mathrm{T}}\left(\hat{B} - B\right) = \tilde{B}^{\mathrm{T}}\hat{B} + B^{\mathrm{T}}\tilde{B}.\end{aligned} \tag{A.4}$$

The substitution of (A.4), (A.3) into (A.2) allows one to obtain:

$$\begin{aligned}u - u^{*} &= -\frac{B^{\mathrm{T}}e_1}{F}\tilde{\vartheta}^{\mathrm{T}}x - \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\tilde{B}^{\mathrm{T}}\hat{B} + \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}}\tilde{B} - \tilde{B}^{\mathrm{T}}\frac{e_1}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right) \\ &= -\frac{B^{\mathrm{T}}e_1}{F}x^{\mathrm{T}}\tilde{\vartheta} - \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\hat{B}^{\mathrm{T}} + \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}} - \frac{e_1^{\mathrm{T}}}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)\tilde{B} \\ &= -\frac{B^{\mathrm{T}}e_1}{F}x^{\mathrm{T}}h_{\vartheta}\Phi_{\vartheta}\tilde{x}_{\vartheta_0} - \left(\frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\hat{B}^{\mathrm{T}} + \frac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}} - \frac{e_1^{\mathrm{T}}}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)h_B\Phi_B\tilde{x}_{B_0} = \tilde{\theta}^{\mathrm{T}}\omega,\end{aligned} \tag{A.5}$$

where

$$\tilde{\theta} = \begin{bmatrix}\tilde{x}_{\vartheta_0}^{\mathrm{T}} & \tilde{x}_{B_0}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \quad \omega = \begin{bmatrix} -\left(\dfrac{B^{\mathrm{T}}e_1}{F}x^{\mathrm{T}}h_{\vartheta}\Phi_{\vartheta}\right)^{\mathrm{T}} \\ -\Phi_B^{\mathrm{T}}h_B^{\mathrm{T}}\left[\left(\dfrac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}\hat{B}^{\mathrm{T}} + \dfrac{\hat{B}^{\mathrm{T}}e_1}{\hat{F}F}B^{\mathrm{T}} - \dfrac{e_1^{\mathrm{T}}}{F}\right)\left(v - \hat{\vartheta}^{\mathrm{T}}x\right)\right]^{\mathrm{T}}\end{bmatrix},$$

which completes proof of Proposition 1.

Proof of Proposition 2. The error $\chi(t) = x(t) - \bar{x}(t)$ is introduced. Differentiating $\chi(t)$ with respect to time, it is obtained that:

$$\begin{aligned}\dot{\chi}(t) &= \dot{x}(t) - \dot{\bar{x}}(t) \\ &= A(t)x(t) + B(t)u(t) - A_K\bar{x}(t) + A_Kx(t) \\ &= A_K\left(x(t) - \bar{x}(t)\right) + A(t)x(t) + B(t)u(t) \\ &= A_K\chi(t) + A_0x(t) + e_1x^{\mathrm{T}}(t)\vartheta(t) + B(t)u(t) \\ &= A_K\chi(t) + A_0x(t) + e_1x^{\mathrm{T}}(t)h_{\vartheta}\Phi_{\vartheta}(t)x_{\vartheta_0} + h_B\Phi_B(t)u(t)x_{B_0}.\end{aligned} \tag{A.6}$$

The solution of the differential equation (A.6), which is multiplied by e_1^{T}, takes the following form:

$$\begin{aligned}e_1^{\mathrm{T}}\left[\chi(t) - \upsilon(t)\right] &= e_1^{\mathrm{T}}\left[x(t) - \bar{x}(t) - \upsilon(t)\right] \\ &= e_1^{\mathrm{T}}e^{A_K\left(t - t_0^{+}\right)}x\left(t_0^{+}\right) + e_1^{\mathrm{T}}\varphi(t)x_{\vartheta_0} + e_1^{\mathrm{T}}\psi(t)x_{B_0} \\ &= e_1^{\mathrm{T}}e^{A_K\left(t - t_0^{+}\right)}x_0 + e_1^{\mathrm{T}}\varphi(t)x_{\vartheta_0} + e_1^{\mathrm{T}}\psi(t)x_{B_0} \\ &= \begin{bmatrix}e_1^{\mathrm{T}}e^{A_K\left(t - t_0^{+}\right)} & e_1^{\mathrm{T}}\varphi(t) & e_1^{\mathrm{T}}\psi(t)\end{bmatrix}\begin{bmatrix}x_0 \\ x_{\vartheta_0} \\ x_{B_0}\end{bmatrix} := \Phi^{\mathrm{T}}(t)\eta.\end{aligned} \tag{A.7}$$

