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LINEAR SYSTEMS

On the External Estimation of Reachable and Null-Controllable Limit Sets for Linear Discrete-Time Systems with a Summary Constraint on the Scalar Control

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Abstract—The problem of constructing reachable and null-controllable sets for stationary linear discrete-time systems with a summary constraint on the scalar control is considered. For the case of quadratic constraints and a diagonalizable matrix of the system, these sets are built explicitly in the form of ellipsoids. In the general case, the limit reachable and null-controllable sets are represented as fixed points of a contraction mapping in the metric space of compact sets. On the basis of the method of simple iteration, a convergent procedure for constructing their external estimates with an indication of the a priori approximation error is proposed. Examples are given.

Keywords: linear discrete-time system, controllable limit set, reachable limit set, Hausdorff distance, contraction mapping principle

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1. INTRODUCTION

It is often necessary to take into account various constraints on control actions when studying dynamic systems, which leads to the unreachability of some terminal states from a given initial state even in infinite time. As a result, the classical Kalman controllability conditions are not sufficient to draw a conclusion about the reachability of one or another terminal state. In this regard, it seems relevant to develop methods for constructing a constructive description of reachable sets, i.e. sets of terminal states to which the system can be transferred from the origin, and null-controllable sets, i.e. sets of initial states from which the system can be transferred to the origin, in a finite number of steps, as well as estimates of the limit reachable and null-controllable sets [1]. The null-controllable and reachable sets can be used in optimal control problems to construct positional control for discrete-time systems [2, 3]. Thus it is possible to judge the solvability of these problems in principle by the apparatus of limit sets.

At the moment, methods for estimating reachable sets of various classes of discrete-time systems [4], hybrid systems [5] and systems with various types of uncertainties [6] are being actively developed. Analytical representations of reachable and null-controllable sets for linear systems with discrete time and constraints on the control function in the sense of the l_∞-norm are known. In particular, it is proved that in the case of linear constraints on the control, the reachable and null-controllable sets in a finite number of steps are polyhedra [2]. For their limit analogues, necessary and sufficient conditions for the boundedness are formulated [7–9]. At the same time, most of the papers are either focused on studying only the general properties of reachable and null-controllable limit sets [8–12], or consider systems with unbounded control [10–14]. Only in a few special cases constructive methods for constructing external estimates have been proposed based on the apparatus of support half-spaces [15, 16] or the maximum principle [17].

For systems with summary control constraints, a description of the reachable and null-controllable limit sets in the form of polytopes is obtained for the case of constraints in the sense of the l_1-norm [18]. When choosing a l_p-norm with an arbitrary value of the parameter $p \in (1; +\infty)$, the general properties of reachable and null-controllable limit sets have been formulated and proven [19]. In particular, their representation in the form of projections of superellipsoidal sets of finite [20, 21] and infinite dimensions has been constructed, which is closely related to strictly convex analysis [22, 23], convex programming [24], and the theory of normed spaces [25] and linear operators [26].

Often in control problems it is necessary to examine a given initial state for reachability and controllability, which reduces to checking whether a some point in the phase space belongs to the reachable or controllable limit set. This procedure can be reduced to calculating the Minkowski functional, but the known results of [19] are not enough to construct it explicitly. Moreover, describing the Minkowski functional of the image of a convex set under a linear mapping in the general case is a non-trivial task. For this reason, it is relevant to develop methods implemented in software that will allow us to calculate exactly the Minkowski functional of the reachable and null-controllable limit sets or their external estimates of an arbitrary accuracy.

The paper studies the issues of constructing the Minkowski functional of reachable and null-controllable sets with a summary control constraint in the sense of the l_p-norm in the case when they are bounded. It is possible to explicitly describe this function under quadratic restrictions on the control and prove that the sets under study are ellipsoids. For the case of arbitrary normed spaces, the reachable and null-controllable limit sets are described as a fixed-point of a contraction mapping in the space of compacta with the Hausdorff distance. This allows us to propose a convergent iterative process for constructing external estimates of these sets with an explicit form of the a priori error. For a number of parameter values, the resulting estimates have a polyhedral structure, which makes it possible to use them in computer calculations.

The contents of the article are as follows. In Section 2 the problem is stated. Section 3 discusses the issues of calculating the Minkowski functional of reachable and null-controllable limit sets. For the special case of l_2-constraints on control and the diagonalizable matrix of the system, the corresponding sets are constructed explicitly. Section 4 describes the apparatus of contraction mappings used to construct reachable and null-controllable limit sets. Section 5 suggests a method for constructing external estimates of these sets of an arbitrary accuracy by the simple iteration method. Section 6 demonstrates the effectiveness of the developed mathematical apparatus through various examples.

2. PROBLEM STATEMENT

We consider a linear discrete-time system with summary constraint on scalar control:

$$\begin{aligned} &x(k+1) = Ax(k) + bu(k),\ \ k \in \mathbb{N} \cup \{0\}, \\ &x(0) = x_0,\ \ \sum_{k=0}^{\infty} |u(k)|^p \leqslant 1, \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector of the system, $u(k) \in \mathbb{R}$ is the scalar control action, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are system matrices, $p > 1$ is a parameter that determines the type of summary control constraint.

For an arbitrary $N \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{Y}_p(N)$ the reachable set of the system (1), i.e. the set of states to which the system (1) can be transferred in N steps from the origin by an admissible control:

$$\mathcal{Y}_p(N) = \begin{cases} \left\{x \in \mathbb{R}^n \colon x = \sum\limits_{k=0}^{N-1} A^{N-k-1} b u(k), \ \ \sum\limits_{k=0}^{N-1} |u(k)|^p \leqslant 1 \right\}, & N \in \mathbb{N}, \\ \{0\}, & N = 0. \end{cases} \tag{2}$$

We denote by $\mathcal{Y}_{p,\infty}$ the reachable limit set of the system (1), i.e. the set of states to which the system (1) can be transferred in any finite number of steps by an admissible control:

$$\mathcal{Y}_{p,\infty} = \bigcup_{N=0}^{\infty} \mathcal{Y}_p(N). \tag{3}$$

For an arbitrary $N \in \mathbb{N} \cup \{0\}$ we denote by $\mathcal{X}_p(N)$ the null-controllable set of the system (1), i.e. the set of initial states from which the system (1) can be transferred to the origin in N steps by an admissible control:

$$\mathcal{X}_p(N) = \begin{cases} \left\{x_0 \in \mathbb{R}^n \colon -A^N x_0 = \sum\limits_{k=0}^{N-1} A^{N-k-1} b u(k), \ \ \sum\limits_{k=0}^{N-1} |u(k)|^p \leqslant 1 \right\}, & N \in \mathbb{N}, \\ \{0\}, & N = 0. \end{cases} \tag{4}$$

We denote by $\mathcal{X}_{p,\infty}$ the set null-controllable limit set of the system (1), i.e. the set of initial states from which the system (1) can be transferred to the origin in any finite number of steps by an admissible control:

$$\mathcal{X}_{p,\infty} = \bigcup_{N=0}^{\infty} \mathcal{X}_p(N). \tag{5}$$

It is required to develop an effective method for constructing an external estimate of the sets (3) and (5) with any predetermined accuracy. The Hausdorff distance ρ_H is considered as an accuracy criterion, and all sets are assumed to be elements of the complete metric space $(\mathbb{K}_n, \rho_H)$ [27]:

$$\mathbb{K}_n = \{\mathcal{X} \subset \mathbb{R}^n \colon \mathcal{X} \text{ is compact}\},$$

$$\rho_H(\mathcal{X}, \mathcal{Y}) = \max\left\{ \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} \|x - y\|_r ; \sup_{y \in \mathcal{Y}} \inf_{x \in \mathcal{X}} \|x - y\|_r \right\},$$

$$\|x\|_r = \begin{cases} \left(\sum\limits_{i=1}^{n} |x_i|^r \right)^{\frac{1}{r}}, & r \geqslant 1, \\ \max\limits_{i=\overline{1,n}} |x_i|, & r = \infty. \end{cases}$$

3. THE PROBLEM OF AN EXACT DESCRIPTION OF THE REACHABLE AND 0-CONTROLLABLE LIMIT SETS

Let us denote by $\mathcal{E}_p(\infty)$ a unit ball in the normed space l_p [25]:

$$\mathcal{E}_p(\infty) = \left\{ u \in l_p \colon \sum_{k=1}^{\infty} |u_k|^p \leqslant 1 \right\}.$$

We also identify the sequence $B = (b_1, b_2, \ldots) \in l_q^n$ with the linear operator $B \colon l_p \to \mathbb{R}^n_r$, acting according to the following rule:

$$Bu = \sum_{k=1}^{\infty} u_k b_k.$$

It is assumed here that the numbers p and q satisfy the relation $\frac{1}{p}+\frac{1}{q}=1$, and the space $\mathbb{R}^n_r$ is a normed space $(\mathbb{R}^n, \|\cdot\|_r)$. Hence, taking into account the Riesz theorem [25], it follows that the operator B is bounded, which allows us to consider it as an element of the normed space l^n_q with an operator norm defined on it:

$$\|B\|_{l^n_q} = \sup_{u\in\mathcal{E}_p(\infty)} \|Bu\|_r.$$

For simplicity, we also identify an arbitrary sequence $y \in l_q$ with the linear and bounded functional $y\colon l_p \to \mathbb{R}$ generated by it according to the Riesz theorem:

$$(y,u) = \sum_{k=1}^{\infty} y_k u_k.$$

The necessary and sufficient conditions for the boundedness of the sets (3) and (5), determined by the system matrices A and b, are essential. The Jordan basis of the matrix A is a set of linearly independent vectors $h_1,\ldots,h_n \subset \mathbb{R}^n$, which specifies the similarity transformation of the matrix A to its real Jordan canonical form [28, Section 3.4 Ch. 3]. Such a basis is unique up to non-zero factors and to the order of the vectors $h_1,\ldots,h_n$, and each basis vector corresponds to some Jordan cell, i.e. to some eigenvalue of the matrix A. If we divide the elements of the Jordan basis into three sets according to the criterion of whether they correspond to an eigenvalue of the matrix A greater than, equal to, or less than 1 in modulus, then we can determine the following three invariant subspaces:

$$\begin{aligned}
\mathbb{L}_{<1} &= \mathrm{Lin}\{h_i\colon h_i \text{ correspond to eigenvalue } \lambda,\ |\lambda|<1\},\\
\mathbb{L}_{=1} &= \mathrm{Lin}\{h_i\colon h_i \text{ correspond to eigenvalue } \lambda,\ |\lambda|=1\},\\
\mathbb{L}_{>1} &= \mathrm{Lin}\{h_i\colon h_i \text{ correspond to eigenvalue } \lambda,\ |\lambda|>1\}.
\end{aligned}$$

In [19] it is demonstrated that $\mathcal{Y}_{p,\infty}$ and $\mathcal{X}_{p,\infty}$ are bounded if and only if the following conditions are true respectively:

$$Y_\infty = (b, Ab, A^2b, \ldots) \in l^n_q \text{ or } b \in \mathbb{L}_{<1}, \tag{6}$$

$$X_\infty = (A^{-1}b, A^{-2}b, \ldots) \in l^n_q \text{ or } b \in \mathbb{L}_{>1}. \tag{7}$$

In these cases the following representations are valid:

$$\overline{\mathcal{Y}}_{p,\infty} = Y_\infty \mathcal{E}_p(\infty) \in \mathbb{K}_n, \tag{8}$$

$$\overline{\mathcal{X}}_{p,\infty} = X_\infty \mathcal{E}_p(\infty) \in \mathbb{K}_n. \tag{9}$$

According to (8) and (9), the limit sets $\mathcal{Y}_{p,\infty}$, $\mathcal{X}_{p,\infty}$ are convex, and therefore, for their description by algebraic inequalities, the Minkowski functional can be used [25, Section 3 §2 ch. III]:

$$\mu(u,\mathcal{U}) = \inf\{t>0\colon u \in t\,\mathcal{U}\}.$$

Let us demonstrate the complexity of calculating the Minkowski functional of the sets (3) and (5) for an arbitrary value of the parameter p, and also give a special case when such description can be constructed.

Lemma 1. *Let $\mathbb{L}_1$, $\mathbb{L}_2$ be normed spaces, $\mathcal{U} \subset \mathbb{L}_1$ is convex and bounded set, $0 \in \mathrm{int}\,\mathcal{U}$, $B\colon \mathbb{L}_1 \to \mathbb{L}_2$ is the linear, surjective and bounded operator.*

Then

$$\mu(x, B\mathcal{U}) = \inf_{u \in B^{-1}(\{x\})} \mu(u,\mathcal{U}).$$

The proof of the Lemma 1 and all other statements is given in the Appendix.

Let us obtain the corollaries of Lemma 1, setting $\mathbb{L}_1 = l_p$, $\mathbb{L}_2 = \mathbb{R}^n$, $\mathcal{U} = \mathcal{E}_p(\infty)$. The choice of norm in the space $\mathbb{R}^n$ is unimportant, since the value of the Minkowski functional does not depend on the norm, and due to the equivalence of all norms in the finite-dimensional space [25]. Hence, the operator B is limited for any norm in $\mathbb{R}^n$. But for brevity of notation we assume that $\mathbb{R}^n$ is Euclidean with the scalar product defined by the following relation:

$$(x, y) = x^{\mathrm{T}} y = \sum_{i=1}^{n} x_i y_i.$$

Let us introduce the nonlinear operator $I_p(u)\colon l_p \to l_q$ according to the following formula:

$$I_p(u) = \left(\operatorname{sgn}(u_1)|u_1|^{p-1}, \operatorname{sgn}(u_2)|u_2|^{p-1}, \ldots\right).$$

The inverse operator to I_p is the operator I_q. Let us denote by $B^*\colon \mathbb{R}^n \to l_q$ the operator adjoint to B.

Lemma 2. *Let $B \in l_q^n$ be the surjection. Then for any $x \in \mathbb{R}^n$ it is true that*

$$\mu(x, B\mathcal{E}_p(\infty)) = \|B^*\lambda\|_{l_q}^{q-1},$$

where $\lambda \in \mathbb{R}^n$ satisfies the following condition:

$$BI_q(B^*\lambda) = x. \tag{10}$$

According to Lemma 2 and the representations (8) and (9), the calculation of the Minkowski functional for the reachable and controllable limit sets can be reduced to solving a system of nonlinear equations of the form (10) when choosing operators Y_∞ and X_∞ as B, respectively, which is a nontrivial problem in the general case. Although, when $p = q = 2$, the solution of the system can be obtained in explicit form.

Corollary 1. *Let $p = q = 2$, $B \in l_q^n$ be the surjection.*

Then for any $x \in \mathbb{R}^n$ it is true that

$$\mu(x, B\mathcal{E}_2(\infty)) = \sqrt{x^{\mathrm{T}}(BB^*)^{-1}x}.$$

The application of Corollary 1 to construct $\mathcal{Y}_{2,\infty}$ and $\mathcal{X}_{2,\infty}$ is determined by the possibility of constructing explicitly the matrix $BB^* \in \mathbb{R}^{n\times n}$, which, according to the definition of the operator B, reduces to the calculation of a convergent series:

$$BB^* = \sum_{k=1}^{\infty} b_k b_k^{\mathrm{T}},$$

where B is assumed to be equal to Y_∞ or X_∞.

Lemma 3. *Let $A \in \mathbb{R}^{n\times n}$ have n linearly independent eigenvectors $h_1, \ldots, h_n \in \mathbb{C}^n$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, $S = (h_1, \ldots, h_n) \in \mathbb{C}^{n\times n}$. The following notations are used:*

$$H = S \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \cdots & \beta_{nn} \end{pmatrix} S^{\mathrm{T}},$$

$$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} = S^{-1} b b^{\mathrm{T}} (S^{-1})^{\mathrm{T}}, \quad \beta_{ij} = \begin{cases} \dfrac{\alpha_{ij}}{1 - \lambda_i \lambda_j}, & \lambda_i \lambda_j \neq 1, \\ 0, & \lambda_i \lambda_j = 1. \end{cases}$$

Then

1) *in the case* $b \in \mathbb{L}_{<1}$ *the following representation is true:*

$$\overline{\mathcal{Y}}_{2,\infty} = \{x \in \mathbb{R}^n \colon x^{\mathrm{T}} H_{\mathcal{Y},\infty} x \leqslant 1\},$$

where $H_{\mathcal{Y},\infty} = H^{-1}$;

2) *in the case* $b \in \mathbb{L}_{>1}$ *and* $\det A \neq 0$ *the following representation is true:*

$$\overline{\mathcal{X}}_{2,\infty} = \left\{x \in \mathbb{R}^n \colon x^{\mathrm{T}} H_{\mathcal{X},\infty} x \leqslant 1\right\},$$

where $H^{-1}_{\mathcal{X},\infty} = -H^{-1}$.

4. REACHABLE AND NULL-CONTROLLABLE LIMIT SETS AS THE FIXED-POINT

Let us present properties of the contraction mapping principle and fixed-points that are useful for representing the sets (3) and (5). It is known that the sets (2) and (4) are convex compact sets and can be represented as the image of $\mathcal{E}_p(\infty)$ under a linear mapping [19, Lemma 9]:

$$\mathcal{Y}_p(N) = Y_N \mathcal{E}_p(\infty),\ Y_N = (b, Ab, \ldots, A^{N-1}b, 0, \ldots) \in l_q^n, \tag{11}$$

$$\mathcal{X}_p(N) = X_N \mathcal{E}_p(\infty),\ X_N = (A^{-1}b, A^{-2}b, \ldots, A^{-N}b, 0, \ldots) \in l_q^n,\ \det A \neq 0. \tag{12}$$

Let us represent the operators X_∞ and Y_∞ as fixed points of the contraction mapping. To do this, we introduce two linear and bounded operators $\mathbf{MULT}_A, \mathbf{R} \colon l_q^n \to l_q^n$:

$$\mathbf{MULT}_A B' = (Ab_1, Ab_2, \ldots),\ A \in \mathbb{R}^{n\times n},$$
$$\mathbf{R} B' = (0, b_1, b_2, \ldots).$$

For arbitrary $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ we define the mapping $\mathbf{F}_{A,b} \colon l_q^n\ tol_q^n$ as follows:

$$\mathbf{F}_{A,b}(B') = \mathbf{R} \circ \mathbf{MULT}_A B' + (b, 0, 0, \ldots) = (b, Ab_1, Ab_2, \ldots). \tag{13}$$

For an arbitrary $M \in \mathbb{N}$ we denote by $\mathbf{F}^{(M)}_{A,b} \colon l_q^n \to l_q^n$ the M-fold composition of the mapping $\mathbf{F}_{A,b}$:

$$\mathbf{F}^{(M)}_{A,b}(B') = (\underbrace{\mathbf{F}_{A,b} \circ \ldots \circ \mathbf{F}_{A,b}}_{M})(B').$$

Lemma 4. *Let* $Y_\infty \in l_q^n$. *Then* Y_∞ *is a fixed-point of the mapping* $\mathbf{F}_{A,b}$.

Lemma 5. *Let* $\det A \neq 0$, $X_\infty \in l_q^n$. *Then* X_∞ *is a fixed-point of the mapping* $\mathbf{F}_{A^{-1},A^{-1}b}$.

Lemma 6. *Let all eigenvalues of the matrix* $A \in \mathbb{R}^{n\times n}$ *be strictly less than 1 in absolute value. Then for any* $b \in \mathbb{R}^n$

1) *there exists* $M \in \mathbb{N}$ *such that* $A^M \colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ *is a contraction mapping with contraction factor* $\alpha_r \in [0; 1)$;

2) $\mathbf{F}^{(M)}_{A,b}$ *is the the contraction mapping with the contraction factor* α_r.

Corollary 2. *Let all eigenvalues of the matrix* $A \in \mathbb{R}^{n\times n}$ *be strictly less than 1 in absolute value. Then* Y_∞ *is the only fixed-point of the mapping* $\mathbf{F}_{A,b}$. *If* $M \in \mathbb{N}$ *is a number such that* $A^M \colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ *is a contraction mapping with contraction factor* $\alpha_r \in [0; 1)$, *then*

$$\|Y_\infty - Y_{NM}\|_{l_q^n} \leqslant \frac{\alpha_r^N}{1-\alpha_r} \|Y_M\|_{l_q^n}.$$

Corollary 3. *Let all eigenvalues of the matrix $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value. Then X_∞ is the only fixed-point of the mapping $\mathbf{F}_{A^{-1},A^{-1}b}$. If $M \in \mathbb{N}$ is a number such that $A^{-M}\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is a contraction mapping with contraction factor $\beta_r \in [0;1)$, then*

$$\|X_\infty - X_{NM}\|_{l_q^n} \leqslant \frac{\beta_r^N}{1-\beta_r}\|X_M\|_{l_q^n}.$$

Estimates for the operators Y_∞ and X_∞, obtained by the simple iteration method in Corollaries 2 and 3, can also be extended to the sets (3) and (5).

Lemma 7. *Let $B', C' \in l_q^n$. Then*

$$\rho_H(B'\mathcal{E}_p(\infty), C'\mathcal{E}_p(\infty)) \leqslant \|B' - C'\|_{l_q^n}.$$

Theorem 1. *Let all eigenvalues of $A \in \mathbb{R}^{n\times n}$ be strictly less than 1 in absolute value, $M \in \mathbb{N}$ such that the mapping $A^M\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is is contraction with contraction factor $\alpha_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inequality holds:*

$$\rho_H\left(\overline{\mathcal{Y}}_{p,\infty}, \mathcal{Y}_p(NM)\right) \leqslant \frac{\alpha_r^N}{1-\alpha_r}\|Y_M\|_{l_q^n}.$$

Theorem 2. *Let all eigenvalues $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, $M \in \mathbb{N}$ such that the mapping $A^{-M}\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is contraction with contraction factor $\beta_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inequality holds:*

$$\rho_H\left(\overline{\mathcal{X}}_{p,\infty}, \mathcal{X}_p(NM)\right) \leqslant \frac{\beta_r^N}{1-\beta_r}\|X_M\|_{l_q^n}.$$

Corollary 2 and Theorem 1 are based on the fact that the constructed operator $\mathbf{F}^{(M)}_{A,b}$ turns out to be contractive if the matrix A^M has the similar property, and it also derives the contraction factor of this matrix. It is possible to ensure that for some $M \in \mathbb{N}$ the mapping A^M turns out to be a contraction if and only if all eigenvalues of A are strictly less than 1 in modulus. It should be noted that this condition is only a sufficient condition for the boundedness of the reachable limit set $\mathcal{Y}_{p,\infty}$, but not necessary. A necessary and sufficient condition for boundedness is the inclusion $b \in \mathbb{L}_{<1}$ [19]. Even if the matrix A has eigenvalues greater than or equal 1 in modulus, satisfying the condition $b \in \mathbb{L}_{<1}$ will lead to the boundedness of the set $\mathcal{Y}_{p,\infty}$. However, it is impossible to directly use the apparatus of contraction mappings for its construction due to the absence of a conctraction factor of matrix A^M for any $M \in \mathbb{N}$.

Nevertheless, for $b \in \mathbb{L}_{<1}$ it is possible to reduce the phase space of the system (1) to an invariant subspace $\mathbb{L}_{<1}$, on which the mapping A have only its eigenvalues that are strictly less than 1 in absolute value. This allow us to use Corollary 2 and Theorem 1 to construct the set $\mathcal{Y}_{p,\infty}$. Similar reasoning is valid for the set $\mathcal{X}_{p,\infty}$, Corollary 3 and Theorem 2 when replacing A with A^{-1}, b with $A^{-1}b$ and $\mathbb{L}_{<1}$ with $\mathbb{L}_{>1}$.

Let us separately note the case, when expanding b in a real Jordan basis A, the components corresponding to $\mathbb{L}_{=1}$ turn out to be different from 0, i.e. A has eigenvalues modulo 1. Then both sets $\mathcal{Y}_{p,\infty}$ and $\mathcal{X}_{p,\infty}$ turn out to be unbounded, which does not allow them to be represented as fixed points of contraction mappings.

5. METHOD FOR CONSTRUCTING EXTERNAL ESTIMATES OF LIMIT SETS

Let us consider the questions of constructing external estimates for the sets $\mathcal{Y}_{p,\infty}$ and $\mathcal{X}_{p,\infty}$. In [19] methods are proposed for constructing the sets $\mathcal{Y}_p(N)$ and $\mathcal{X}_p(N)$ for any arbitrary $N \in \mathbb{N}$

based on an exact description of their support functions. Moreover, Theorems 1 and 2 give an a priori estimate of the accuracy when considering the sets (2) and (4) as an internal approximation of the limit sets (3) and (5) respectively. In combination with the properties of the Hausdorff distance, this also allows us to construct an external approximation.

To do this, let us denote by $\mathcal{B}^r_R(x_0) \subset \mathbb{R}^n_r$ a ball of radius R with centers at x_0 in the space $\mathbb{R}^n_r$.

Theorem 3. *Let all eigenvalues of $A \in \mathbb{R}^{n\times n}$ be strictly less than 1 in absolute value, $M \in \mathbb{N}$ be such that the mapping $A^M\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is contraction with contraction factor $\alpha_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inclusion is true:*

$$\mathcal{Y}_{p,\infty} \subset \mathcal{Y}_p(NM) + \mathcal{B}^r_{R_N}(0),$$
$$R_N = \frac{\alpha_r^N}{1-\alpha_r}\|Y_M\|_{l_q^n}.$$

Theorem 4. *Let all eigenvalues $A \in \mathbb{R}^{n\times n}$ be strictly greater than 1 in absolute value, $M \in \mathbb{N}$ be such that the mapping $A^{-M}\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$ is contraction with contraction factor $\beta_r \in [0;1)$. Then for any $N \in \mathbb{N}$ the following inclusion is true:*

$$\mathcal{X}_{p,\infty} \subset \mathcal{X}_p(NM) + \mathcal{B}^r_{R_N}(0),$$
$$R_N = \frac{\beta_r^N}{1-\beta_r}\|X_M\|_{l_q^n}.$$

Since the value R_N converges to 0 under the assumptions of Theorems 3 and 4, these statements make it possible to construct, with an arbitrary accuracy, external estimates of the reachable limit sets $\mathcal{Y}_{p,\infty}$ and null-controllable limit sets $\mathcal{X}_{p,\infty}$ of system (1) under the assumption that reachable sets $\{\mathcal{Y}_p(N)\}_{N=0}^{\infty}$ and null-controllable sets $\{\mathcal{X}_p(N)\}_{N=0}^{\infty}$ in a finite number of steps were constructed. To construct them, we can use the results presented in [19, Theorem 1], where for Kallman-controlled systems for the sets (2) and (4) the descriptions of an arbitrary support hyperplane and tangent points are explicitly indicated depending on the support vector.

It is somewhat difficult to calculate the values of M, α_r, β_r, $\|Y_M\|_{l_q^n}$ and $\|X_M\|_{l_q^n}$. In general, α_r is the operator norm of $A^M\colon \mathbb{R}^n_r \to \mathbb{R}^n_r$:

$$\alpha_r = \max_{\|x\|_r \leqslant 1} \|A^M x\|_r. \tag{14}$$

The convex programming problem (14) can be solved numerically for a chosen value of the parameter $r \in [1;\infty]$. Moreover, for the values $r \in \{1, 2, \infty\}$ the analytical representations are known [25]:

$$\alpha_1 = \max_{1\leqslant j\leqslant n}\sum_{i=1}^n |a_{ij}|, \quad \alpha_2 = \sqrt{\sum_{i=1}^n\sum_{j=1}^n a_{ij}^2}, \quad \alpha_\infty = \max_{1\leqslant i\leqslant n}\sum_{j=1}^n |a_{ij}|,$$

where a_{ij} denotes the components of the matrix A^M. The value of β_r is determined in a similar way by replacing the matrix A with A^{-1}.

The value of M can be determined by sequentially calculating α_r or β_r until the condition $\alpha_r \in [0;1)$ is satisfied when constructing $\mathcal{Y}_{p,\infty}$ or the condition $\beta_r \in [0;1)$ is satisfied when constructing $\mathcal{X}_{p,\infty}$. A priori estimates of M are unknown, although for the case, where A has n linearly independent eigenvectors, $M = 1$.

We present the methods for calculating $\|Y_M\|_{l_q^n}$ and $\|X_M\|_{l_q^n}$ in the form of the following theorem.

Theorem 5. *Let for some $M \in \mathbb{N}$ the following representation be true:*

$$B' = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} = \begin{pmatrix} b_{11} & \dots & b_{1M} & 0 & \dots \\ \vdots & & \vdots & \vdots & \\ b_{n1} & \dots & b_{nM} & 0 & \dots \end{pmatrix} \in l_q^n.$$

Then the following estimates are valid:

1) *for all $r \in [1; \infty)$ and $p > 1$, the calculation of $\|B'\|_{l_q^n}$ reduces to solving the following convex programming problem:*

$$\left(\|B'\|_{l_q^n}\right)^r = \max_{\sum\limits_{k=1}^{M} |u_k|^p = 1} \sum_{i=1}^{n} \left| \sum_{k=1}^{M} b_{ik} u_k \right|^r;$$

2) *for all $r \in [1; \infty)$ and $p > 1$*

$$\|B'\|_{l_q^n} \leqslant \left(\sum_{i=1}^{n} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}};$$

3) *if $M = 1$, then*

$$\|B'\|_{l_q^n} = \left(\sum_{i=1}^{n} |b_{i1}|^r \right)^{\frac{1}{r}};$$

4) *if $r = p = 2$, then*

$$\|B'\|_{l_q^n} = \sqrt{\max_{\lambda \in \sigma(B^{\mathrm{T}} B)} |\lambda|},$$

$$B = \begin{pmatrix} b_{11} & \dots & b_{1M} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nM} \end{pmatrix} \in \mathbb{R}^{n \times M},$$

$$\sigma(A) = \{\lambda \in \mathbb{C} \colon \lambda \text{ is eigenvalue of } A\};$$

5) *if $r = \infty$, then*

$$\|B'\|_{l_q^n} = \max_{i=\overline{1,n}} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{1}{q}};$$

if $r = 1$, then

$$\|B'\|_{l_q^n} = \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \left(\sum_{k=1}^{M} \left| \sum_{i=1}^{n} \gamma_i b_{ik} \right|^q \right)^{\frac{1}{q}}.$$

Theorem 5 allows us to use external estimates obtained in Theorems 3 and 4 for any value of the parameter r. However the most suitable values are $r = 1$ and $r = \infty$, since in these cases analytical expressions for α_r, β_r, $\|Y_M\|_{l_q^n}$ and $\|X_M\|_{l_q^n}$ are known, and the ball $\mathcal{B}_R^r(0)$ is a polyhedron:

$$\mathcal{B}_R^1(0) = \operatorname{conv}\left\{\begin{pmatrix}\pm1\\0\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\\pm1\\\vdots\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\vdots\\\pm1\end{pmatrix}\right\},$$

$$\mathcal{B}_R^\infty(0) = \operatorname{conv}\left\{\begin{pmatrix}\gamma_1\\\gamma_2\\\vdots\\\gamma_n\end{pmatrix} : \gamma_i \in \{-1;1\}, i = \overline{1,n}\right\}.$$

6. EXAMPLES

Let us demonstrate the results of Theorems 3–5 by examples.

Example 1. Let the system (1) have the following matrices for $p = \frac{4}{3}$:

$$A = 0.9\begin{pmatrix}\cos(2) & \sin(2)\\ -\sin(2) & \cos(2)\end{pmatrix}, \ b = \begin{pmatrix}2\\2\end{pmatrix}. \tag{15}$$

Consider two values of $r \in \{1, \infty\}$. Let's put $M = 4$. Then the corresponding operator norms A^M, defined by the relations (14), take the fiollowing values:

$$\alpha_1 = \alpha_\infty = 0.4648.$$

Taking into account items 5 and 6 of Theorem 5 the following equalities are true:

$$\|Y_M\|_{l_4^n} = \begin{cases}4.0976, & r = 1,\\ 2.4962, & r = \infty.\end{cases}$$

Let us calculate, according to Theorem 3, the value of R_N for different $N \in \{1, 2, 3, 4\}$. In the case $r = 1$ we obtain the following results:

$$R_1 = 3.5592,\ R_2 = 1.6545,\ R_3 = 0.7691,\ R_4 = 0.3575.$$

In the case $r = \infty$ we obtain the following results:

$$R_1 = 2.1681,\ R_2 = 1.0078,\ R_3 = 0.4685,\ R_4 = 0.2178.$$

Then we can construct estimates for the set $\mathcal{Y}_{\frac{4}{3},\infty}$ according to Theorem 3. The results are presented in Figs. 1 and 2. Sets $\mathcal{Y}_{\frac{4}{3}}(4)$, $\mathcal{Y}_{\frac{4}{3}}(8)$, $\mathcal{Y}_{\frac{4}{3}}(12)$, $\mathcal{Y}_{\frac{4}{3}}(16)$ are constructed by the method presented in [19]. The dotted lines indicate external estimates of the set $\mathcal{Y}_{\frac{4}{3},\infty}$ for different $N \in \{1, 2, 3, 4\}$ and for $r = 1$ in Fig. 1 and for $r = \infty$ in Fig. 2.

Example 2. Let the system (1) have the following matrices for $p = 4$:

$$A = \begin{pmatrix}0.8 & 0.2\\ 0 & 0.7\end{pmatrix}, \quad b = \begin{pmatrix}1\\2\end{pmatrix}. \tag{16}$$

Consider two values of $r \in \{1, \infty\}$. Let's put $M = 4$. Then the corresponding operator norms A^M, defined by the relations (14), take the following values:

$$\alpha_1 = 0.5791,\ \alpha_\infty = 0.7486.$$

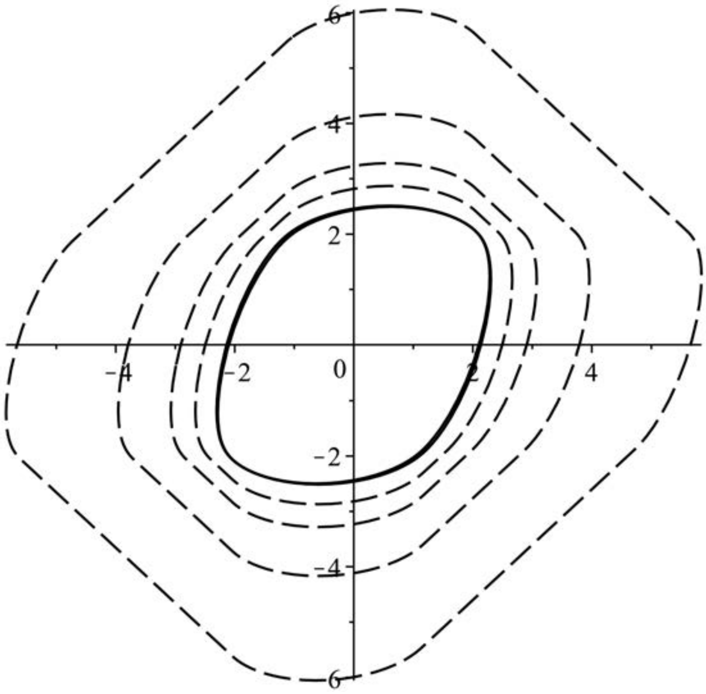

Fig. 1. Reachable sets $\mathcal{Y}_{\frac{4}{3}}(4)$, $\mathcal{Y}_{\frac{4}{3}}(8)$, $\mathcal{Y}_{\frac{4}{3}}(12)$, $\mathcal{Y}_{\frac{4}{3}}(16)$ (solid lines) and estimates $\mathcal{Y}_{\frac{4}{3},\infty}$ (dotted lines), obtained based on Theorem 3 for $r = 1$ and system matrices (15).

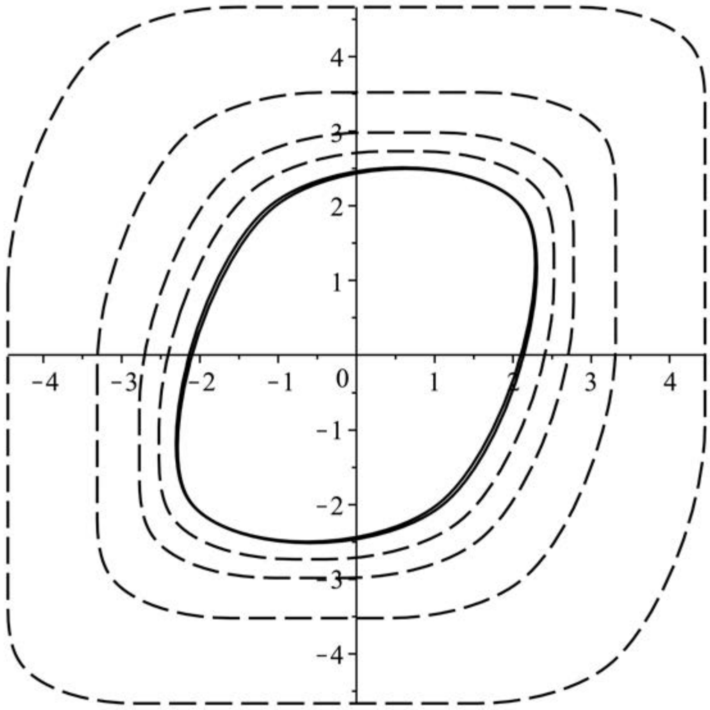

Fig. 2. Reachable sets $\mathcal{Y}_{\frac{4}{3}}(4)$, $\mathcal{Y}_{\frac{4}{3}}(8)$, $\mathcal{Y}_{\frac{4}{3}}(12)$, $\mathcal{Y}_{\frac{4}{3}}(16)$ (solid lines) and estimates $\mathcal{Y}_{\frac{4}{3},\infty}$ (dotted lines), obtained based on Theorem 3 for $r = \infty$ and system matrices (15).

Taking into account items 5 and 6 of Theorem 5 the following equalities are true:

$$\|Y_M\|_{l_4^n} = \begin{cases} 6.8891, & r = 1, \\ 3.6717, & r = \infty. \end{cases}$$

Let us calculate, according to Theorem 3, the value of R_N for different $N \in \{1, 2, 3, 4\}$. In the case $r = 1$ we obtain the following results:

$$R_1 = 9.4784,\ R_2 = 5.4890,\ R_3 = 3.1887,\ R_4 = 1.8408.$$

In the case $r = \infty$ we obtain the following results:

$$R_1 = 10.9333,\ R_2 = 8.1847,\ R_3 = 6.1271,\ R_4 = 4.5867.$$

Then we can construct estimates for the set $\mathcal{Y}_{4,\infty}$ according to Theorem 3. The results are presented in Figs. 3 and 4. The sets $\mathcal{Y}_4(4)$, $\mathcal{Y}_4(8)$, $\mathcal{Y}_4(12)$, $\mathcal{Y}_4(16)$ are constructed by the method presented in [19]. The dotted lines indicate external estimates of the set $\mathcal{Y}_{4,\infty}$ for different $N \in \{1, 2, 3, 4\}$ for $r = 1$ in Fig. 3 and for $r = \infty$ in Fig. 4.