Having applied to the regression equation (A.7) the procedures of normalization (3.2.2), dynamic extension (3.2.3a), (3.2.3b) and mixing (3.2.3c), and using the property $\operatorname{adj}\{\Delta(t)\}\Delta(t) = \det\{\Delta(t)\}I_{(n_{\vartheta}+n_B+n)\times(n_{\vartheta}+n_B+n)}$, the measurable regression equation (3.2.4) is obtained.

Proof of the fact that for all $t \geqslant t_e$ the inequality $\Omega_{\mathrm{UB}}(t) \geqslant \Omega(t) \geqslant \Omega_{LB} > 0$ holds if $\Phi(t) \in \mathrm{FE}$ has been obtain in Proposition 4 of [23].

Proof of Theorem 1. Proof of the first part of theorem coincides with the one of the first part of the theorem from [19].

To prove the second part of theorem, the eigenvalue decomposition is applied to the matrix $G(t)$:

$$\begin{gathered}\forall t \geqslant t_0^+ \; F(t) = x_{B_0}^{\mathrm{T}} V(t)\Lambda(t)V^{\mathrm{T}}(t)x_{B_0} = x_{B_0}^{\mathrm{T}} V_1(t)\Lambda_1(t)V_1^{\mathrm{T}}(t)x_{B_0} = \Theta^{\mathrm{T}}(t)\Lambda_1(t)\Theta(t),\\ V(t) = \begin{bmatrix} V_1(t) & V_2(t)\end{bmatrix}, \quad \Lambda(t) = \begin{bmatrix} \Lambda_1(t) & 0_{r_G(t)\times \bar{r}_G(t)} \\ 0_{\bar{r}_G(t)\times r_G(t)} & 0_{\bar{r}_G(t)}\end{bmatrix},\\ \Lambda_1(t) = \operatorname{diag}\left\{\lambda_1(t), \lambda_2(t), \ldots, \lambda_{r_G(t)}(t)\right\}, \quad \lambda_{\min}\left(\Lambda_1(t)\right) > 0,\end{gathered}$$

where $V_1(t) \in \mathbb{R}^{n_B \times r_G(t)}$, $V_2(t) \in \mathbb{R}^{n_B \times \bar{r}_G(t)}$, $\Lambda(t) \in \mathbb{R}^{n_B \times n_B}$, $r_G(t) = \operatorname{rank}\{G(t)\}$, $\bar{r}_G(t) = n_B - r_G(t)$.

Using the above-introduced decomposition, the lower bound of $\hat{F}(t)$ is written:

$$\begin{aligned}\forall t \geqslant t_0^+ \; \hat{F}(t) &= \hat{B}^{\mathrm{T}}(t)\hat{B}(t) = \hat{x}_{B_0}^{\mathrm{T}}(t)V_1(t)\Lambda_1(t)V_1^{\mathrm{T}}(t)\hat{x}_{B_0}(t)\\ &= \hat{\Theta}^{\mathrm{T}}(t)\Lambda_1(t)\hat{\Theta}(t) \geqslant \lambda_{\min}\left(\Lambda_1(t)\right)\left\|\hat{\Theta}(t)\right\|^2 > 0.\end{aligned} \tag{A.8}$$

Based on (A.8), it is necessary and sufficient to satisfy the following inequality to ensure that $\hat{F}(t) > 0$

$$\begin{gathered}\forall t \geqslant t_0^+ \; \left\|\hat{\Theta}(t)\right\|^2 = \sum_{i=1}^{r_G(t)} \left(\hat{\Theta}_i(t)\right)^2 \neq 0,\\ \Updownarrow \\ \forall i \in \overline{1,\ r_G(t)} \; \left|\hat{\Theta}_i(t)\right| \neq 0,\end{gathered} \tag{A.9}$$

where $\hat{\Theta}^i(t)$ denotes the i^{th} element of the vector $\hat{\Theta}(t) \in \mathbb{R}^{r_G(t)}$.