In Example 1 the approximation accuracy turned out to be higher for $r = \infty$, while in Example 2 better results are obtained for $r = 1$. In the general case, it is possible to calculate estimates for various values of the parameter $r \in [1; \infty]$, and consider the final estimate in the form of their intersection.

Example 3. Let us consider separately the case $p = 2$, for which reachable limit sets can be constructed explicitly based on Lemma 3. Note that from the point of view of Lemma 3 it does not matter whether the eigenvalues of the matrix A are real or complex. In intermediate calculations when constructing the matrix $H_{\mathcal{Y},\infty}$, complex numbers may be used, but the resulting matrix of quadratic form, which defines the structure of $\mathcal{Y}_{2,\infty}$, will be real in any case.

For the case (15) it is true that

$$\lambda_1 = -0.3329 + 0.7274\mathbf{i},\ \lambda_2 = -0.3329 - 0.7274\mathbf{i},\ S = \begin{pmatrix} 0.7071 & 0.7071 \\ 0.7071\mathbf{i} & -0.7071\mathbf{i} \end{pmatrix},$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} -4\mathbf{i} & 4 \\ 4 & 4\mathbf{i} \end{pmatrix},$$

$$\begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} -0.8625 - 2.5257\mathbf{i} & 11.1111 \\ 11.1111 & -0.8625 + 2.5257\mathbf{i} \end{pmatrix}.$$

From here we finally get that

$$H_{\mathcal{Y},\infty} = \begin{pmatrix} 0.1029 & -0.0217 \\ -0.0217 & 0.0881 \end{pmatrix}.$$

The results are presented in Fig. 5. The sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ were constructed by the method from [19].

Similarly, for the case (16) it is true that

$$\lambda_1 = -0.8,\ \lambda_2 = -0.7,\ S = \begin{pmatrix} 1 & -0.8944 \\ 0 & 0.4472 \end{pmatrix},$$

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 25 & 22.3607 \\ 22.3607 & 20 \end{pmatrix},\quad \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} 69.4444 & 50.8197 \\ 50.8197 & 39.5197 \end{pmatrix}.$$

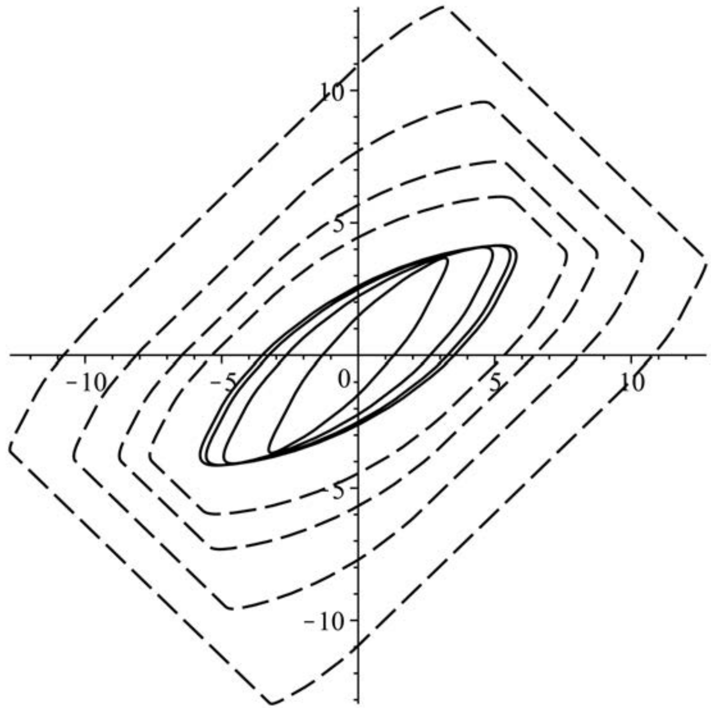

Fig. 3. Reachable sets $\mathcal{Y}_4(4)$, $\mathcal{Y}_4(8)$, $\mathcal{Y}_4(12)$, $\mathcal{Y}_4(16)$ (solid lines) and estimates $\mathcal{Y}_{4,\infty}$ (dotted lines), obtained based on Theorem 3 for $r = 1$ and system matrices (16).

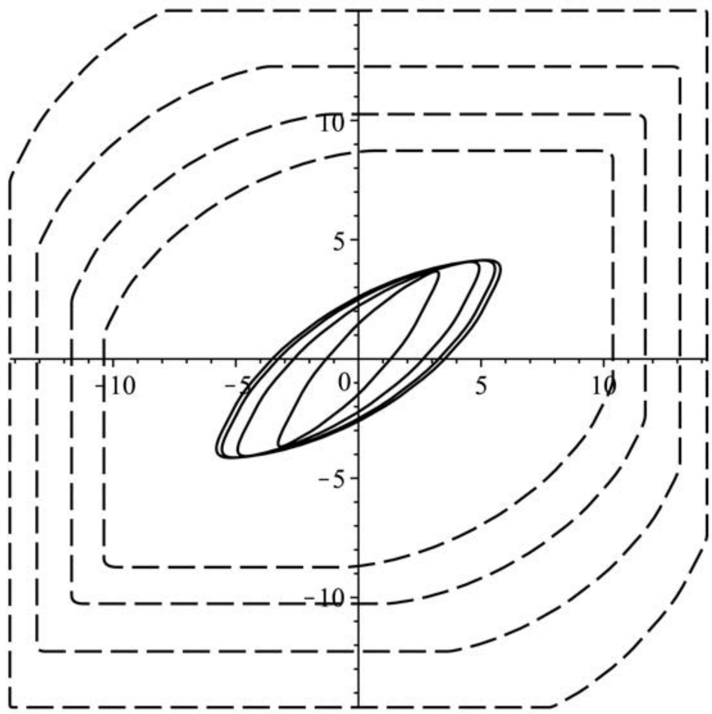

Fig. 4. Reachable sets $\mathcal{Y}_4(4)$, $\mathcal{Y}_4(8)$, $\mathcal{Y}_4(12)$, $\mathcal{Y}_4(16)$ (solid lines) and estimates $\mathcal{Y}_{4,\infty}$ (dotted lines), obtained based on Theorem 3 for $r = \infty$ and system matrices (16).

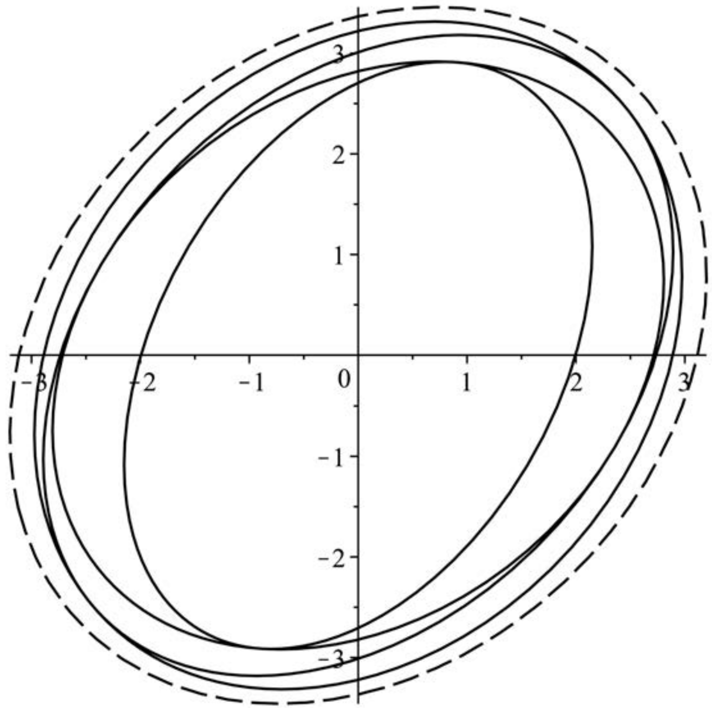

Fig. 5. Reachable sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ (solid lines) and $\mathcal{Y}_{2,\infty}$ (dotted line) obtained by Lemma 3 for the matrices of the system (15).

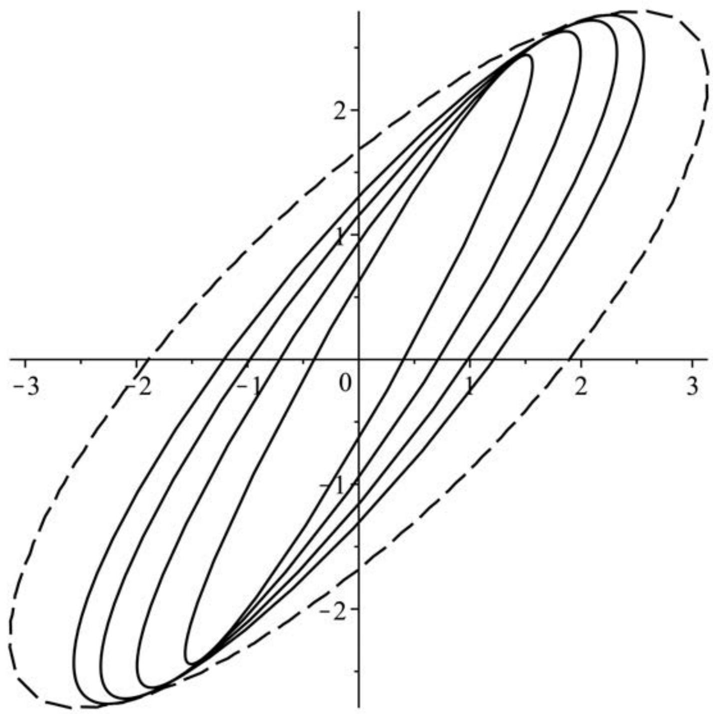

Fig. 6. Reachable sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ (solid lines) and $\mathcal{Y}_{2,\infty}$ (dotted line) obtained by Lemma 3 for the matrices of the system (16).

From here we finally get that

$$H_{\mathcal{Y},\infty} = \begin{pmatrix} 0.2788 & -0.2503 \\ -0.2503 & 0.3522 \end{pmatrix}.$$

The results are presented in Fig. 6. The sets $\mathcal{Y}_2(2)$, $\mathcal{Y}_2(3)$, $\mathcal{Y}_2(4)$, $\mathcal{Y}_2(5)$ are constructed by the method from [19].

Example 4. Taking into account the relations (12) and (11) the results of Example 1 and Figs. 1 and 2 correspond to similar constructions for the null-controllable sets $\{\mathcal{X}_{\frac{4}{3}}(N)\}_{N=0}^{\infty}$ and $\mathcal{X}_{\frac{4}{3},\infty}$ when replacing A with A^{-1} and b with $A^{-1}b$:

$$A = \frac{10}{9}\begin{pmatrix} \cos(2) & -\sin(2) \\ \sin(2) & \cos(2) \end{pmatrix}, \; b = \begin{pmatrix} \dfrac{10}{9}(\cos(2) - \sin(2)) \\ \dfrac{10}{9}(\cos(2) + \sin(2)) \end{pmatrix}.$$

Also the results of the Example 2 and Figs. 3 and 4 correspond to similar constructions for the null-controllable sets $\{\mathcal{X}_4(N)\}_{N=0}^{\infty}$ and $\mathcal{X}_{4,\infty}$ when replacing A with A^{-1} and b with $A^{-1}b$:

$$A = \begin{pmatrix} \dfrac{5}{4} & -\dfrac{5}{14} \\ 0 & \dfrac{10}{7} \end{pmatrix}, \quad b = \begin{pmatrix} \dfrac{15}{28} \\ \dfrac{20}{7} \end{pmatrix}.$$

7. CONCLUSION

The paper considers the problem for constructing reachable and 0-controllable limit sets for stationary linear discrete-time systems with a summary constraint on scalar control. It is demonstrated that in the general case, the calculation of the Minkowski functional for given sets reduces to the operation of projecting a ball from the normed space l_p onto a finite-dimensional phase space and to solving systems of nonlinear equations. For the case of quadratic constraints and a diagonalizable matrix of the system, these equations can be solved analytically, which makes it possible to describe the Minkowski functional in explicit form. In the general case, the reachable and null-controllable limit sets can be represented as fixed points of a contraction mapping in the metric space of compacta, the contraction coefficient of which can be calculated numerically. This fact allows us to estimate the error of the simple iteration method when constructing these sets, which, in combination with the properties of the Hausdorff distance, leads to the possibility of constructing their external estimates of an arbitrary order of accuracy. In a particular case, when choosing the Minkowski or Chebyshev norm as a norm in the phase space, the resulting estimates have a polyhedral structure.

It is planned to generalize the obtained results to systems with vector control in the future. In particular, this requires developing a model for taking into account summary restrictions, for example, by using the Minkowski functional with respect to some cost set. Another problem is the construction of a recurrent description for the reachable and null-controllable sets, which would allow us to search for their limit analogues in the form of fixed points, similar to the results obtained for the case of separate control restrictions at each moment of time [15].

FUNDING

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APPENDIX

Proof of Lemma 1. By the definition of the Minkowski functional, for any $x \in \mathbb{L}_2$ the following relations are true:

$$\mu(x, B\mathcal{U}) = \inf\{t > 0\colon x \in tB\mathcal{U}\} = \inf\{t > 0\colon \exists u \in t\mathcal{U},\ x = Bu\}$$
$$= \inf\{t > 0\colon \exists u \in B^{-1}(\{x\}),\ u \in t\mathcal{U}\} = \inf_{u \in B^{-1}(\{x\})} \inf\{t > 0\colon u \in t\mathcal{U}\} = \inf_{u \in B^{-1}(\{x\})} \mu(u, \mathcal{U}).$$

Lemma 1 is proven.

Proof of Lemma 2. Since, due to the Riesz theorem, the operator B is linear and bounded, then according to Lemma 1

$$(\mu(x, B\mathcal{E}_p(\infty)))^p = \left(\inf_{u \in B^{-1}(\{x\})} \mu(u, \mathcal{E}_p(\infty))\right)^p = \inf_{\substack{Bu=x \\ u \in l_p}} \sum_{k=1}^{\infty} |u_k|^p. \tag{A.1}$$

To solve the optimization problem (A.1) we will use the Lagrange multiplier method for infinite-dimensional spaces [29]. The Lagrange function for $\lambda \in \mathbb{R}^n$ has the following form

$$L(u, \lambda) = \sum_{k=1}^{\infty} |u_k|^p + \lambda^{\mathrm{T}}(x - Bu).$$

Then the search for a minimum in the problem (A.1) is reduced to solving the system of equations

$$\begin{cases} \dfrac{\partial L}{\partial u_k} = 0,\ k \in \mathbb{N}, \\ Bu = x, \end{cases} \qquad \begin{cases} pI_p(u) - B^*\lambda = 0, \\ Bu = x, \end{cases} \qquad \begin{cases} u = I_p^{-1}\left(\dfrac{1}{p}B^*\lambda\right) = I_q\left(\dfrac{1}{p}B^*\lambda\right), \\ Bu = x. \end{cases}$$

Hence, taking into account (A.1) and the identity $\sum\limits_{k=1}^{\infty} |u_k|^p = (u, I_p(u))$ it follows that

$$(\mu(x, B\mathcal{E}_p(\infty))^p = \left(I_q\left(\frac{1}{p}B^*\lambda\right), I_p\left(I_q\left(\frac{1}{p}B^*\lambda\right)\right)\right) = \left(I_q\left(\frac{1}{p}B^*\lambda\right), \frac{1}{p}B^*\lambda\right) = \left\|\frac{1}{p}B^*\lambda\right\|_{l_q}^q,$$
$$BI_q\left(\frac{1}{p}B^*\lambda\right) = x.$$

Redesignating $\frac{1}{p}\lambda$ by λ, we finally obtain

$$\mu(x, B\mathcal{E}_p(\infty)) = \|B^*\lambda\|_{l_q}^{\frac{q}{p}} = \|B^*\lambda\|_{l_q}^{q-1},$$

where $\lambda \in \mathbb{R}^n$ is determined from (10).

Lemma 2 is proven.

Proof of Corollary 1. By definition, the operator I_2 is identical. Then the condition (10) take the form

$$BB^*\lambda = x.$$

Since B is surjective, the operator $BB^*\colon \mathbb{R}^n \to \mathbb{R}^n$ and the matrix generating it are invertible, which leads to the relation

$$\lambda = (BB^*)^{-1}x.$$

Taking into account Lemma 2 we obtain

$$\mu(x, B\mathcal{E}_2(\infty)) = \|B^*(BB^*)^{-1}x\|_2 = \sqrt{(B^*(BB^*)^{-1}x, B^*(BB^*)^{-1}x)} = \sqrt{x^{\mathrm{T}}(BB^*)^{-1}x}.$$

Corollary 1 is proven.

Proof of Lemma 3. Let $B = Y_\infty$. Then, taking into account the spectral decomposition of the matrix $A = S\Lambda S^{-1}$, the following equalities are true for all $k \in \mathbb{N}$:

$$\begin{aligned} b_k b_k^{\mathrm{T}} &= A^{k-1}bb^{\mathrm{T}}(A^{k-1})^{\mathrm{T}} = S\Lambda^{k-1}S^{-1}bb^{\mathrm{T}}(S^{-1})^{\mathrm{T}}\Lambda^{k-1}S^{\mathrm{T}} \\ &= S\mathrm{diag}(\lambda_1^{k-1}, \ldots, \lambda_n^{k-1}) \begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \ldots & \alpha_{nn} \end{pmatrix} \mathrm{diag}(\lambda_1^{k-1}, \ldots, \lambda_n^{k-1})S^{\mathrm{T}} \\ &= S \begin{pmatrix} (\lambda_1\lambda_1)^{k-1}\alpha_{11} & \ldots & (\lambda_1\lambda_n)^{k-1}\alpha_{1n} \\ \vdots & \ddots & \vdots \\ (\lambda_n\lambda_1)^{k-1}\alpha_{n1} & \ldots & (\lambda_n\lambda_n)^{k-1}\alpha_{nn} \end{pmatrix} S^{\mathrm{T}}. \end{aligned}$$

Due to the inclusion $b \in \mathbb{L}_{<1}$, the coefficients α_{ij} are equal to zero for all $i, j = \overline{1, n}$ such that at least one of the two eigenvalues λ_i or λ_j turns out to be greater than or equal to 1:

$$\alpha_{ij} = 0, \text{ if } |\lambda_i| \geqslant 1 \text{ or } |\lambda_j| \geqslant 1. \tag{A.2}$$

Hence, taking into account the expression for the sum of an infinite decreasing geometric progression, we get the following equality:

$$\sum_{k=1}^{\infty} (\lambda_i\lambda_j)^{k-1}\alpha_{ij} = \begin{cases} \dfrac{\alpha_{ij}}{1 - \lambda_i\lambda_j}, & |\lambda_i| < 1 \text{ and } |\lambda_j| < 1, \\ 0, & |\lambda_i| \geqslant 1 \text{ or } |\lambda_j| \geqslant 1, \end{cases}$$

which, due to the (A.2), coincides with the definition of β_{ij}.

Then the following chain of equalities is true:

$$\begin{aligned} BB^* = \sum_{k=1}^{\infty} b_k b_k^{\mathrm{T}} &= S \begin{pmatrix} \sum\limits_{k=1}^{\infty}(\lambda_1\lambda_1)^{k-1}\alpha_{11} & \ldots & \sum\limits_{k=1}^{\infty}(\lambda_1\lambda_n)^{k-1}\alpha_{1n} \\ \vdots & \ddots & \vdots \\ \sum\limits_{k=1}^{\infty}(\lambda_n\lambda_1)^{k-1}\alpha_{n1} & \ldots & \sum\limits_{k=1}^{\infty}(\lambda_n\lambda_n)^{k-1}\alpha_{nn} \end{pmatrix} S^{\mathrm{T}} = \\ &= S \begin{pmatrix} \beta_{11} & \ldots & \beta_{1n} \\ \vdots & \ddots & \vdots \\ \beta_{n1} & \ldots & \beta_{nn} \end{pmatrix} S^{\mathrm{T}} = H. \end{aligned}$$

This, taking into account Corollary 1, implies the equality $H_{\mathcal{Y},\infty} = H^{-1}$.

The second item of Lemma 3 is proven in a similar way under the redesignation $B = X_\infty$.

Lemma 3 is proven.

Proof of Lemma 4. The proof follows directly from (6) and (13).

Proof of Lemma 5. The proof follows directly from (7) and (13).

Proof of Lemma 6. Since all eigenvalues of the matrix A are strictly less than 1 in absolute value, according to [22, Theorem 5.6.12] the relation $\|A^k\| \xrightarrow{k\to\infty} 0$ is true. Then, by definition of the limit there is $M\in\mathbb{N}$ for $\alpha_r\in[0;1)$ such that $\|A^M\| = \sup_{\|x\|_r\leqslant 1}\|A^M x\|_r < \alpha_r$. Since the inequality

$$\|A^M(x-y)\|_r \leqslant \|A^M\|\|x-y\|_r \leqslant \alpha_r\|x-y\|_r$$

is true, A^M is a contraction with the contraction factor $\alpha_r\in[0;1)$.

Let $B' = (b_1, b_2, \ldots)\in l_q^n$, $C' = (c_1, c_2, \ldots)\in l_q^n$. Then

$$\begin{aligned}
\left\|\mathbf{F}_{A,b}^{(M)}(B') - \mathbf{F}_{A,b}^{(M)}(C')\right\|_{l_q^n} &= \Big\|(b, Ab, \ldots A^{M-1}b, A^M b_1, A^M b_2, \ldots) \\
&\quad - (b, Ab, \ldots A^{M-1}b, A^M c_1, A^M c_2, \ldots)\Big\|_{l_q^n} \\
&= \left\|(0, \ldots, 0, A^M(b_1-c_1), A^M(b_2-c_2), \ldots)\right\|_{l_q^n} \\
&= \left\|(A^M(b_1-c_1), A^M(b_2-c_2), \ldots)\right\|_{l_q^n} \\
&= \sup_{u\in\mathcal{E}_p(\infty)} \|(A^M(b_1-c_1), A^M(b_2-c_2), \ldots)u\|_r
\end{aligned}$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\left\|\sum_{k=1}^{\infty} A^M(b_k-c_k)u_k\right\|_r = \sup_{u\in\mathcal{E}_p(\infty)}\left\|A^M\sum_{k=1}^{\infty} b_k u_k - A^M\sum_{k=1}^{\infty} c_k u_k\right\|_r$$

$$\leqslant \sup_{u\in\mathcal{E}_p(\infty)} \alpha_r\left\|\sum_{k=1}^{\infty} b_k u_k - \sum_{k=1}^{\infty} c_k u_k\right\|_r = \alpha_r\|B'-C'\|_{l_q^n}.$$

Lemma 6 is proven.

Proof of Corollary 2. The mapping $\mathbf{F}_{A,b}^{(M)}$ is a contraction due to Theorem 6, and the space l_q^n is complete. Then, by virtue of the Banach contraction mapping principle [25], there is a unique fixed-point of $\mathbf{F}_{A,b}^{(M)}$. Moreover, by construction, a fixed-point of the mapping $\mathbf{F}_{A,b}$ must also be a fixed-point of the mapping $\mathbf{F}_{A,b}^{(M)}$. Taking into account Lemma 4 the unique fixed-point of $\mathbf{F}_{A,b}^{(M)}$ is $Y_\infty\in l_q^n$. Hence, Y_∞ is the unique fixed-point of $\mathbf{F}_{A,b}$.

Note that the representation $\mathbf{F}_{A,b}^{(NM)}(O) = Y_{NM}$ is valid, where by $O\colon l_p\to\mathbb{R}_r^n$ denotes the zero operator, which is identified with the zero sequence $(0, 0, \ldots)\in l_q^n$. Then according to Theorem 6

$$\begin{aligned}
\|Y_\infty - Y_{NM}\|_{l_q^n} &= \left\|\mathbf{F}_{A,b}^{(M)}(Y_\infty) - \mathbf{F}_{A,b}^{(M)}\left(\mathbf{F}_{A,b}^{(NM-M)}(O)\right)\right\|_{l_q^n} \\
&\leqslant \alpha_r\left\|Y_\infty - \mathbf{F}_{A,b}^{(NM-M)}(O)\right\|_{l_q^n} \\
&\leqslant \alpha_r\left\|Y_\infty - \mathbf{F}_{A,b}^{(NM)}(O)\right\|_{l_q^n} + \alpha_r\left\|\mathbf{F}_{A,b}^{(NM)}(O) - \mathbf{F}_{A,b}^{(NM-M)}(O)\right\|_{l_q^n} \\
&\leqslant \alpha_r\left\|Y_\infty - \mathbf{F}_{A,b}^{(NM)}(O)\right\|_{l_q^n} + \alpha_r^N\left\|\mathbf{F}_{A,b}^{(M)}(O) - O\right\|_{l_q^n} \\
&= \alpha_r\|Y_\infty - Y_{NM}\|_{l_q^n} + \alpha_r^N\|Y_M\|_{l_q^n},
\end{aligned}$$

$$\|Y_\infty - Y_{NM}\|_{l_q^n} \leqslant \frac{\alpha_r^N}{1-\alpha_r}\|Y_M\|_{l_q^n}.$$

Corollary 2 is proven.

Proof of Corollary 3. The proof follows from Corollary 2 by replacing A with A^{-1}, b with $A^{-1}b$ in conjunction with Lemma 5 and the fact that the eigenvalues of A^{-1} are inverse of the eigenvalues of A [28].

Proof of Lemma 7. Let's consider the value

$$\rho_H(B'\mathcal{E}_p(\infty), C'\mathcal{E}_p(\infty))$$

$$= \max\left\{\sup_{x\in B'\mathcal{E}_p(\infty)}\inf_{y\in C'\mathcal{E}_p(\infty)}\|x-y\|_r;\ \sup_{x\in C'\mathcal{E}_p(\infty)}\inf_{y\in B'\mathcal{E}_p(\infty)}\|x-y\|_r\right\}$$

$$= \max\left\{\sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|B'u-C'v\|_r;\ \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|C'u-B'v\|_r\right\},$$

$$\sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|C'u-B'v\|_r = \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|B'u-C'v\|_r$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}\|B'u-C'u+C'u-C'v\|_r$$

$$\leqslant \sup_{u\in\mathcal{E}_p(\infty)}\inf_{v\in\mathcal{E}_p(\infty)}(\|B'u-C'u\|_r+\|C'u-C'v\|_r)$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\left(\|(B'-C')u\|_r+\inf_{v\in\mathcal{E}_p(\infty)}\|C'(u-v)\|_r\right)$$

$$= \sup_{u\in\mathcal{E}_p(\infty)}\|(B'-C')u\|_r = \|B'-C'\|_{l_q^n}.$$

Finally we get that

$$\rho_H(B'\mathcal{E}_p(\infty), C'\mathcal{E}_p(\infty)) \leqslant \|B'-C'\|_{l_q^n}.$$

Lemma 7 is proven.

Proof of Theorem 1. The proof follows directly from Corollary 2, the representations (11) and (8), and Lemma 7.

Proof of Theorem 2. The proof follows directly from Corollary 3, the representations (12) and (9), and Lemma 7.

Proof of Theorem 3. As is known [27], for any $\mathcal{X},\mathcal{Y}\in\mathbb{K}_n$ satisfying the condition $\rho_H(\mathcal{X},\mathcal{Y})\leqslant R$, the following inclusion is true:

$$\mathcal{X}\subset\mathcal{Y}+\mathcal{B}_R^r(0).$$

From here, taking into account Theorem 1, Theorem 3 follows.

Proof of Theorem 4. The proof is similar to the proof of Theorem 3, replacing Theorem 1 with Theorem 2.

Proof of Theorem 5. 1. Item 1 follows from the definition of the operator norm and the fact that the maximum of a convex function is achieved on the boundary of the convex set [22]:

$$\|B'\|_{l_q^n} = \sup_{u\in\mathcal{E}_p(\infty)}\|B'u\|_r = \sup_{\sum\limits_{k=1}^{\infty}|u_k|^p=1}\left(\sum_{i=1}^{n}\left|\sum_{k=1}^{\infty}b_{ik}u_k\right|^r\right)^{\frac{1}{r}}$$

$$= \left(\sup_{\sum\limits_{k=1}^{\infty}|u_k|^p=1}\sum_{i=1}^{n}\left|\sum_{k=1}^{M}b_{ik}u_k\right|^r\right)^{\frac{1}{r}} = \left(\max_{\sum\limits_{k=1}^{M}|u_k|^p=1}\sum_{i=1}^{n}\left|\sum_{k=1}^{M}b_{ik}u_k\right|^r\right)^{\frac{1}{r}}.$$

2. By virtue of Hölder's inequality, item 2 follows from item 1:

$$\|B'\|_{l_q^n} \leqslant \left(\max_{\sum\limits_{k=1}^{M} |u_k|^p = 1} \sum_{i=1}^{n} \left[\left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{1}{q}} \left(\sum_{k=1}^{M} |u_k|^p \right)^{\frac{1}{p}} \right]^r \right)^{\frac{1}{r}} = \left(\sum_{i=1}^{n} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{r}{q}} \right)^{\frac{1}{r}}.$$

3. Also, for $M = 1$, item 3 follows from item 1:

$$\|B'\|_{l_q^n} = \left(\max_{|u_1|^p = 1} \sum_{i=1}^{n} |b_{i1} u_1|^r \right)^{\frac{1}{r}} = \left(\sum_{i=1}^{n} |b_{i1}|^r \right)^{\frac{1}{r}}.$$

4. For $r = p = 2$ the operator norm can be represented in terms of the scalar product in $\mathbb{R}_2^n$:

$$(x, y) = \sum_{i=1}^{n} x_i y_i.$$

Then, taking into account item 1, the following representation is true:

$$\Big(\|B'\|_{l_q^n} \Big)^2 = \max_{\substack{u \in \mathbb{R}^M: \\ (u,u)=1}} (Bu, Bu) = \max_{\substack{u \in \mathbb{R}^M: \\ (u,u)=1}} (u, B^{\mathrm{T}} B u).$$

Due to the Lagrange multiplier method [29], the maximum point of the optimization problem under consideration $u^* \in \mathbb{R}^M$ satisfies the following conditions:

$$\begin{cases} \nabla \Big((u, B^{\mathrm{T}} B u) + \lambda (1 - (u,u)) \Big) = 0, \\ (u,u) = 1, \end{cases} \quad \begin{cases} 2B^{\mathrm{T}} B - 2\lambda u = 0, \\ (u,u) = 1, \end{cases} \quad \begin{cases} (B^{\mathrm{T}} B - \lambda I) u = 0, \\ (u,u) = 1. \end{cases}$$

Then, by definition, u^* is a normed eigenvector of the matrix $B^{\mathrm{T}} B$ corresponding to the eigenvalue λ^*, i.e.

$$\Big(\|B'\|_{l_q^n} \Big)^2 = (u^*, B^{\mathrm{T}} B u^*) = (u^*, \lambda^* u^*) = \lambda^*.$$

Item 4 is completely proven.

5. Item 5 follows from the representation of the operator norm B' and the Riesz theorem on the norm of a linear and bounded functional in l_p [25]:

$$\|B'\|_{l_q^n} = \sup_{u \in \mathcal{E}_p(\infty)} \max_{i=\overline{1,n}} \left| \sum_{k=1}^{\infty} b_{ik} u_k \right| = \max_{i=\overline{1,n}} \left(\sum_{k=1}^{\infty} |b_{ik}|^q \right)^{\frac{1}{q}} = \max_{i=\overline{1,n}} \left(\sum_{k=1}^{M} |b_{ik}|^q \right)^{\frac{1}{q}}.$$

6. To prove item 6, we take into account the representation $|\gamma| = \max\{\gamma, -\gamma\}$ for any $\gamma \in \mathbb{R}$ and consider the following chain of equalities:

$$\|B'\|_{l_q^n} = \sup_{u \in \mathcal{E}_p(\infty)} \sum_{i=1}^{n} \left| \sum_{k=1}^{\infty} b_{ik} u_k \right| = \sup_{u \in \mathcal{E}_p(\infty)} \sum_{i=1}^{n} \max_{\gamma_i \in \{-1;1\}} \left(\gamma_i \sum_{k=1}^{\infty} b_{ik} u_k \right)$$

$$= \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \sup_{u \in \mathcal{E}_p(\infty)} \left| \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} \gamma_i b_{ik} \right) u_k \right| = \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \left(\sum_{k=1}^{\infty} \left| \sum_{i=1}^{n} \gamma_i b_{ik} \right|^q \right)^{\frac{1}{q}}$$

$$= \max_{\substack{\gamma_i \in \{-1;1\} \\ i=\overline{1,n}}} \left(\sum_{k=1}^{M} \left| \sum_{i=1}^{n} \gamma_i b_{ik} \right|^q \right)^{\frac{1}{q}}.$$

Theorem 5 is completely proven.

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LINEAR SYSTEMS

Composite Observer of a Linear Time-Varying Singularly Perturbed System with Quasidifferentiable Coefficients

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Abstract—For a linear time-varying singularly perturbed system with a small parameter μ for a part of derivatives and quasi-differentiable coefficients, existence conditions are established and μ-asymptotic composite full- and reduced-order observers are constructed. The error in estimating a state with an arbitrary predetermined exponential decay rate converges to an infinitesimal value of the same order of smallness as the small parameter. The observer gain vector are expressed in terms of the gain vectors of subsystems of smaller dimension than the original one and independent of the small parameter, and the parameters of the original system are subject to weaker requirements than those previously known. A constructive algorithm for calculating the gain vector of a composite observer is presented.

Keywords: : robust observer, time-varying singularly perturbed system, quasi-differentiability

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1. INTRODUCTION

The problem of estimating the states of dynamic systems using available information is relevant due to its importance for various positioning systems (location determination) of control objects. However, in real situations, direct measurement of the state vector may be difficult (for cost reasons, due to technological limitations, etc.). In this case, states can be estimated using a specially constructed dynamic system called an observer (estimator). The observer's input is the output function of the original system, and its state must, in one sense or another, approximate the state of the original system [1–6]. If, with increasing time, the state of the observer converges to the phase vector of the system, then the observer is called *asymptotic*. If, in addition, there is exponential convergence, then such an observer is called *exponential*.

The definition of an observer was first introduced by Luenberger in his dissertation in 1963 (see also [7, 8]). He showed that for every linear system being observed, an observer that is itself a linear system can be designed with the estimation error tending to zero at a given rate. In this case, the task of constructing an observer comes down to choosing the gain, which is calculated from the system parameters and does not depend on the output [8].

Singularly perturbed systems (SPS) with a small parameter μ for a part of derivatives are widely used in applications in aviation, chemistry, electrical engineering, mechanics, etc. as models of multi-tempo processes when simulating the dynamics of aircraft, chemical reactions, movements of robotic manipulators, etc. (see reviews [9–13] and references there). For SPS, depending on the information about the small parameter and the needs of the applications, different formulations of observation problems can be considered: with a known value of the small parameter μ [14], for a known closed interval of values of the small parameter $\mu \in [\underline{\mu}, \overline{\mu}] \subset (0, \mu^0]$ [15], for an unknown value of the small parameter μ [16]. Formulations are also distinguished depending on the composition of the components being evaluated: estimation of both slow and fast components, or only slow components (see [10] and references there). In applications, the exact value of the small parameter μ can be unknown. Therefore, it is important that observers provide a "good" state estimate for the entire μ-parametric family of SPSs for various values of the parameter μ. Observers provide estimates of the system states in conditions where the system model is not precisely known, are called *robust* observers.

Depending on the dimension of the observer *full-order observers*, the state of which has the same dimension as the state of the observed system, as well as *reduced-order observers*, the dimension of whose states is smaller (by the dimension of the output), are distinguished [1, p. 379]. Identification in the SPS of fast (parasitic) dynamics, when describing which a small parameter appears in the model, allows us to further reduce observers.

The use of the multi-tempo structure of the SPS allows, when constructing observers, to operate with systems of a smaller dimension than the original one (see [14, 16, 17])—subsystems of slow and fast movements separated by time scales. In this case, the gain for the observer of the original SPS can be calculated as a composition of the gains of separately designed observers of the slow and fast subsystems. This approach for constructing a composite [14] observer is used in [17] for linear time-invariant SPS (LTISPS). In [16] for the LTISPS the concept was introduced and the construction of μ-asymptotic observers was justified, for which the state estimation error with an arbitrary predetermined exponential decay rate converges to an infinitesimal value of the same order of smallness as the small parameter. In [14] the construction of a μ-asymptotic full-order observer for a linear time-varying SPS (LTVSPS) is described, but the rules for constructing such observers are not given there. Research on the design of observers of slow states of nonlinear SPS can be found in [18–21] and the works cited there.

Many real dynamic systems are described by models whose parameters depend on time. Constructive methods for the analysis and synthesis of time-varying systems can be obtained for systems that can be reduced to time-invariant [22].Time-varying systems can arise during the linearization of time-invariant nonlinear systems. Linearization of time-invariant systems can lead to a decrease in the smoothness of the system parameters. The use of the quasi-differentiation apparatus [23, 24] allows us to expand the class of time-varying systems for which it is possible to obtain constructive results.

When we consider deterministic observation systems, state estimation assumes that the system has a certain type of observability. For a time-invariant system, complete observability guarantees the existence of an asymptotic observer [2]. For a time-varying system, its uniformly complete observability is required. However, this property is difficult to verify in terms of the coefficients of the original observation system, and therefore, from a constructive point of view, it is not very effective. The approach proposed in [5] based on the quasi-differentiation technique makes it possible to constructively construct observers for uniformly observed time-varying systems, while weakening the known requirements for smoothness of coefficients.

Issues of SPS observability were previously studied by the author in [25–30]. The contribution of this work is that, in contrast to [25–30], the problem of constructing observers is studied; in contrast to [5], a singularly perturbed system is considered; in comparison with [14], a constructive algorithm for constructing a composite exponential LTVSPS observer has been developed, while the requirements for the smoothness of the system parameters have been relaxed; in contrast to [16, 17], time-varying SPS is considered.