The next aim is to obtain the functional definition of the estimate $\hat{\Theta}_i(t)$ for all $t \geqslant t_0^+$. For this the differential equation (3.2.5) is solved

$$\forall t \geqslant t_0^+ \; \tilde{x}_{B_0}(t) = \phi\left(t,\ t_0^+\right)\tilde{x}_{B_0}\left(t_0^+\right), \tag{A.10}$$

then equation (A.10) is multiplied by $V_1^{\mathrm{T}}(t)$ and $\Theta(t)$ is added to both left- and right-hand sides of the obtained multiplication:

$$\begin{gathered} V_1^{\mathrm{T}}\tilde{x}_{B_0}(t) + \Theta(t) = \hat{\Theta}(t) = \phi\left(t,\ t_0^+\right)V_1^{\mathrm{T}}(t)\tilde{x}_{B_0}\left(t_0^+\right) + \Theta(t),\\ \Updownarrow \\ \hat{\Theta}_i(t) = \phi\left(t,\ t_0^+\right)\tilde{\Theta}_i^0(t) + \Theta_i(t),\end{gathered} \tag{A.11}$$

where $\phi\left(t,\ t_0^+\right) = e^{-\beta_{\max}\int\limits_{t_0^+}^{t}\begin{cases}0, \text{ if } t<t_e,\\ \gamma_0\lambda_{\max}\left(\hat{\omega}(\tau)\hat{\omega}^{\mathrm{T}}(\tau)\right)+\gamma_1 \text{ otherwise}\end{cases}d\tau}$, $\tilde{\Theta}_i^0(t) = \hat{\Theta}_i^0(t) - \Theta_i(t)$, $\hat{\Theta}_i^0(t)$ is the i^{th} element of the vector $V_1^{\mathrm{T}}\hat{x}_{B_0}\left(t_0^+\right)$.

Then (A.9) is met, if it holds that

$$\begin{gathered}
\hat{\Theta}_i(t) = \phi\left(t,\ t_0^+\right)\tilde{\Theta}_i^0(t) + \Theta_i(t) \neq 0 \Rightarrow \phi\left(t,\ t_0^+\right)\tilde{\Theta}_i^0(t) \neq -\Theta_i(t) \\
\Rightarrow \underbrace{\operatorname{sgn}\left(\phi\left(t,\ t_0^+\right)\right)}_{=1} \neq \operatorname{sgn}\left(\frac{-\Theta_i(t)}{\tilde{\Theta}_i^0(t)}\right) = \operatorname{sgn}\left(\frac{-\Theta_i(t)}{\hat{\Theta}_i^0(t) - \Theta_i(t)}\right) \\
\Rightarrow \operatorname{sgn}\left(\hat{\Theta}_i^0(t) - \Theta_i(t)\right) \neq -\operatorname{sgn}\left(\Theta_i(t)\right) \\
\Rightarrow \left\{\begin{array}{l} \operatorname{sgn}\left(\hat{\Theta}_i^0(t)\right) = \operatorname{sgn}\left(\Theta_i(t)\right) \\ \left|\hat{\Theta}_i^0(t)\right| > \left|\Theta_i(t)\right| \end{array}\right\} \\
\Rightarrow \left\{\begin{array}{l} \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right) = \operatorname{sgn}\left(V_1^{\mathrm{T}}(t)x_{B_0}\right) \\ \left|V_1^{\mathrm{T}}(t)\hat{x}_{B_0}\left(t_0^+\right)\right| > \left|V_1^{\mathrm{T}}(t)x_{B_0}\right| \end{array}\right\},
\end{gathered} \tag{A.12}$$

which completes proof of the second part of theorem.

The next step is to prove the third part of theorem. According to the Kalman–Yakubovich–Popov lemma, for the pair $(A_{ref}, I_{n\times n})$ and any constant matrix $D > 0$ one can find matrices $Q \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{n\times n}$ and a constant $\mu > 0$ such that there exists a solution of the following set of equations

$$\begin{gathered}
A_{ref}^{\mathrm{T}}P + PA_{ref} = -QQ^{\mathrm{T}} - \mu P, \quad PI_{n\times n} = QK, \\
K^{\mathrm{T}}K = D + D^{\mathrm{T}},
\end{gathered} \tag{A.13}$$

or a solution of the following Riccati equation, which is an equivalent of the above-given set of equations in a particular case $D = 0.5k^2 I_{n\times n}$, $K = k^2 I_{n\times n}$, $k = 1$:

$$A_{ref}^{\mathrm{T}}P + PA_{ref} + PP^{\mathrm{T}} + \mu P = 0. \tag{A.14}$$

The following quadratic form is introduced to analyze the stability:

$$\begin{gathered}
V = \xi^{\mathrm{T}}H\xi = \gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \frac{\beta_{\max}^2}{2}\tilde{\theta}^{\mathrm{T}}\tilde{\theta}, \\
H = \operatorname{blockdiag}\left\{\gamma_0 P, \quad \frac{\beta_{\max}^2}{2} I_{(n_\vartheta + n_B)\times(n_\vartheta + n_B)}\right\}, \\
\underbrace{\lambda_{\min}(H)}_{\lambda_m}\|\xi\|^2 \leqslant V(\|\xi\|) \leqslant \underbrace{\lambda_{\max}(H)}_{\lambda_M}\|\xi\|^2,
\end{gathered} \tag{A.15}$$

where the matrix P is a solution of the set (A.13) when $K = k^2 I_{n\times n}$, $D = 0.5k^2 I_{n\times n}$, $k = 1$ or equivalent Riccati equation (A.14).

Owing to equations (3.1.6) and (3.2.5), the derivative of the quadratic form (A.15) is written as

$$\begin{aligned}
\dot{V} &= \gamma_0\left[e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}}P + PA_{ref}\right)e_{ref} + 2\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}}PB\right] - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta} \\
&= \gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} - e_{ref}^{\mathrm{T}}QQ^{\mathrm{T}}e_{ref} + 2\tilde{\theta}^{\mathrm{T}}\omega e_{ref}^{\mathrm{T}}PI_{n\times n}B\right] - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta} \\
&= \gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} - e_{ref}^{\mathrm{T}}QQ^{\mathrm{T}}e_{ref} + 2\tilde{\theta}^{\mathrm{T}}\omega B^{\mathrm{T}}Q^{\mathrm{T}}e_{ref}\right] - \beta_{\max}^2\tilde{\theta}^{\mathrm{T}}\gamma\Omega^2\tilde{\theta}.
\end{aligned} \tag{A.16}$$

Completing the square in (A.16), it is obtained:

$$
\begin{aligned}
\dot V &= \gamma_0\Big[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} - e_{ref}^{\mathrm{T}}QQ^{\mathrm{T}}e_{ref} + 2e_{ref}^{\mathrm{T}}QB\omega^{\mathrm{T}}\tilde\theta \pm 2\tilde\theta^{\mathrm{T}}\omega B^{\mathrm{T}}B\omega^{\mathrm{T}}\tilde\theta\Big] - \beta_{\max}^2\tilde\theta^{\mathrm{T}}\gamma\Omega^2\tilde\theta \\
&= \gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} - \left(e_{ref}^{\mathrm{T}}Q - B\omega^{\mathrm{T}}\tilde\theta\right)^2 + \tilde\theta^{\mathrm{T}}\omega B^{\mathrm{T}}B\omega^{\mathrm{T}}\tilde\theta\right] - \beta_{\max}^2\tilde\theta^{\mathrm{T}}\gamma\Omega^2\tilde\theta \\
&\leqslant \gamma_0\left[-\mu e_{ref}^{\mathrm{T}}Pe_{ref} + \tilde\theta^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde\theta\right] - \beta_{\max}^2\tilde\theta^{\mathrm{T}}\gamma\Omega^2\tilde\theta.
\end{aligned}
\tag{A.17}
$$

Two situations need to be considered: $t < t_e$ and $t \geqslant t_e$. As for the first one, according to proposition 1, in the most conservative case it holds that $\Omega(t) = 0$ and $\left\|\tilde\theta(t)\right\| = \left\|\tilde\theta\left(t_0^+\right)\right\|$.

Then for all $t < t_e$ equation (A.17) is rewritten as

$$
\begin{aligned}
\dot V &\leqslant -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \gamma_0\tilde\theta^{\mathrm{T}}\left(t_0^+\right)\omega F\omega^{\mathrm{T}}\tilde\theta\left(t_0^+\right) \pm \beta_{\max}^2\tilde\theta^{\mathrm{T}}\tilde\theta \\
&\leqslant -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} - \beta_{\max}^2\tilde\theta^{\mathrm{T}}\tilde\theta + \gamma_0\tilde\theta^{\mathrm{T}}\left(t_0^+\right)\omega F\omega^{\mathrm{T}}\tilde\theta\left(t_0^+\right) + \beta_{\max}^2\tilde\theta^{\mathrm{T}}\left(t_0^+\right)\tilde\theta\left(t_0^+\right).
\end{aligned}
\tag{A.18}
$$

The notion of the maximum eigenvalue of the matrix $\omega(t)B^{\mathrm{T}}(t)B(t)\omega^{\mathrm{T}}(t)$ over the time range $[0;\, t_e)$ is introduced:

$$
\delta = \sup\max_{\forall t<t_e}\lambda_{\max}\left(\omega(t)F(t)\omega^{\mathrm{T}}(t)\right). \tag{A.19}
$$

The function $F(t)$ is bounded according to Assumption 2, the rate of the regressor $\omega(t)$ change is no grater than exponential when Assumption 1 is met, therefore, it holds that $\delta \in L_\infty$.

Considering (A.19), equation (A.18) is rewritten as follows for $t < t_e$

$$
\dot V \leqslant -\mu\gamma_0\lambda_{\min}(P)\left\|e_{ref}\right\|^2 - \beta_{\max}^2\left\|\tilde\theta\right\|^2 + \left(\gamma_0\delta + \beta_{\max}^2\right)\left\|\tilde\theta\left(t_0^+\right)\right\|^2 \leqslant -\eta_1 V + r_B, \tag{A.20}
$$

where $\eta_1 = \min\left\{\frac{\mu\lambda_{\min}(P)}{\lambda_{\max}(P)};\, 2\right\}$, $r_B = \left(\gamma_0\delta + \beta_{\max}^2\right)\left\|\tilde\theta\left(t_0^+\right)\right\|^2$.

Having solved the differential equation (A.20), we have:

$$
\forall t < t_e\colon V(t) \leqslant e^{-\eta_1\left(t - t_0^+\right)}V\left(t_0^+\right) + \frac{r_B}{\eta_1}. \tag{A.21}
$$

Taking into consideration $\lambda_m\|\xi(t)\|^2 \leqslant V(t)$ and $V\left(t_0^+\right) \leqslant \lambda_M\left\|\xi\left(t_0^+\right)\right\|^2$, the following upper bound of the augmented tracking error is obtained for all $t < t_e$ from (A.21):

$$
\|\xi(t)\| \leqslant \sqrt{\frac{\lambda_M}{\lambda_m}e^{-\eta_1\left(t-t_0^+\right)}\left\|\xi\left(t_0^+\right)\right\|^2 + \frac{r_B}{\lambda_m\eta_1}} \leqslant \sqrt{\frac{\lambda_M}{\lambda_m}\left\|\xi\left(t_0^+\right)\right\|^2 + \frac{r_B}{\lambda_m\eta_1}}, \tag{A.22}
$$

from which it follows that $\xi(t)$ is bounded for all $t < t_e$.

As for the second situation, considering the definition of the adaptive gain γ and the fact that, owing to Proposition 1, for all $t \geqslant t_e$ the inequality $0 < \Omega_{\mathrm{LB}} \leqslant \Omega(t) \leqslant \Omega_{\mathrm{UB}}$ holds, it is obtained from (A.18) for $t \geqslant t_e$ that:

$$
\begin{aligned}
\dot V &\leqslant -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \gamma_0\tilde\theta^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde\theta - \beta_{\max}^2\tilde\theta^{\mathrm{T}}\frac{\left(\gamma_0\lambda_{\max}\left(\hat\omega\hat\omega^{\mathrm{T}}\right) + \gamma_1\right)\Omega^2}{\Omega^2}\tilde\theta \\
&= -\mu\gamma_0 e_{ref}^{\mathrm{T}}Pe_{ref} + \gamma_0\tilde\theta^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde\theta - \beta_{\max}^2\tilde\theta^{\mathrm{T}}\left[\gamma_0\lambda_{\max}\left(\hat\omega\hat\omega^{\mathrm{T}}\right) + \gamma_1\right]\tilde\theta.
\end{aligned}
\tag{A.23}
$$