2. FORMULATION OF THE PROBLEM

We consider a linear time-varying singularly perturbed system (LTVSPS):

$$\begin{aligned} &\dot{x}(t) = A_1(t)x(t) + A_2(t)y(t), \quad x \in \mathbb{R}^{n_1}, \quad y \in \mathbb{R}^{n_2}, \\ &\mu\dot{y}(t) = A_3(t)x(t) + A_4(t)y(t), \quad t \in T = [t_0, +\infty), \end{aligned} \tag{1}$$

with scalar output

$$v(t) = c_1(t)x(t) + c_2(t)y(t), \quad v \in \mathbb{R}, \quad t \in T = [t_0, +\infty). \tag{2}$$

Here μ is a small parameter, $\mu \in (0, \mu^0]$, $\mu^0 \ll 1$, $x(t)$ is an unknown vector of slow variables, $y(t)$ is an unknown vector of fast variables, $A_i(t)$, $i = \overline{1,4}$, $c_j(t)$, $j = 1,2$ are continuous matrix functions of compartible orders and row vector functions bounded on T, respectively, $v(t)$ is a measured output function.

Let in the LTVSPS (1) some fixed value of the parameter $\mu \in (0, \mu^0]$ was implemented and an unknown initial state

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad x_0 \in \mathbb{R}^{n_1}, \quad y_0 \in \mathbb{R}^{n_2}, \tag{3}$$

which, due to the system (1), (2), generated a process $\{(x(t), y(t)),\ t \in T\}$, inaccessible to direct observation and a error-free measurable output function $v(t) = v(t, \mu, x_0, y_0)$, $t \in T$. It is required to estimate the unknown state $(x(t), y(t))$, $t \in T$ using the known function $v(t)$, $t \in T$. To solve this problem, we will construct an asymptotic observer.

Let's denote $n = n_1 + n_2$, $z' = (x', y')$, $z_0' = (x_0', y_0')$, the symbol "$'$" (prime) indicates transposition.

Using the LTVSPS parameters (1), (2) we define the vector function $c(t) = (c_1(t),\ c_2(t))$, as well as the matrix function depending on the parameter $\mu > 0$

$$A(t, \mu) = \begin{pmatrix} A_1(t) & A_2(t) \\ \dfrac{A_3(t)}{\mu} & \dfrac{A_4(t)}{\mu} \end{pmatrix}. \tag{4}$$

Then the system (1)–(3) can be represented in the state space $\mathbb{R}^n$

$$(A_\mu, c): \quad \begin{aligned} &\dot{z}(t) = A(t, \mu) z(t),\ z \in \mathbb{R}^n,\ t \in T, \\ &v(t) = c(t) z(t),\ v \in \mathbb{R},\ t \in T, \\ &z(t_0) = z_0. \end{aligned} \tag{5}$$

Let us identify the system (5) with the pair (A_μ, c), consisting of the matrix functions $A(t, \mu)$ and $c(t)$. Let us represent the matrix $A(t, \mu)$ (4) in the form

$$A(t, \mu) = \operatorname{diag}\left\{E_{n_1}, \frac{1}{\mu}E_{n_2}\right\} A(t), \quad A(t) = \begin{pmatrix} A_1(t) & A_2(t) \\ A_3(t) & A_4(t) \end{pmatrix}. \tag{6}$$

Here and below, E_k denotes the unit $k \times k$-matrix.

By (6), the system (5), defined by the pair of matrix functions $A(t), c(t)$ and a small parameter $\mu \in (0, \mu^0]$, is also identified with set $\{A, c, \mu\}$. If the parameter μ takes all possible values from the interval $(0, \mu^0]$, then we obtain a μ-parametric family of systems $\{A, c\}_{\mu^0}$, which is considered as a single mathematical object defined on $T \times (0, \mu^0]$. Fixed $\mu \in (0, \mu^0]$ distinguishes a specific system (A_μ, c) from the family $\{A, c\}_{\mu^0}$.

Let ρ be some positive number.

Definition 1. System of differential equations

$$\dot{w}(t) = A(t,\mu)w(t) + K(t,\mu)\left(v(t) - c(t)w(t)\right), \quad w \in \mathbb{R}^n, \tag{7}$$

is called a full-order ρ-exponential observer of the family of systems $\{A,c\}_{\mu^0}$ with a gain vector $K(t,\mu)$, and an estimation coefficient $c_\rho(\mu) > 0,\ \mu \in (0,\mu^0]$, if for any $\bar{t} > t_0$ the estimation error $\varepsilon(t,\mu) = z(t,\mu) - w(t,\mu)$ satisfies the inequality

$$||\varepsilon(t,\mu)|| \leqslant c_\rho(\mu)\exp(-\rho(t-\bar{t})),\ \forall t \geqslant \bar{t}, \forall \mu \in (0,\mu^0].$$

A method for constructing ρ-exponential observers for uniformly observed time-varying systems with quasi-differentiable coefficients was proposed in [5]. For any fixed μ, this method can be used to construct the LTVSPS observer (5). However, the following problems may arise: in the general case, the observer's parameters will depend on a small parameter that may be unknown in advance; when using the method from [5], the existence of the canonical Frobenius form and the construction of a corresponding parameter dependent transformation matrix for a high-dimensional SPS are required. In this case, for $\mu \to 0$ the transformation matrix in the general case will be ill-conditioned, and the elements of the canonical Frobenius form tend to infinity. Therefore, it is relevant to develop methods for synthesizing observers that are robust with respect to a small parameter, which do not use knowledge of the value of the small parameter and provide "good" estimates of the state for any sufficiently small values.

The conditions for robust P-uniform observability of a linear time-varying two-time-scale LTVSPS, necessary for constructing ρ-exponential observers, were obtained in [31].

We denote by $O\left(\mu\right)$ a vector function $f\left(t,\mu\right)$ on the interval $\left[t^1,\infty\right)$ such that there exist constants $\mu^* > 0$, $d > 0$ such that the Euclidean norm $||f\left(t,\mu\right)||$ satisfies the inequality $||f\left(t,\mu\right)|| \leqslant d\mu$ $\forall \mu \in\ (0,\mu^*]$, $\forall t \in\ \left[t^1,\infty\right)$.

Let $r(t,\mu) > 0$ be a given function bounded on $T \times (0,\mu^0]$.

Definition 2. The system (7) is called a full-order ρ-exponential observer with a bounded on $T \times (0,\mu^0]$ error $r(t,\mu)$ for the family of systems $\{A,c\}_{\mu^0}$ if for any $\bar{t} > t_0$ the estimation error satisfies the inequality $||\varepsilon(t,\mu)|| \leqslant c_\rho(\mu)\exp(-\rho(t-\bar{t})) + r(t,\mu)$, $\forall t \geqslant \bar{t}$, $\forall \mu \in (0,\mu^0]$.

If in (7) $K(t,\mu) = \operatorname{diag}\{E_{n_1}, \frac{1}{\mu}E_{n_2}\}K(t)$, $c_\rho(\mu) \equiv c_\rho$, $\mu \in (0,\mu^0]$, and for some n, $n = 1,2,3,\ldots$, the equality $r(t,\mu) = O\left(\mu^n\right), t \in T$, is true then we call (7) a robust μ-asymptotic ρ-exponential observer of the LTVSPS family (5).

The definitions introduced above are consistent with the concepts from [5, 14, 16, 34].

Robust μ-asymptotic ρ-exponential observer of the LTVSPS family $\{A,c\}_{\mu^0}$ (5) performs uniform (on μ) asymptotic (on t) estimation of the state vector (x,y) of any system of the LTVSVS family (5) with a bounded error, which has an order of smallness no less than μ. Its gain vector is calculated independently of the small parameter and provides such estimates of the states of systems of the LTVSVS family (5) that the estimation error with an arbitrary predetermined exponential decay rate tends to an infinitesimal value of the order of smallness no less than the small parameter.

Problem 1. To construct for the LTVSPS (1)–(2) a robust μ-asymptotic ρ-exponential observer. In this case, the gain vector must be expressed by gain vectors of parameter-independent subsystems, constructed according to the LTVSPS (1)–(2) and having a dimension smaller than the original one.

3. LTVSPS SUBSYSTEMS, THEIR OBSERVABILITY AND OBSERVERS

3.1. LTVSPS Subsystems and Their Connection with LTVSPS

Solving problems of analysis and synthesis of SPS is often simplified by using the asymptotic decomposition of SPS into subsystems of smaller dimension. This paper describes a constructive method for constructing observers for the LTVSPS, using the asymptotic decomposition of the LTVSPS and implemented through the construction of observers of associated with the LTVSPS (1)–(2) independent of the parameter μ a degenerate system (DS) and a boundary layer systems (BLS) [32], which are obtained from a singularly perturbed system if we consider it separately in the "fast" and "slow" time scales at $\mu = 0$.

Let $\det A_4(t) \neq 0,\ t \in T$, i.e., a standard LTVSPS is being considered. Then the DS (slow subsystem) has the form

$$(A_s, c_s): \quad \begin{array}{l} \dot{x}_s(t) = A_s(t)\, x_s(t),\, x_s(0) = x_0,\ v_s(t) = c_s(t)\, x_s(t),\ t \in T, \\ A_s(t) \triangleq A_1(t) - A_2(t)A_4^{-1}(t)A_3(t),\ c_s(t) \triangleq c_1(t) - c_2(t)A_4^{-1}(t)A_3(t), \end{array} \tag{8}$$

and is a time-varying n_1-dimensional system. Let us identify it with the pair (A_s, c_s).

The BLS (fast subsystem) for the LTVSPS (1)–(2) has the form:

$$(A_4(t_0), c_2(t_0)): \quad \begin{array}{l} \dfrac{dy_f(\tau)}{d\tau} = A_4(t_0)y_f(\tau),\ v_f(\tau) = c_2(t_0)y_f(\tau),\ \tau = \dfrac{t-t_0}{\mu} \in T_\mu \triangleq \left[0, \dfrac{t_1 - t_0}{\mu}\right], \\ y_f(\tau) = y(t_0 + \mu\tau) - A_4^{-1}(t_0)A_3(t_0)x_0,\, y_f(0) = y_0 + A_4^{-1}(t_0)A_3(t_0)x_0, \end{array} \tag{9}$$

and is a linear time-invariant n_2-dimensional system. Let us identify it with the pair $(A_4(t_0), c_2(t_0))$.

Along with the time-invariant BLS (9), we introduce a *t-family of fast subsystems* $(A_4, c_2)(t)$ of the form (9) with $A_4(t), c_2(t)$, (where $t \in T$ is a fixed value, considered as a family parameter) instead of $A_4(t_0), c_2(t_0)$. The BLS (9) is isolated from the t-family of fast subsystems $(A_4, c_2)(t)$ when $t = t_0$.

Note that the DS (A_s, c_s) (8) and the t-family of fast subsystems $(A_4, c_2)(t)$ (9) are defined for the entire family $\{A, c\}_{\mu^0}$ immediately.

The following statement, which follows from [32, Theorem 6.1, p. 227], establishes a connection between the solutions of the LTVSPS (1), (3) and its subsystems (8), (9).

Statement 1. *Let the roots* $\lambda(A_4(t))$ *of the characteristic equation* $\det(\lambda E_{n_2} - A_4(t)) = 0$ *of matrix* $A_4(t)$ *satisfy the inequality* $\operatorname{Re}\lambda(A_4(t)) < -\gamma < 0\ \forall t \in T,\ \gamma = \text{const} > 0$; $A_i(t),\ i = \overline{1,4}$ *are continuously differentiable on* T, $\dot{A}_k(t), k = \overline{2,4}$, *are bounded on* T. *Then there exists* $\mu^* > 0$ *such that for all* $\mu \in (0, \mu^*]$ *the functions*

$$x^1(t) = x_s(t),\quad y^1(t) = y_f\left(\frac{t-t_0}{\mu}\right) - A_4^{-1}(t)A_3(t)x_s(t),\ t \in T,$$

where $x_s(t), y_f(t)$ *are solutions of the DS* (8) *and the BLS* (9), *are 1st-order asymptotic approximations of solution of the problem* (1), (3), *uniform on* $t \in T$ *:*

$$x(t) = x^1(t) + O(\mu),\ y(t) = y^1(t) + O(\mu),\ t \in T.$$

3.2. Observability of Subsystems

In this work, when constructing observers for the DS (8), we use a method that imposes weaker requirements on the smoothness of functions, compared to previously known ones, and uses the concept of quasi-differentiability with respect to some lower triangular matrix P_s, a system of class $\{P_s, n_1 - 1\}$ and uniform observability of the DS. Let us introduce the concepts related to this.

For an arbitrary non-negative integer k, we denote by $\mathcal{U}_k(T)$ the set of all lower triangular matrices $P(t)$ of dimension $((k+1)\times(k+1))$ with elements $p_{ji}(t)$ continuous on T $(j,i=0,1,\dots,k)$ satisfying the condition $p_{jj}(t)\neq 0$ $(t\in T)$, $(j=0,1,\dots,k)$. For an arbitrary matrix $P(t)$ from the set $\mathcal{U}_k(T)$ and a continuous function $w: T\to\mathbb{R}$ quasi-derivatives ${}^{j}_{P}w(t)$ of order j $(j=0,1,\dots,k)$ with respect to the matrix $P(t)$ are determined according to recurrent rules [23]:

$$\begin{gathered}{}^{0}_{P}w(t)=p_{00}(t)w(t),\quad {}^{1}_{P}w(t)=p_{11}(t)\frac{d({}^{0}_{P}w(t))}{dt}+p_{10}(t)({}^{0}_{P}w(t)),\dots,\\ {}^{j}_{P}w(t)=p_{jj}(t)\frac{d({}^{j-1}_{P}w(t))}{dt}+\sum_{i=0}^{j-1}p_{ji}(t)({}^{i}_{P}w(t))\quad (j=2,3,\dots,k).\end{gathered}\tag{10}$$

It is assumed that the differentiation operations in the formulas (10) are feasible and lead to continuous functions.

Let some matrix $P_s\in\mathcal{U}_{n_1}(T)$ be given. Following [24, page. 31], we introduce the definition for the DS (8).

Definition 3. The DS (A_s,c_s) (8) has a P_s-class m if each of its output functions $v_s(t)=v_s(t,x_0)$, $t\in T$, has continuous quasi-derivatives with respect to the matrix P_s up to order m inclusive.

Applying Lemma 2.1 from [24, p. 32] to the system (8) we obtain

Statement 2. *The DS* (8) *has the P_s-class n_1-1 if and only if for any $k=1,\dots,n_1-1$ following row vectors exist and are continuous*

$$s_{s0}(t)=p_{s,00}(t)c_s(t),\ \ s_{sj}(t)=p_{s,jj}(t)\big(s_{s,j-1}(t)A_s(t)+\dot{s}_{s,j-1}(t)\big)+\sum_{i=0}^{j-1}p_{s,ji}(t)s_{si}(t).\tag{11}$$

In particular, the DS (8) has the class $\{P_s,n_1-1\}$ with respect to the $n_1\times n_1$-matrix P_s of the form (18) from [5], constructed according to the parameters of the DS (8).

Definition 4. The DS (A_s,c_s) (8) of class $\{P_s,n_1-1\}$ is called *P_s-uniformly observable on the interval T* if for any $x_0\in\mathbb{R}^{n_1}$ mapping

$$x_s(t)\to\big({}^{0}_{P_s}v_s(t),\,{}^{1}_{P_s}v_s(t),\dots,\,{}^{n_1-1}_{P_s}v_s(t)\big),\ v_s(t)=v_s(t,x_0)$$

is injective for each $t\in T$.

Definition 5. The t-family of fast subsystems $(A_4,c_2)(t)$ (9) is called completely observable on T_μ, if any subsystem from the t-family $(t\in T)$ is completely observable [35, p. 68; 24, p. 29].

Let the DS (A_s,c_s) (8) have P_s-class n_1-1. Let us define the $(n_1\times n_1)$ observability matrix of the DS (A_s,c_s):

$$S_{P_s}(t)=\begin{pmatrix}s_{s0}(t)\\ s_{s1}(t)\\ \dots\dots\\ s_{s,n_1-1}(t)\end{pmatrix},\quad (t\in T),$$

where n_1-row vectors $s_{sj}(t)$, $j=1,2,\dots$ are determined by the formulas (11) and $(n_2\times n_2)$-observability matrix of t-family of fast subsystems $(A_4,c_2)(t)$:

$$S_f(t)=\begin{pmatrix}s_{f0}(t)\\ s_{f1}(t)\\ \dots\dots\\ s_{f,n_2-1}(t)\end{pmatrix},\quad (t\in T),\tag{12}$$

where n_2-row vectors $s_{f0}(t),\ s_{f1}(t),\ \dots$ are defined by the formulas

$$s_{fj}(t)=s_{f,j-1}(t)A_4(t),\quad s_{f0}(t)=c_2(t).\tag{13}$$

Applying the conditions from [35, p. 68; 36, p. 89] to the DS (8) and to the t-family of fast subsystems (9), we obtain

Statement 3. *The DS* (8) *of P_s-class $n_1 - 1$ is P_s-uniformly observable on T if and only if* rank $S_{P_s}(t) = n_1 \quad \forall\, t \in T$.

Statement 4. *The t-family of fast subsystems $(A_4, c_2)(t)$* (9) *is completely observable if and only if* $\operatorname{rank} S_f(t) = n_2 \quad \forall\, t \in T$.

3.3. Observers for Subsystems

Let ρ_s, ρ_f be some positive numbers.

Definition 6. A system of differential equations

$$\dot{w}_s(t) = A_s(t)w_s(t) + k_s(t)\left(v_s(t) - c_s(t)w_s(t)\right), \quad w_s \in \mathbb{R}^{n_1}, \;\; t > t_0, \tag{14}$$

is called a ρ_s-exponential observer of the DS (A_s, c_s) (8) with the gain vector $k_s(t)$ and the estimation coefficient $c_{\rho s} > 0$, if the estimation error $\varepsilon_s(t) = x_s(t) - w_s(t)$ satisfies the inequality $||\varepsilon_s(t)|| \leqslant c_{\rho s} \exp(-\rho_s(t - \bar{t}))$, $t \geqslant \bar{t}$, for any $\bar{t} > t_0$.

Definition 7. A system of differential equations

$$\frac{dw_f(\tau)}{d\tau} = A_4(t)w_f(\tau) + k_f(t)\left(v_f(\tau) - c_2(t))w_f(\tau)\right), \quad w_f \in \mathrm{R}^{n_2}, \;\; \tau > 0, \quad t \in T, \tag{15}$$

is called a ρ_f-exponential observer of the t-family of fast subsystems $(A_4, c_2)(t)$ with the gain vector $k_f(t)$ and the estimation coefficient $c_{\rho f} > 0$, if for any system of the t-family ($\forall t \in T$) the error $\varepsilon_f(\tau) = y_f(\tau) - w_f(\tau)$ satisfies the inequality $||\varepsilon_f(\tau)|| \leqslant c_{\rho f} \exp\left(-\rho_f(\tau - \bar{\tau})\right)$, $\tau \geqslant \bar{\tau}$, for any $\bar{\tau} > 0$.

We denote by $\mathcal{L}(n_1)$ the set of all invertible $n_1 \times n_1$ matrices that are continuously differentiable and bounded on T together with their inverses. The set $\mathcal{L}(n_1)$ is a Lyapunov group [37]. The action of the group $\mathcal{L}(n_1)$ on the pair (A, c), consisting of a $n_1 \times n_1$-matrix function and a n_1-row vector with elements continuous on T, we define according to the following rule [24, p. 42]

$$G * (A, c) = \left(G^{-1}AG - G^{-1}\dot{G},\, cG\right), \quad G \in \mathcal{L}(n_1). \tag{16}$$

Using [5, 31], it is easy to verify the validity of the following statements.