The regressor $\hat{\omega}(t)$ is defined as follows:

$$\hat{\omega}(t)=\begin{bmatrix} -\left(\frac{\beta_{\max}}{\beta_{\min}^2}x^{\mathrm{T}}h_{\vartheta}\Phi_{\vartheta}\right)^{\mathrm{T}} \\ -\Phi_B^{\mathrm{T}}h_B^{\mathrm{T}}\left[\left(\frac{\hat{B}^{\mathrm{T}}e_1}{\beta_{\min}^2\hat{F}e_1^{\mathrm{T}}e_1}\hat{B}^{\mathrm{T}}+\frac{\hat{B}^{\mathrm{T}}e_1}{\beta_{\min}^2\hat{F}e_1^{\mathrm{T}}e_1}\beta_{\max}e_1^{\mathrm{T}}-\frac{e_1^{\mathrm{T}}}{\beta_{\min}^2e_1^{\mathrm{T}}e_1}\right)\left(v-\hat{\vartheta}^{\mathrm{T}}x\right)\right]^{\mathrm{T}}\end{bmatrix}.$$

It also holds for any $\omega(t)$ that

$$\begin{gathered}\gamma_0\tilde{\theta}^{\mathrm{T}}\omega F\omega^{\mathrm{T}}\tilde{\theta}-\tilde{\theta}^{\mathrm{T}}\gamma_0\beta_{\max}^2\lambda_{\max}\left(\hat{\omega}\hat{\omega}^{\mathrm{T}}\right)\tilde{\theta} \\ =\tilde{\theta}^{\mathrm{T}}\underbrace{\left(\gamma_0\omega F\omega^{\mathrm{T}}-\gamma_0\beta_{\max}^2\lambda_{\max}\left(\hat{\omega}\hat{\omega}^{\mathrm{T}}\right)I_{(n_\vartheta+n_B)\times(n_\vartheta+n_B)}\right)}_{\leqslant-\kappa I_{(n_\vartheta+n_B)\times(n_\vartheta+n_B)}}\tilde{\theta}\leqslant 0,\end{gathered}\tag{A.24}$$

so equation(A.23) is rewritten as

$$\begin{gathered}\dot{V}\leqslant-\gamma_0\mu e_{ref}^{\mathrm{T}}Pe_{ref}-\tilde{\theta}^{\mathrm{T}}\left(\kappa+\beta_{\max}^2\gamma_1\right)\tilde{\theta} \\ \leqslant-\mu\gamma_0\lambda_{\min}(P)\left\|e_{ref}\right\|^2-\left(\kappa+\beta_{\max}^2\gamma_1\right)\left\|\tilde{\theta}\right\|^2\leqslant-\eta_2V,\end{gathered}\tag{A.25}$$

where $\eta_2=\min\left\{\frac{\mu\lambda_{\min}(P)}{\lambda_{\max}(P)};\, 2\left(\frac{\kappa}{\beta_{\max}^2}+\gamma_1\right)\right\}$.

Having solved the inequality (A.25), it is obtained that $V(t)\leqslant e^{-\eta_2(t-t_e)}V(t_e)$ for $t\geqslant t_e$.

Taking into account $\lambda_m\|\xi(t)\|^2\leqslant V(t)$, $V(t_e)\leqslant\lambda_M\|\xi(t_e)\|^2$ and equation (A.22), the upper bound of the augmented tracking error is obtained for $t\geqslant t_e$:

$$\|\xi(t)\|\leqslant\sqrt{\frac{\lambda_M}{\lambda_m}e^{-\eta_2(t-t_e)}\|\xi(t_e)\|^2}\leqslant\sqrt{\frac{\lambda_M}{\lambda_m}\left(\frac{\lambda_M}{\lambda_m}\left\|\xi\left(t_0^+\right)\right\|^2+\frac{r_B}{\lambda_m\eta_1}\right)},\tag{A.26}$$

from which together with (A.22) it follows that $\xi(t)\in L_\infty$, as well as exponential convergence of the error $\xi(t)$ to zero for all $t\geqslant t_e$ with the rate, which is directly proportional to the parameters γ_0, γ_1, Q.E.D.

REFERENCES

1. Ioannou, P. and Sun, J., *Robust Adaptive Control*, New York: Dover, 2013.
2. Narendra, K.S. and Annaswamy, A.M., *Stable Adaptive Systems*, Courier Corporation, 2012.
3. Chen, K. and Astolfi, A., Adaptive control for systems with time-varying parameters, *IEEE Transactions on Automatic Control*, 2020, vol. 66, no. 5, pp. 1986–2001.
4. Chen, K. and Astolfi, A., Identification-based Adaptive Control for Systems with Time-varying Parameters, *2021 60th IEEE Conference on Decision and Control (CDC)*, 2021, pp. 1083–1088.
5. Krstic, M., Kanellakopoulos, I., and Kokotovic, P.V., *Nonlinear and Adaptive Control Design*, New York, USA: Wiley, 1995.
6. Patil, O.S., Sun, R., Bhasin, S., and Dixon, W.E., Adaptive Control of Time-Varying Parameter Systems with Asymptotic Tracking, *IEEE Transactions on Automatic Control*, 2022, vol. 67, no. 9, pp. 1–7.
7. Na, J., Xing, Y., and Costa-Castello, R., Adaptive estimation of time-varying parameters with application to roto-magnet plant, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2018, vol. 51, no. 2, pp. 731–741.

8. Hu, Y., Wu, J., and Zeng, C., Robust adaptive identification of linear time-varying systems under relaxed excitation conditions, *IEEE Access*, 2020, vol. 8, no. 1, pp. 8268–8274.

9. Goel, R. and Roy, S.B., Composite Adaptive Control for Time-varying Systems with Dual Adaptation, *arXiv preprint arXiv:2206.01700*, 2022, pp. 1–6.

10. Glushchenko, A. and Lastochkin, K., Robust Time-Varying Parameters Estimation Based on I-DREM Procedure, *FAC-PapersOnLine*, 2022, vol. 55, no. 12, pp. 91–96.

11. Gerasimov, D.N., Lyzlova, M.V., and Nikiforov, V.O., Simple algorithms of adaptive and robust control over a class of linear time-varying objects, *Journal of Instrument Engineering*, 2015, vol. 58, no. 5, pp. 351–361. In Russian.

12. Miroshnik, I.V. and Nikiforov, V.O., An adaptation algorithm with astatism of the second order, *Autom. Remote Control*, 1995, no. 7, pp. 99–107.

13. Tsykunov, A.M., Robust control of multidimensional nonstationary linear plants, *Autom. Remote Control*, 2009, vol. 70, no. 2, pp. 109–121.

14. Tsykunov, A.M., An algorithm of robust control of a non-stationary linear plant with perturbation compensation, *Journal of computer and systems sciences international*, 2008, vol. 47, no. 4, pp. 527–534.

15. Furtat, I.B. and Tsykunov, A.M., Robust control of unsteady-state nonlinear structurally undefined objects, *Control Sciences*, 2008, no. 5, pp. 2–7. In Russian.

16. Nikiforov, V.O. and Gerasimov, D.N., *Adaptive Regulation: Reference Tracking and Disturbance Rejection*, Springer-Verlag, 2022.

17. Pyrkin, A., Bobtsov, A., Ortega, R., and Isidori, A., An adaptive observer for uncertain linear time-varying systems with unknown additive perturbations, *Automatica*, 2023, vol. 147, p. 110677.

18. Bobtsov, A., Ortega, R., Yi, B., and Nikolaev, N., Adaptive state estimation of state-affine systems with unknown time-varying parameters, *Int. J. Control*, 2022, vol. 95, no 9, pp. 2460–2472.

19. Glushchenko, A., Lastochkin, K., and Petrov, V., Exponentially Stable Adaptive Control. Part I. Time-Invariant Plants, *Autom. Remote Control*, 2022, vol. 83, no. 4, pp. 548–578.

20. Leiva, H. and Siegmund, S., A necessary algebraic condition for controllability and observability of linear time-varying systems, *IEEE Transactions on Automatic Control*, 2003, vol. 48, no. 12, pp. 2229–2232.

21. Nussbaum, R.D., Some remarks on a conjecture in parameter adaptive control, *Syst. Control Lett.*, 1983, vol. 3, no. 5, pp. 243–246.

22. Glushchenko, A. and Lastochkin, K., Exponentially Stable Adaptive Control. Part II. Switched Systems, *Autom. Remote Control*, 2023, vol. 84, no. 3, pp. 285–316.

23. Glushchenko, A. and Lastochkin, K., Unknown Piecewise Constant Parameters Identification with Exponential Rate of Convergence, *Int. J. Adapt. Control Signal Proc.*, 2023, vol. 37, no. 1, pp. 315–346.

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