Statement 5. *If for some $P_s \in \mathcal{U}_{n_1}(T)$ the DS* (8) *is P_s-uniformly observable and for it there exists a canonical Frobenius form (A_s^0, c_s^0),*

$$A_s^0(t) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & \alpha_0 \\ 1 & 0 & 0 & \ldots & 0 & \alpha_1 \\ 0 & 1 & 0 & \ldots & 0 & \alpha_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 1 & \alpha_{n_1-1} \end{pmatrix}, \quad c_s^0(t) = (0, 0, \ldots, 0, 1)', \tag{17}$$

relative to the actions of the Lyapunov group $\mathcal{L}(n_1)$, then for any $\rho_s > 0$ there exists a ρ_s-exponential observer (14).

Statement 6. *If the t-family of fast subsystems* (9) *is completely observable, then for any $\rho_f > 0$ there exists a ρ_f-exponential observer* (15).

Observers for the DS (8) and the t-family (9) can be constructed by applying for them the method of constructing ρ-exponential observers from [5, Theorem 5].

3.3.1. Scheme for Constructing a ρ-Exponential Observer for a Time-Varying DS (8).

1) Let the DS (8) be uniformly observable (Statement 3) and for it there exists a canonical Frobenius form (A_s^0, c_s^0), with respect to the actions of the Lyapunov group $\mathcal{L}(n_1)$ [24, p. 69]. Let us denote by $\alpha_s(t) = (\alpha_0, \alpha_1, \ldots, \alpha_{n_1-1})'$ n_1-vector column of coefficients of the matrix $A_s^0(t)$.

2) Find the transformation $G_s(t) \in \mathcal{L}(n_1)$ for which $G_s * (A_s, c_s) = (A_s^0, c_s^0)$.

Note that to construct the canonical Frobenius form and the corresponding transformation matrix $G_s(t)$ from the Lyapunov group $\mathcal{L}(n_1)$ you can use the method described in [5; 24, p. 81].

3) Choose real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$, satisfying for the given $\rho > 0$ the inequality $\lambda_i < -\rho$, $i = 1, 2, \ldots, n_1$.

4) Construct a polynomial $(\xi - \lambda_1)(\xi - \lambda_2) \cdots (\xi - \lambda_{n_1}) = \xi^{n_1} - \beta_{n_1-1}\xi^{n-1} - \cdots - \beta_1\xi - \beta_0$ and let $\beta_s = (\beta_0, \beta_1, \ldots, \beta_{n_1-1})'$ be n_1-vector column..

5) Calculate the gain vector $k_s(t)$ for the observer of DS (14):

$$k_s(t) = G_s(t)(\alpha_s(t) - \beta_s). \tag{18}$$

3.3.2. Scheme for Constructing a ρ-Exponential Observer for a t-Family of Time-Invariant Fast Subsystems.

1) Let the t-family of fast subsystems (9) be completely observable (Statement 4). By the canonical Frobenius form (A_f^0, c_f^0) of the t-family (9) we mean a set of systems, each of which is the canonical Frobenius form of the corresponding fast subsystem. To find it, we construct a characteristic polynomial of the t-family (9). For each $t \in T$ its coefficients define the vector $\alpha_f(t) = (\alpha_0(t), \alpha_1(t), \ldots, \alpha_{n_2-1}(t))'$ coefficients of the canonical Frobenius form of the corresponding fast subsystem.

2) Calculate the transition matrix to the canonical Frobenius form for the t-family of fast subsystems: $G_f(t) = S_f^{-1}(t)S_f^0(t)$, where $S_f(t)$ and $S_f^0(t)$ are the observability matrices of the t-family of fast subsystems and its canonical Frobenius form, calculated using the formulas (12), (13) with matrices $A_4(t), c_2(t)$ and $A_f^0(t), c_f^0(t)$, respectively.

3) Choose real numbers $\lambda_1, \lambda_2, \ldots, \lambda_{n_2}$ that, for a given positive number ρ, satisfy the inequality $\lambda_i < -\rho,\ i = 1, 2, \ldots, n_2$.

4) Form a n_2-vector $\beta_f = (\beta_0, \beta_1, \ldots, \beta_{n_2-1})'$ with constant elements such that $(\xi - \lambda_1)(\xi - \lambda_2) \cdots (\xi - \lambda_{n_2}) = \xi^{n_1} - \beta_{n_1-1}\xi^{n-2} - \cdots - \beta_1\xi - \beta_0$.

5) Calculate the gain vector $k_f(t)$ for the observer of the t-family of fast subsystems:

$$k_f(t) = G_f(t)\left(\alpha_f(t) - \beta_f\right). \tag{19}$$

Note that the function $k_f(t)$, $t \in T$, defined in this way inherits the properties of smoothness and continuity of the functions $\alpha_f(t)$, and hence the coefficients $A_4(t), c_2(t)$ of the t-family of fast subsystems.

4. COMPOSITE OBSERVER OF LTVSVS

4.1. Robust μ-Asymptotic ρ-Exponential Observer of LTVSPS

Let $\rho > 0$ be given.

Theorem 1. *Let*

(i) *for some matrix $P_s \in \mathcal{U}_{n_1}(T)$ the conditions of Statements 2, 3 are satisfied and DS (A_s, c_s)* (8) *has the canonical Frobenius form under the action of the Lyapunov group $\mathcal{L}(n_1)$;*

(ii) *for the DS* (8) *a ρ-exponential observer with a gain vector $k_s(t)$* (18) *and an estimation coefficient c_{ρ_s} was constructed;*

(iii) *the conditions of the Statement 4 are satisfied;*

(iv) *for the t-family of fast subsystems* $(A_4, c_2)(t)$ (9) *a* $\mu^0\rho$*-exponential observer with a gain vector* $k_f(t)$ (19) *and an estimation coefficient* c_{ρ_f} *was constructed;*

(v) *matrix functions*

$$\begin{aligned}\tilde{A}_1(t) = A_1(t) - k_1(t)c_1(t), \quad \tilde{A}_2(t) = A_2(t) - k_1(t)c_2(t),\\ \tilde{A}_3(t) = A_3(t) - k_2(t)c_1(t), \quad \tilde{A}_4(t) = A_4(t) - k_2(t)c_2(t),\end{aligned} \tag{20}$$

where

$$k_1(t) = A_2(t)A_4^{-1}(t)k_f(t) + k_s(t)[E_{n_2} - c_2(t)A_4^{-1}(t)k_f(t)], \quad k_2(t) = k_f(t), \tag{21}$$

are continuously differentiable and bounded, derivatives of functions $\tilde{A}_i(t)$, $i = 2, 3, 4$, *are bounded on* T*;*

(vi) $\operatorname{Re}\lambda(\tilde{A}_4(t)) \leqslant -\gamma_1 < 0$, $\gamma_1 = \text{const} > 0$, $\forall t \geqslant t_0$.

Then there exists $\mu^* \in (0, \mu^0]$ *such that the system*

$$\begin{gathered}\dot{w}_x(t) = \tilde{A}_1(t)w_x(t) + \tilde{A}_2(t)w_y(t) + k_1(t)v(t), \quad w_x \in \mathbb{R}^{n_1}, \quad w_y \in \mathbb{R}^{n_2},\\ \mu\dot{w}_y(t) = \tilde{A}_3(t)w_x(t) + \tilde{A}_4(t)w_y(t) + k_2(t)v(t), \quad t > t_0,\\ w_x(0) = 0, \quad w_y(0) = 0,\end{gathered} \tag{22}$$

is a robust μ*-asymptotic* ρ*-exponential observer of the LTVSPS* (5) *family* $\{A, c\}_{\mu^*}$*.*

The proof of Theorem 1 is given in Appendix.

Note that the requirement for the existence of a canonical Frobenius form for the DS weakens the requirement for the existence of a canonical Frobenius form for the original LTVSPS. For example, a system like $\dot{x}(t) = |t-1|x(t) + (1 - |t-1|)y(t)$, $\mu\dot{y}(t) = tx(t) - ty(t)$, $v(t) = y(t)$ does not have the canonical Frobenius form [24, p. 73, Theorem 3.4]. At the same time, the DS $\dot{x}_s(t) = x_s(t)$, $v_s(t) = x_s(t)$, for this system has the form of the canonical Frobenius form. For a t-family of time-invariant fast subsystems, the canonical Frobenius form always exists if the condition of complete observability is satisfied (Statement 4). For the example given here, the DS has the form $\frac{d}{d\tau}y_f(\tau) = -t\,y_f(\tau), v_f(\tau) = y_f(\tau)$ and is completely observable.

4.2. Algorithm for Constructing a Robust μ*-Asymptotic* ρ*-Exponential Observer of the LTVSPS* (5)

The following algorithm follows from the proof of Theorem 1 (see Appendix).

1. Construct the DS (8) and check the fulfillment of the conditions (i) of Theorem 1. If they are not satisfied for any P_s, then the DS is not uniformly observable and, therefore, there is no canonical Frobenius form for it. Notice, that one of the matrices P_s for which the conditions of the Statement 2 are satisfied is a matrix P_s of the form (18) from [5], which is constructed according to the parameters of the DS (8).

2. Construct the t-family of fast subsystems (9) and check the fulfillment of the conditions (iii) of Theorem 1.

3. Set the desired value for the rate of exponential decrease in observation errors $\rho > 0$.

4. Calculate the gain vector $k_s(t)$ of ρ-exponential observer for the DS (8) according to (18) (item 3.3.1).

5. Calculate the gain vector $k_f(t)$ of $\mu^0\rho$-exponential observer for the t-family of fast subsystems (9) using (19) (item 3.3.2).

6. Calculate the coefficients $k_1(t)$, $k_2(t)$ using (21).

7. Check the fulfillment of the conditions (v) and (vi) of Theorem 1.

8. Form composite observer (22).

5. REDUCED OBSERVERS OF LTVSPS

The observer's state (22) approaches the $O(\mu)$-approximation of the LTVSPS state (5) with an exponential rate ρ, which can be chosen arbitrarily. However, if the value of the small parameter μ is very small or unknown, then it is difficult to practically implement the observer (22). In this regard, it is advisable to evaluate the state of the original system using a system that does not have "fast" observer modes (22).

Similar to [16], we introduce two reduced observers of the LTVSPS.

According to the first approach, an asymptotic Luenberger observer (14) is constructed for the DS (8) of the original LTVSPS (5). In [16] for time-invariant SPS it is proven that if matrices of the DS (8) and the BLS (9) are Hurwitz and the DS is observable, then the asymptotic observer of the DS is a μ-asymptotic observer of the original LTISPS. Following the scheme of proof of Theorem 4 from [16] using Theorem 6.1. [32, p. 227], a similar result can be proven for the LTVSPS.

Theorem 2. *Let the conditions of Statements 1, 5 be satisfied. Then there exists $\mu^* > 0$ such that the system*

$$\begin{gathered}\dot{w}_{sx}(t) = (A_s(t) - k_s(t)c_s(t))\, w_{sx}(t) + k_s(t)v(t),\\ w_{sy} = -A_4^{-1}(t)A_3(t)w_{sx}(t), \quad t > t_0,\\ w_{sx} \in \mathbb{R}^{n_1}, \quad w_{sy} \in \mathbb{R}^{n_2}, \quad w_{sx}(0) = 0,\end{gathered} \tag{23}$$

is a robust μ-asymptotic ρ-exponential observer of the LTVSPS (5) *family* $\{A, c\}_{\mu^*}$.

According to the second approach, the degenerate system is constructed for the observer (22), which is taken to be the observer of the original LTVSPS (5).

Theorem 3. *Let the conditions of Theorem 1 be satisfied. Then there exists $\mu^* > 0$ such that the system*

$$\begin{gathered}\dot{w}_{xs}(t) = \left(\tilde{A}_1(t) - \tilde{A}_2(t)\tilde{A}_4^{-1}(t)\tilde{A}_3(t)\right) w_{xs}(t) + \left(k_1(t) - \tilde{A}_2(t)\tilde{A}_4^{-1}(t)k_2(t)\right) v(t),\\ w_{ys}(t) = -\tilde{A}_4^{-1}(t)\tilde{A}_3(t)w_{xs}(t) - \tilde{A}_4^{-1}(t)k_2(t)v(t), \quad t > t_0,\\ w_{xs} \in \mathbb{R}^{n_1}, \quad w_{ys} \in \mathbb{R}^{n_2}, \quad w_{xs}(0) = 0,\end{gathered} \tag{24}$$

is a robust μ-asymptotic ρ-exponential observer of the LTVSPS (5) *family* $\{A, c\}_{\mu^*}$.

6. EXAMPLES

Let us consider numerical examples illustrating the application of the proposed method for constructing robust μ-asymptotic ρ-exponencial LTVSPS observers. The practical implementation of the method uses the algorithm for constructing a robust μ-asymptotic ρ-exponential observer of the LTVSPS (5) outlined in Section 4.2 (items 1–8) and schemes for constructing ρ-exponential observers for subsystems from sections 3.3.1 and 3.3.2.

Example 1. Let's consider LTVSPS

$$\begin{gathered}\dot{x}_1(t) = (\alpha(t) - 1)x_2(t) + (2 - \alpha(t))y(t), \quad \dot{x}_2(t) = -x_1(t) - x_2(t),\\ \mu\dot{y}(t) = x_2(t) - y(t), \quad v(t) = y(t), \quad t \in T,\end{gathered} \tag{25}$$

whose matrices have the form: $A_1(t) = \begin{pmatrix} 0 & \alpha(t) - 1 \\ -1 & -1 \end{pmatrix}$, $A_2(t) = \begin{pmatrix} 2 - \alpha(t) \\ 0 \end{pmatrix}$, $A_3(t) = \begin{pmatrix} 0 & 1 \end{pmatrix}$, $A_4(t) = \begin{pmatrix} -1 \end{pmatrix}$, $c_1(t) = \begin{pmatrix} 0 & 0 \end{pmatrix}$, $c_2(t) = \begin{pmatrix} 1 \end{pmatrix}$ and the function $\alpha(t)$ is bounded and continuously differentiable on T with bounded derivative.

1. The degenerate system for the LTVSPS (25):

$$\dot{x}_{s1}(t) = x_{s2}(t), \quad \dot{x}_{s2}(t) = -x_{s1}(t) - x_{s2}(t), \quad v_s(t) = x_{s2}(t), \tag{26}$$

where $A_s = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $c_s = \begin{pmatrix} 0, & 1 \end{pmatrix}$, is time-invariant and has a non-singular observability matrix $S_s(t) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, this means that for the DS (26) there is a canonical Frobenius form and, according to Statement 3, the condition (i) of Theorem 1 is satisfied.

The transformation (16) of the DS (26) by using matrix $G_s(t) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ leads to the canonical Frobenius form (A_s^0, c_s^0), $A_s^0(t) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $c_s^0(t) = (0,\, 1)$, where $\alpha_s(t) = (-1,\, -1)'$.

2. The t-family of fast subsystems for LTVSPS (25)

$$\frac{d\tilde{y}(\tau)}{d\tau} = -\tilde{y}(\tau), \quad \tilde{v}_f(\tau) = \tilde{y}(\tau), \quad t \in T, \tag{27}$$

has an observability matrix $S_f(t) = (1)$, rank $S_f(t) = 1$, $\forall t \in T$, then, according to Statement 4, the condition (iii) of Theorem 1 is satisfied.

3. Let us set the rate of exponential decrease in observer errors: $\rho = 2$.

4. Let's take $\lambda_i = -3, i = 1, 2$ and calculate $\beta_s = (-9, -6)'$ and $k_s(t) = (-8,\, 5)'$.

5. The t-family of fast subsystems (27) already has a Frobenius form with $\alpha_f = (-1)$, so $G_f(t) = 1$. Let's choose $\lambda_f = -3$, then $\beta_f = (-3)$ and $k_f(t) =(2)$.

6. Calculate the coefficients (21): $k_1(t) = (2\alpha(t) - 28,\, 15)'\,, k_2 = (2)$.

7. Matrix functions (20) $\tilde{A}_1(t) = A_1(t)$, $\tilde{A}_2(t) = (-3\alpha(t) + 30,\, -15)'$, $\tilde{A}_3(t) = (0, 1)$, $\tilde{A}_4(t) = (-3)$ for LTVSPS (25) satisfy the conditions (v), (vi) of Theorem 1.

8. Finally, the robust μ-asymptotic ρ-exponential composite full-order observer (22) for the LTVSPS (25) for $\rho = 2$ will take the form:

$$\begin{gathered}\dot{w}_{x1}(t) = (\alpha(t) - 1)w_{x2}(t) - 3(\alpha(t) - 10)w_y(t) + 2(\alpha(t) - 14)v(t), \\ \dot{w}_{x2}(t) = -w_{x1}(t) - w_{x2}(t) - 15w_y(t) + 15v(t), \\ \mu\dot{w}_y(t) = w_{x2}(t) - 3w_y(t) + 2v(t).\end{gathered} \tag{28}$$

Figures 1, 2 and Table 1 show the results of numerical experiments (performed using Wolfram Mathematica) with the model (25) with initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $y(0) = 0$, with $\alpha(t) = \sin(t)$,

Figure 1 shows the dynamics of errors ε_{x1} (thick), ε_{x2} (thin), ε_y (dashed line) of the composite observer (22) for the LTVSPS (25) at $\mu = 0.01$. Figure 1a corresponds to the choice of $\lambda_i = -3$, Fig. 1b corresponds to $\lambda_i = -6$ and demonstrates the change in the dynamics of observer errors with increasing exponential decay rate.

To compare the quality of estimation for different values of the small parameter, we calculate the integral norm of the observer (28) errors on the interval $[0, 30]$ (Table 1).

Table 1. Integral norm of the observer (28) errors, $\alpha(t) = \sin(t)$

	$\mu = 0.5$	$\mu = 0.1$	$\mu = 0.01$
$\|\|\varepsilon_{x1}\|\|_1$	0.6755937	0.668459	0.666458
$\|\|\varepsilon_{x2}\|\|_1$	0.112656	0.111410	0.111141
$\|\|\varepsilon_y\|\|_1$	0.037552	0.0371366	0.030469

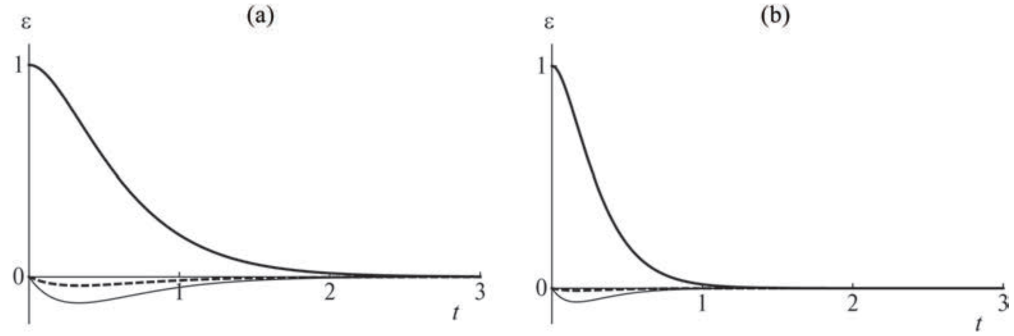

Fig. 1. Dynamics of errors of the composite observer (22) for the LTVSPS (25).

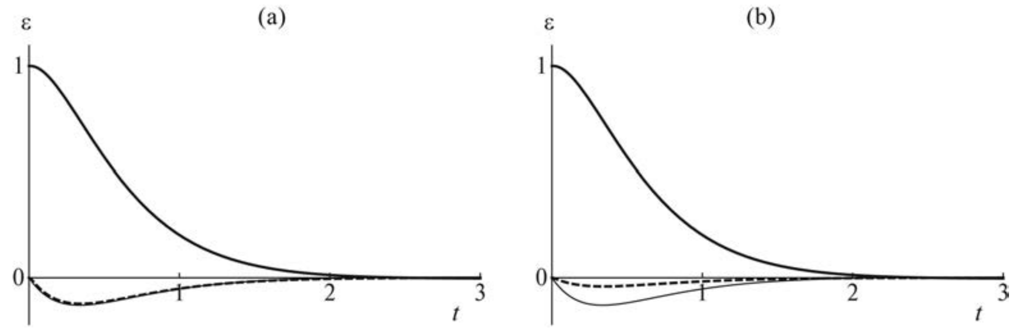

Fig. 2. Dynamics of errors of reduced observers (29) and (30) for the LTVSPS (25).

Comparison of error estimates from Table 1 confirms that as μ decreases, the estimation error decreases.

The reduced observer (23) for the LTVSPS (25) has the form

$$\begin{gathered}\dot{w}_{sx1}(t) = 9w_{sx2}(t) - 8v(t),\\ \dot{w}_{sx2}(t) = -w_{sx1}(t) - 6w_{sx2}(t) + 5v(t),\\ w_{sy} = w_{sx2}(t), \quad w_{sx1}(0) = 0, \quad w_{sx2}(0) = 0.\end{gathered} \tag{29}$$

The reduced observer (24) for the LTVSPS (25) has the form

$$\begin{gathered}\dot{w}_{xs1}(t) = 9w_{xs2}(t) - 8v(t), \quad w_{xs1}(0) = 0,\\ \dot{w}_{xs2}(t) = -w_{xs1}(t) - 6w_{xs2}(t) + 5v(t), \quad w_{xs2}(0) = 0,\\ w_{ys}(t) = \frac{1}{3}w_{xs2}(t) + \frac{2}{3}v(t).\end{gathered} \tag{30}$$

Error dynamics ε_{x1} (thick), ε_{x2} (thin), ε_y (dashed line) of reduced observers (29) and (30) for the LTVSPS (25) at $\alpha(t) = \sin(t)$, $\mu = 0.01$, $\lambda_i = -3$ shown in Fig. 2: Fig. 2a for observer (29) and Fig. 2b for observer (30).

Example 2. Let's consider LTVSPS

$$\begin{gathered}\dot{x}_1(t) = \left(\varphi(t) - \frac{\dot{\gamma}(t)}{\gamma(t)} - \delta(t)\right) x_1(t) + \zeta(t)x_2(t) + \delta(t)y(t),\\ \dot{x}_2(t) = (\gamma(t) - \alpha(t))\, x_1(t) + \xi(t)x_2(t) + \alpha(t)y(t),\\ \mu\dot{y}(t) = x_1(t) - y(t),\\ v(t) = -x_1(t) + x_2(t) + y(t),\ t \in T,\end{gathered} \tag{31}$$

where $\gamma(t) = \sin(t) + 2,\ \varphi(t) = \sin(t) + 1,\ \zeta = \cos(t),\ \xi = -\sin(t) - 1,\ \delta(t) = \sin(t) + 1 - \frac{\cos(t)}{\sin(t)+2}$, $\alpha(t)$ is not twice continuously differentiable at at least one point $t \in T$.

The system (31) in the form (1)–(2) has parameters $n_1 = 2,\ n_2 = m = 1$ and matrices:

$$A_1 = \begin{pmatrix} \varphi(t) - \dfrac{\dot\gamma(t)}{\gamma(t)} - \delta(t) & \zeta(t) \\ \gamma(t) - \alpha(t) & \xi(t) \end{pmatrix}, \quad A_2 = \begin{pmatrix} \delta(t) \\ \alpha(t) \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} -1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 1 \end{pmatrix}.$$

For LTVSPS (31) the matrix function $A(t,\mu)$ (4) is not twice continuously differentiable and for such a system there is no classical observability matrix. Therefore, constructing an observer according to the scheme [5] directly for the LTVSPS (31) is impossible. However, as shown below, it is possible to construct an observer (22) for the system (31).

1. The DS (8) for the LTVSPS (31)

$$\begin{aligned} \dot{\bar x}_1(t) &= \left(\varphi(t) - \frac{\dot\gamma(t)}{\gamma(t)}\right)\bar x_1(t) + \zeta(t)\bar x_2(t), \\ \dot{\bar x}_2(t) &= \gamma(t)\bar x_1(t) + \xi(t)\bar x_2(t), \\ \bar v_s(t) &= \bar x_2(t), \quad t \in T, \end{aligned} \tag{32}$$

has the class $\{E_2, 2\}$. For the DS (32) the classical observability matrix is defined: $S_{E_2}(t) = \begin{pmatrix} 0 & 1 \\ \gamma & \xi \end{pmatrix}$. Since $rank\, S_{E_2}(t) = 2 = n_1,\ \forall t \in T$, then according to the Statement 3 the DS (32) is uniformly observable and the condition (i) of Theorem 1 is satisfied.

2. The t-family of fast subsystems for the LTVSPS (31) coincides with (27), which means that condition (iii) of Theorem 1 is satisfied.

3. Let us set the rate of exponential decrease in observer errors: $\rho = 2$.

4. Choose $\lambda_i = -3$ and calculate $\beta_s = (-9, -6)'$.

5. The transformation (16) of the DS (32) using matrix $G_s(t) = \begin{pmatrix} \frac{1}{\gamma} & \frac{\varphi}{\gamma} \\ 0 & 1 \end{pmatrix}$ leads to the canonical Frobenius form (17) with $\alpha_s(t) = (-\dot\varphi - \xi\varphi + \zeta\gamma,\ \varphi + \xi)'$. Calculated gain vectors for subsystems: $k_s(t) = \begin{pmatrix} \gamma^{-1}\left(9 + 6\varphi + \varphi^2 - \dot\phi\right) + \zeta \\ 6 + \xi + \varphi \end{pmatrix}$, $k_f = (2)$.

6. By (21) we have: $k_1(t) = \begin{pmatrix} -2\delta + 3\zeta + 3\gamma^{-1}\left((\varphi+3)^2 - \dot\varphi\right) \\ -2\alpha + 3(6 + \xi + \varphi) \end{pmatrix}$, $k_2(t) = (2)$.

7. Matrix functions (20)

$$\tilde A_1(t) = \begin{pmatrix} 3(\zeta - \delta) + \varphi + \gamma^{-1}\left(3(\varphi+3)^2 - \dot\gamma - 3\dot\varphi\right) & 2(\delta - \zeta) - 3\gamma^{-1}\left((\varphi+3)^2 - \dot\varphi\right) \\ -3\alpha + \gamma + 3(6 + \xi + \varphi) & 2\alpha + \xi - 3(6 + \xi + \varphi) \end{pmatrix},$$

$$\tilde A_2(t) = \begin{pmatrix} 3\delta - 3\zeta - 3\gamma^{-1}\left((\varphi+3)^2 - \dot\varphi\right) \\ 3\alpha - 3(6 + \xi + \varphi) \end{pmatrix}, \ \tilde A_3(t) = \begin{pmatrix} 3 & -2 \end{pmatrix}, \ \tilde A_4(t) = \begin{pmatrix} -3 \end{pmatrix}$$

for LTVSPS (31) satisfy the conditions *(v), (vi)* of Theorem 1.

8. According to Theorem 1, the robust μ-asymptotic ρ-exponential composite full-order observer for LTVSPS (31) for $\rho = 2$ has the form (22) with the coefficients $k_1(t)$, $k_2(t)$ calculated in item 6.

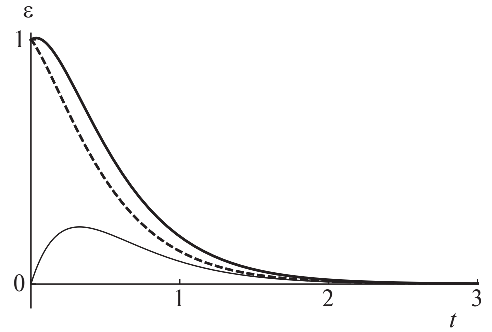

Fig. 3. Dynamics of composite observer (22) errors for the LTVSPS (31).

Figure 3 shows the dynamics of errors ε_{x1} (thick), ε_{x2} (thin), ε_y (dashed line) of composite observer (22) for the LTVSPS (31) with initial conditions $x_1(0) = 1$, $x_2(0) = 0$, $y(0) = 1$ at $\mu = 0.01$.

7. CONCLUSION

The method proposed in this work for synthesizing observers of the LTVSPS states allows us to split the problem into solving independent subproblems of synthesizing observers for systems of smaller dimension, some of which are time-invariant, to ensure the robustness of observers with respect to the small parameter and to significantly weaken the known requirements for the smoothness of coefficients. The vector of gains of the composite observer is expressed through the gains of subsystems independent of the small parameter, corresponding to the separation of time scales. The state estimation error with an arbitrary predetermined exponential decay rate converges to an infinitesimal value of the same order of smallness as the small parameter.

Theorem 1 provides sufficient conditions for the existence of a robust μ-asymptotic ρ-exponential LTVSPS observer. The μ-asymptotic composite full-order (22) and reduced-order (23), (24) observers are constructed. A constructive algorithm for calculating the gain vector (18), (19), (21) of the composite observer is presented, and illustrative examples are given.

When constructing the robust μ-asymptotic ρ-exponential observer for the LTVSPS ρ should be chosen so as to ensure the desired rate of convergence of observation errors into the $O(\mu)$-neighborhood of zero.

Note that when constructing the μ-asymptotic ρ-exponential LTVSPS observer, the existence of a canonical Frobenius form and quasi-differentiability of the output functions of the original LTVSPS are not required.

The results obtained can be used in the design of control systems, identification and diagnostics of dynamic systems described by linear time-varying singularly perturbed systems.

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APPENDIX

Proof of Theorem 1.

It follows from (i) that the DS (A_s, c_s) (8) is uniformly observable and for it there exists a ρ-exponential observer. Under assumption (iii), there exists a $\mu^0\rho$-exponential observer for the t-family of fast subsystems (9).

We will look for a full-order observer for the LTVSPS (5) in the form of the system (7). We will look for the gain vector $K(t,\mu)$ in the form

$$K(t,\mu) = \operatorname{diag}\left\{E_{n_1}, \frac{1}{\mu}E_{n_2}\right\}\begin{pmatrix} k_1(t) \\ k_2(t) \end{pmatrix}, \quad k_1(t) \in \mathbb{R}^{n_1}, \quad k_2(t) \in \mathbb{R}^{n_2},$$

where $k_1(t), k_2(t)$ are not yet defined. Then the observer (7) will take the form (22), and the error dynamics equations $\varepsilon_x(t,\mu) = x(t,\mu) - w_x(t,\mu)$, $\varepsilon_y(t,\mu) = y(t,\mu) - w_y(t,\mu)$ for the observer (22) will look like:

$$\begin{gathered} \dot{\varepsilon}_x(t) = \tilde{A}_1(t)\varepsilon_x(t) + \tilde{A}_2(t)\varepsilon_y(t), \quad \varepsilon_x \in \mathbb{R}^{n_1}, \\ \mu\dot{\varepsilon}_y(t) = \tilde{A}_3(t)\varepsilon_x(t) + \tilde{A}_4(t)\varepsilon_y(t), \quad \varepsilon_y \in \mathbb{R}^{n_2}, \quad t > t_0. \end{gathered} \tag{A.1}$$

Since the observation error dynamics system (A.1) has the form of the LTVSPS (1)–(2), then when the assumptions (v), (vi) of Theorem 1 for the error system (A.1) the conditions of Theorem 3.1 from [32, p. 212] and, therefore, there is a decoupling Lyapunov transformation of the form (3.4) from [32, p. 210] with continuously differentiable matrices $\tilde{L}(t,\mu), \tilde{H}(t,\mu)$ bounded on T, which satisfy the following system (in order not to clutter the notation, we will omit the dependence of functions on the argument t in some places):

$$\tilde{A}_3 - \tilde{A}_4\tilde{L}(\mu) + \mu\tilde{L}(\mu)\left(\tilde{A}_1 - \tilde{A}_2\tilde{L}(\mu)\right) = \mu\dot{\tilde{L}}(\mu), \tag{A.2}$$

$$\mu\left[\tilde{A}_1 - \tilde{A}_2\tilde{L}(\mu)\right]\tilde{H}(\mu) - \tilde{H}(\mu)\left[\tilde{A}_4 + \mu\tilde{L}(\mu)\tilde{A}_2\right] + \tilde{A}_2 = \mu\dot{\tilde{H}}(\mu). \tag{A.3}$$

From (A.2), (A.3) taking into account (i), (vi) we have the approximation:

$$\begin{gathered} \tilde{L}(t,\mu) = A_4^{-1}(t)\,k_2(t)c_2(t)\tilde{L}(t,\mu)\tilde{A}_3(t) + O(\mu), \\ \tilde{L}(t,\mu) = \tilde{A}_4^{-1}(t)\tilde{A}_3(t) + O(\mu), \\ \tilde{H}(t,\mu) = \left(\tilde{H}(t,\mu)\,k_2(t)c_2(t) + \tilde{A}_2(t)\right)A_4^{-1}(t) + O(\mu), \\ \tilde{H}(t,\mu) = \tilde{A}_2(t)\tilde{A}_4^{-1}(t) + O(\mu). \end{gathered} \tag{A.4}$$

As a result of the decoupling transformation, the error dynamics system (A.1) will take the form of a system separated by time scales:

$$\begin{gathered} \dot{\varepsilon}_\xi(t) = A_\xi(t,\mu)\varepsilon_\xi(t), \quad \varepsilon_\xi \in \mathbb{R}^{n_1}, \\ \mu\dot{\varepsilon}_\eta(t) = A_\eta(t,\mu)\varepsilon_\eta(t), \quad \varepsilon_\eta \in \mathbb{R}^{n_2}, \quad t > t_0, \end{gathered} \tag{A.5}$$

where

$$A_\xi(t,\mu) = \tilde{A}_1(t) - \tilde{A}_2(t)\tilde{L}(t,\mu), \quad A_\eta(t,\mu) = \tilde{A}_4(t) + \mu\tilde{L}(t,\mu)\tilde{A}_2(t). \tag{A.6}$$

Moreover, according to Statement 1, the solutions (A.1) and (A.5) satisfy the following equalities:

$$\begin{gathered} \varepsilon_\xi(t) = \varepsilon_x(t) + O(\mu),\ \varepsilon_x(t) = \varepsilon_\xi(t) + O(\mu), \\ \varepsilon_\eta(t) = \tilde{A}_4^{-1}(t)\tilde{A}_3(t)\varepsilon_x(t) + \varepsilon_y(\tau) + O(\mu), \\ \varepsilon_y(t) = -\tilde{A}_4^{-1}(t)\tilde{A}_3(t)\varepsilon_\xi(t) + \varepsilon_\eta(\tau) + O(\mu). \end{gathered} \tag{A.7}$$

Let's put

$$k_2(t) = k_f(t) \tag{A.8}$$

and we will look for $k_1(t)$ in the form:

$$k_1(t) = k_s(t) + \tilde{H}^0(t)k_2(t), \quad \tilde{H}^0(t) = (A_2(t) - k_s(t)c_2(t))A_4^{-1}(t). \tag{A.9}$$

Let's substitute (A.9), (A.8) into (A.6) and perform sequential transformations:

$$\begin{aligned}
A_\xi(\mu) &\overset{(A.9)}{=} A_1 - \left(k_s + \tilde{H}^0 k_2\right) c_1 - \left(A_2 - \left(k_s + \tilde{H}^0 k_2\right) c_2\right) \tilde{L}(\mu) \\
&= \left(A_1 - A_2\tilde{L}(\mu)\right) - \left(k_s + \tilde{H}^0 k_2\right)\left(c_1 - c_2\tilde{L}(\mu)\right) \\
&\overset{(A.4)}{=} A_1 - A_2A_4^{-1}\left(k_2c_2\tilde{L}(\mu) + A_3 - k_2c_1\right) \\
&- \left(k_s + \tilde{H}^0 k_2\right)\left[c_1 - c_2A_4^{-1}\left(k_2c_2\tilde{L}(\mu) + A_3 - k_2c_1\right)\right] + O(\mu) \\
&\overset{(As,cs)}{=} A_s - k_sc_s + \left(-A_2 + k_sc_2 + \tilde{H}^0 k_2c_2\right) A_4^{-1}k_2c_2\tilde{L}(\mu) \\
&+ \left(A_2 - k_sc_2 - \tilde{H}^0 A_4\right) A_4^{-1}k_2c_1 + \tilde{H}^0 k_2c_2A_4^{-1}\left(A_3 - k_2c_1\right) + O(\mu) \\
&\overset{(A.9)}{=} A_s - k_sc_s + +\tilde{H}^0 k_2c_2A_4^{-1}\left(A_4\tilde{L}(\mu) + k_2c_2\tilde{L}(\mu) - k_2c_1 + A_3\right) + O(\mu) \\
&\overset{(A.4)}{=} A_s - k_sc_s + O(\mu).
\end{aligned}$$

Thus, for $A_\xi(t,\mu)$ with $k_1(t), k_2(t)$ of the form (A.9), (A.8) the approximation is valid:

$$A_\xi(t,\mu) = A_s(t) - k_s(t)c_s(t) + O(\mu). \tag{A.10}$$

Further, from (A.6) it follows

$$A_\eta(t,\mu) = (A_4(t) - k_2(t)c_2(t)) + O(\mu). \tag{A.11}$$

Thus, combining (A.10) and (A.11) from (A.5) we get:

$$\begin{aligned}
\dot{\varepsilon}_\xi(t) &= \left(A_s(t) - k_s(t)c_s(t) + O(\mu)\right)\varepsilon_\xi(t), \\
\mu\dot{\varepsilon}_\eta(t) &= \left(A_4(t) - k_2(t)c_2(t) + O(\mu)\right)\varepsilon_\eta(t), \quad t > t_0.
\end{aligned} \tag{A.12}$$

Since in (A.12) $k_s(t)$, $k_2(t)$ are the gain vectors for the observer (14) of the DS (8) and the observer (15) of the t-family of fast subsystems, then the parameters of the error system (A.12) are $O(\mu)$-close to the parameters of the error dynamics system for the DS and the t-family of fast subsystems observers with gain vectors k_s and k_f, respectively. Therefore, due to the continuous dependence of the solution (A.12) on additive perturbations of the system coefficients, the following estimates are valid: $||\varepsilon_\xi(t)|| \leqslant c_{\rho_s}\exp\left(-\rho(t-\bar{t})\right) + O(\mu)$, $t \geqslant \bar{t}$, $||\varepsilon_\eta(t)|| \leqslant c_{\rho_f}\exp\left(-\mu^0\rho\frac{(t-\bar{t})}{\mu}\right) + O(\mu)$, $t \geqslant \bar{t}$, whence, taking into account (A.7), it follows that the estimates are fair

$$\begin{aligned}
&||\varepsilon_x(t)|| \leqslant c_{\rho_s}\exp(-\rho(t-\bar{t})) + O(\mu), \quad t \geqslant \bar{t}, \\
&||\varepsilon_y(t)|| \leqslant c_{\rho_s}||\tilde{A}_4^{-1}(t)\tilde{A}_3(t)||\exp(-\rho(t-\bar{t})) + c_{\rho_f}\exp\left(-\mu^0\rho\left(\frac{t-\bar{t}}{\mu}\right)\right) + O(\mu), \quad t \geqslant \bar{t}.
\end{aligned}$$

Let $c_\rho = \max\{c_{\rho_s}, c_{\rho_f}, c_{\rho_s}||\tilde{A}_4^{-1}(t)\tilde{A}_3(t)||\}$. For $\mu \in (0,\mu^0]$ the estimate $\exp\left(-\mu^0\rho\left(\frac{t-\bar{t}}{\mu}\right)\right) < \exp\left(-\rho\left(t-\bar{t}\right)\right)$, $t \geqslant \bar{t}$, which implies the validity of the estimates $||\varepsilon(t,\mu)|| \leqslant c_\rho \exp(-\rho(t-\bar{t})) + O(\mu)$, $t \geqslant \bar{t}$, and, according to the Definition 7 and from the coincidence of (A.9) and (21) imply fairness of Theorem 1.

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Global Stabilization of a Chain of Two Integrators by a Feedback in the Form of Nested Saturators

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Abstract—Stability of a switching system, which comes to existence when stabilizing a chain of two integrators by a feedback in the form of nested saturators, is studied. The use of the feedback in the form of nested saturators makes it possible to take into account boundedness of the control resource and to ensure the fulfillment of the phase constraint on the velocity of approaching the equilibrium state, which is especially important in the case of large initial deviations. A Lyapunov function is constructed by means of which global asymptotic stability of the closed-loop system is proved for any positive feedback coefficients.

Keywords: : global asymptotic stabilization, switching system, Lyapunov function, nested saturators

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1. INTRODUCTION

Stabilization of chains of integrators is one of the topical control problems. The interest to this problem is due to the fact that original models in many applications (e.g., models of mechanical planar systems) are specified as chains of integrators. Moreover, controls developed for chains of integrators are easily extended to other classes of systems. In last decades, to stabilize such systems, an approach based on the use of special feedbacks in the form of nested nonsmooth saturation functions (saturators) is widely used. This work is a sequel of the paper [1] studying stability of the second-order integrator by a feedback in the form of nested saturators.

The interest to the feedback in the form of nested saturators is explained by the number of remarkable properties of the closed-loop system obtained. Advantages of such feedbacks, as well as topicality of the problem of stabilizing chains of integrators are discussed in many publications, for example, in [1–8]. The use of a feedback in the form of nested saturators, however, results in a quite complex nonlinear switching system, stability analysis of which presents a nontrivial task. The proofs of global stability available in the literature refer basically to the second-order systems [1, 2, 5] and to feedbacks of special forms. For example, the proof of global stability of the second-order system in [2], which is based on the construction of a Lyapunov function, is not applicable to the system with the inverse order of the saturator nesting considered in [1] (see [1] for more detail). The general case of the n-dimensional integrator stabilized by a feedback in the form of n nested saturators is discussed in [3, 4]. However, global stability of the closed-loop system has been proved only for the case where limit values of the nested saturation functions satisfy special conditions, which are seldom met in practice [3, Theorem 2.1].

In [1], global asymptotic stability of the second-order integrator closed by the feedback in the form of nested saturators has been proved for one particular case where the feedback coefficients are selected from a one-parameter family. The goal of this work is to present a simpler proof of global stability of this system, which is based on the construction of a Lyapunov function and is valid in the general case of arbitrary choice of the feedback coefficients.

2. PROBLEM STATEMENT

We consider the problem of stabilizing the second-order integrator

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = U(x), \quad x \equiv [x_1, x_2]^{\mathrm{T}}, \tag{1}$$

by the feedback in the form of nested saturators:

$$U(x) = -\mathrm{sat}_{k_4}(k_3(x_2 + \mathrm{sat}_{k_2}(k_1 x_1))), \tag{2}$$

where $\mathrm{sat}_d(\cdot)$, $d = \{k_2, k_4\}$ is the non-smooth saturation function, $\mathrm{sat}_d(w) = w$ when $|w| \leqslant d$ and $\mathrm{sat}_d(w) = d\,\mathrm{sgn}(w)$ when $|w| > d$; k_4 is the control resource, and k_2 is the maximum velocity of approaching the equilibrium. The right-hand side of (2) determines partition of the phase plane into sets D_1, D_2, and D_3 (Fig. 1). The set D_1 includes the points where both saturators are not saturated (the inclined strip bounded by the dashed lines in Fig. 1); the set $D_2 = D_2^- \cup D_2^+$, the points where only the internal saturator is saturated; and $D_3 = D_3^- \cup D_3^+$, all points where the external saturator is saturated (see [1, 7] for more detail). It can be seen from formula (2) that $U(x)$ is a piecewise linear function and that (1), (2) is a switching system consisting of five linear subsystems the switchings between them depend on the system state and occur when the trajectory intersects the boundaries between the sets.

The original problem depending on the four parameters reduces to the study of two-parameter problem if we turn to the dimensionless variables $\tilde{x}_1 = k_4 x_1/k_2^2$, $\tilde{x}_2 = x_2/k_2$ and time $\tilde{t} = k_4 t/k_2$ [1]. In the dimensionless model, two coefficients turn to ones, $\tilde{k}_4 = \tilde{k}_2 = 1$, and the two others are defined by the formulas $\tilde{k}_1 = k_1 k_2^2/k_4$ and $\tilde{k}_3 = k_2 k_3/k_4$. In what follows, we assume that all variables and constants are dimensionless and use the same (without tilde) notation for them. In the dimensionless model, feedback (2) takes the form [1]

$$U(x_1, x_2) = -\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1 x_1))), \tag{3}$$

where the designation $\mathrm{sat}(\cdot)$ without lower index is used for the saturation functions with the unit limit: $\mathrm{sat}(\cdot) \equiv \mathrm{sat}_1(\cdot)$.

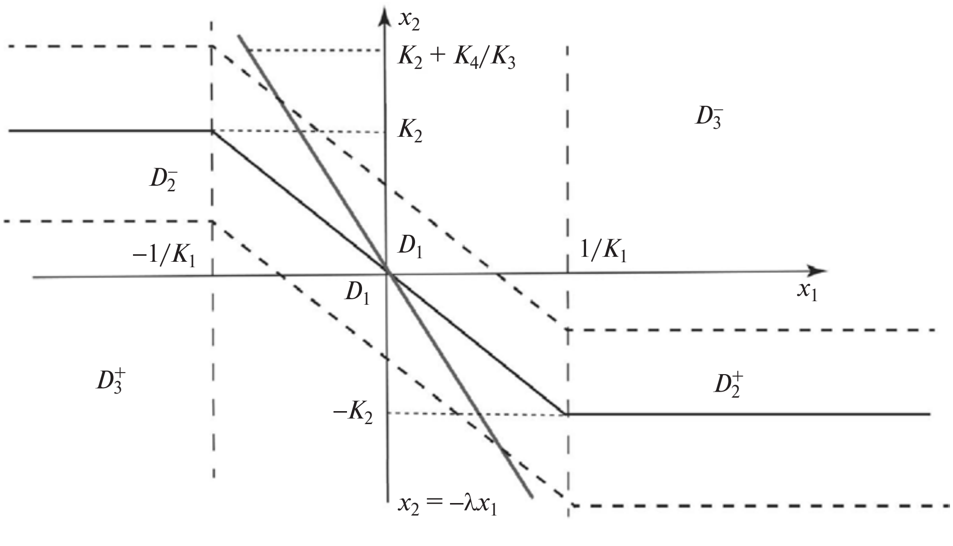

Fig. 1. Partition of the phase plane into the sets D_1, D_2, and D_3.

Note that feedback (3) guarantees the fulfillment of the phase constraint $|x_2(t)| \leqslant 1$ for any initial deviation $x_1(0)$ as longs as $|x_2(0)| \leqslant 1$ [7]; i.e., the domain $\Pi = \{x : |x_2| \leqslant 1\}$ is an invariant set of system (1), (3). Moreover, a desired type of the equilibrium state (a node or focus) and any desired value of the exponential rate of deviation decrease in the neighborhood of the equilibrium point can be ensured by an appropriate selection of the coefficients k_1 and k_3 [1, 7].

It is easy to show that the above phase constraint is fulfilled for any initial conditions starting from a certain finite time. Indeed, consider the function $v(x) = x_2^2$, which is positive definite for all $x_2 \neq 0$, and differentiate it by virtue of system (1), (3): $\dot{v}(x) = -2x_2\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1x_1)))$. Function $\dot{v}(x)$ is negative definite in the domain $|x_2| > 1$; i.e., $|x_2| \leqslant 1$ is an attracting set for solution of system (1), (3). Then, taking into account that no entire trajectory can belong to the set $|x_2| = 1$, it follows that any trajectory enters the invariant set $|x_2| \leqslant 1$ in a finite time.

In [1], it was proved that system (1), (3) is globally asymptotically stable in the particular case of selecting coefficients k_1 and k_3 from a one-parameter family

$$k_1 = \lambda/2, \quad k_3 = 2\lambda, \quad \lambda > 0, \tag{4}$$

where λ is the exponential rate of deviation decrease in the neighborhood of the origin. The proof is not difficult but rather cumbersome and heavily relies on the fact that k_1 and k_3 are given by (4); therefore, it cannot be extended to the case of independent choice of the coefficients.

The proof of global asymptotic stability presented below is based on the construction of a Lyapunov function of system (1), (3) and is valid for arbitrary positive feedback coefficients.

3. PROOF OF GLOBAL ASYMPTOTIC STABILITY

Theorem 1. *System* (1), (3) *is globally asymptotically stable for any positive feedback coefficients.*

Proof. Let us prove that the function

$$V(x) = \frac{1}{2}x_2^2 + \int\limits_0^{x_1} \mathrm{sat}(k_3\mathrm{sat}(k_1s))ds \tag{5}$$

is the Lyapunov function of system (1), (3). From the definition of the saturation function, it follows that the inequalities $\mathrm{sat}(s)s > 0$ and $\mathrm{sat}(f(s))s > 0$ $\forall s \neq 0$ hold, where $f(s)$ is an arbitrary continuous nondecreasing function such that $f(0) = 0$. Then, it follows that the integral term in (5) and, hence, the function $V(x)$ are positive in the entire R^2. It is also evident that $V(x)$ tends to infinity as $\|x\| \to \infty$. Differentiating $V(x)$ by virtue of system (1), (3), we obtain

$$\begin{aligned}\dot{V} &= -x_2\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1x_1))) + \mathrm{sat}(k_3\mathrm{sat}(k_1x_1))x_2 \\ &= -[\mathrm{sat}(k_3(x_2 + \mathrm{sat}(k_1x_1))) - \mathrm{sat}(k_3\mathrm{sat}(k_1x_1))]x_2.\end{aligned}$$

Since the saturator is a nondecreasing function, $\forall s \neq 0$ and $\forall s_0$, the inequality $[\mathrm{sat}(s + s_0) - \mathrm{sat}(s_0)]s \geqslant 0$ holds, from which it follows that $\dot{V}(x) \leqslant 0$.

When $k_3 < 1$, the expression in the square brackets and, hence, the derivative vanish only on the set $x_2 = 0$, which contains no entire trajectories except for $x = 0$. If $k_3 \geqslant 1$, the derivative vanishes also on the subsets of sets D_3^+ and D_3^- on which both addends in the square brackets are simultaneously equal to $+1$ and -1, respectively. It is easy to see that these subsets cannot contain entire trajectories either. Indeed, in D_3^- and D_3^+, trajectories of the system are parabolas

$$x_1 = \mp\frac{1}{2}x_2^2 + C.$$

Since none of these parabolas can lie entirely in D_3^- or D_3^+ (see Fig. 1) and the system moves with a constant acceleration, the trajectory inevitablely enters either D_1 or D_2 in a finite time.

Thus, function $V(x)$ satisfies all conditions of the Barbashin–Krasovskii theorem [9], and, hence, the origin is an asymptotically stable equilibrium state of system (1), (3) in the whole. The theorem is proved.

4. NUMERICAL ILLUSTRATION

To illustrate the above discussion, we constructed level lines of the Lyapunov function (5) for system (1), (3) with the coefficients $k_1 = 1$ and $k_3 = 3$. Figure 2 shows one of the level lines (the solid curve) and several phase trajectories (the dotted curves) with the initial points (marked by circles) lying on the level line. As can be seen from the figure, none of the trajectories leaves the invariant set bounded by the level line. The trajectory segments going along the boundary of the set lie in the subsets of sets D_3^- and D_3^+ in which the derivative of the Lyapunov function by virtue of the system is equal to zero. After intersecting the boundary with set D_1 or D_2, the derivative becomes negative, and the trajectory goes inside the invariant set. Other numerical examples illustrating efficiency of the feedback in the form of nested saturators can be found in [6, 7].

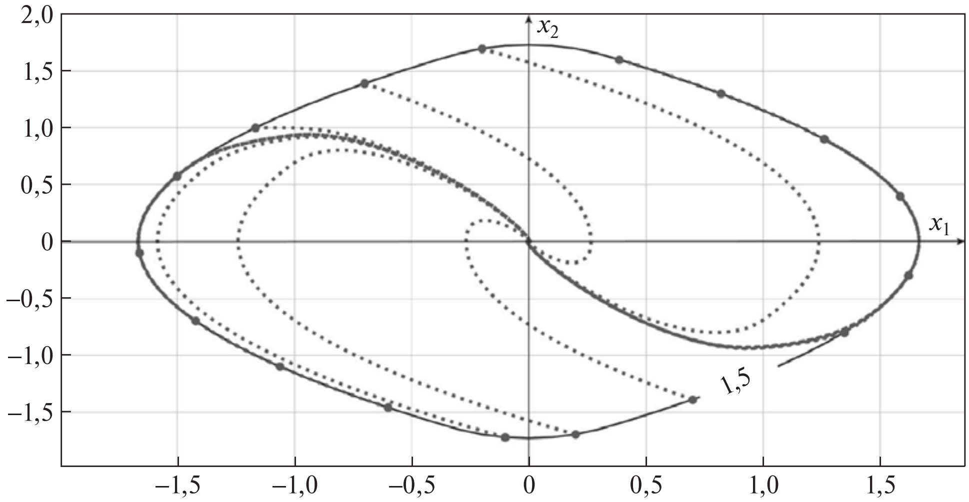

Fig. 2. A level line of the Lyapunov function and phase trajectories.

5. CONCLUSIONS

The problem of stabilizing a chain of two integrators by a feedback in the form of two nested saturators has been considered. By turning to dimensionless variables, the original problem depending on four feedback coefficients has been reduced to study of stability of a two-parameter system. Advantages of the feedback in the form of nested saturators have been discussed. The main result of the work is construction of a Lyapunov function of the system, by means of which global stability of the closed-loop system for any positive feedback coefficients has been proved.

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An Extension of the Feedback Linearization Method in the Control Problem of an Inverted Pendulum on a Wheel

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Abstract—This paper continues previous studies on designing stabilizing control laws for a mechanical system consisting of a wheel and a pendulum suspended on its axis. The control objective is to simultaneously stabilize the vertical position of the pendulum and a given position of the wheel. The difficulty of this problem is that the same control is used to achieve two targets, i.e., stabilize the pendulum angle and the wheel rotation angle. Previously, the output feedback linearization method was applied to this problem. The sum of the pendulum angle and the wheel rotation angle was taken as the output. For the closed loop system to be not only asymptotically stable in the output but also to have asymptotically stable zero dynamics, a dissipative term was added to the output-stabilizing control law. Below, a two-parameter modification of this law is described. Along with the dissipative term, we introduce a positive factor. The more general parameterization allows stabilizing this system in the cases where the control law proposed previously appeared ineffective. The properties of the new control law are investigated, and the attraction domain is estimated. The estimation procedure is reduced to checking the feasibility of linear matrix inequalities.

Keywords: asymptotic stabilization, inverted pendulum, estimation of the attraction domain, linear matrix inequalities

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1. INTRODUCTION

The mechanical system considered in this paper consists of a wheel and a pendulum suspended on its axis. The wheel rolls on a flat surface whose intersection with the vertical plane forms the ξ-axis (Fig. 1).

This system, as well as its sister system known as cart and pole apparatus (a cart with an inverted pendulum mounted on it), was studied in many publications on control theory; for example, see [1–11]. A list of related research works and an analysis of the state-of-art in this area can be found in [1]. Note that this system attracts interest as a nonlinear, unstable, and nonminimum-phase system when examining different control design methods.

Many researchers investigated control design to stabilize the vertical position of a pendulum in the linearized system; for example, see [2, 3, 8]. A nonlinear controller is easily designed using the output feedback linearization method with the pendulum angle as the output [4]. However, this solution does not settle the complete state stabilization problem since the zero dynamics remain unstable and the position of the wheel center is not stabilized. An approach to constructing the

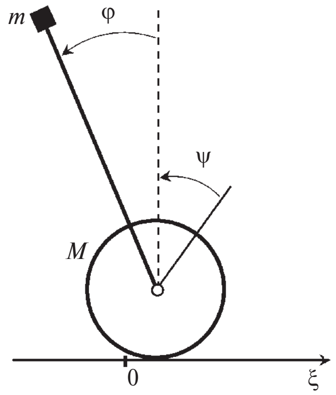

Fig. 1. The diagram of a pendulum on a wheel.

so-called virtual outputs was developed in [12]: stabilization of the virtual outputs ensures state stabilization. This approach is difficult to apply in a general form.

Some approaches to solving the problem in a nonlinear statement were briefly reviewed in [1]. Teel used the theory of small gains [9]. A solution based on the time-optimal control design was described in [10]. For the global stabilization problem, Srinivasan et al. proposed a combined control law under which, for large initial deviations, the pendulum control ensures reaching the local stabilizability domain [11]. Estimation of such a domain is of great importance. Obviously, an estimate that depends on the constructed control law and the chosen Lyapunov function can be conservative. It is topical to obtain a maximum-size estimate in the class defined by the Lyapunov function parameters.

This problem is addressed below, in continuation of the previous studies [1]. As was demonstrated therein, with the sum of the pendulum angle and the reduced wheel rotation angle taken as the output, the control stabilizing this output will make the closed loop system stable, albeit not asymptotically. The zero state (the trivial equilibrium) is stabilized by adding to the control law a term proportional to the difference in the angular velocities of the wheel and the pendulum; this term is interpreted as the viscous friction torque in the wheel axle. Also, the attraction domain of the equilibrium state was estimated using a specially constructed Lyapunov function consisting of a quadratic part and a nonlinear term [1]. The parameters of this Lyapunov function were calculated by solving a sequence of linear matrix inequalities (LMIs).

In this paper, we propose a two-parameter modification of the control law obtained by the output feedback linearization method. Along with the dissipative term, interpreted as the viscous friction torque at the suspension point, we introduce a positive factor at the control law. The more general parameterization allows stabilizing the system in the cases where the control law proposed previously appeared ineffective. An example is given to illustrate this fact.

2. MODEL OF THE SYSTEM

We use the mathematical model described in [2] with the notations introduced in [1], including the change of the time variable. The positive value of the angles is counted counterclockwise.

The following notations are used:

ξ is the position of the wheel center on the horizontal axis in Fig. 1;

φ is the angular deviation of the pendulum from the vertical axis (the pendulum angle);

l is the pendulum length;

ψ is the angle between the vertical and some distinguished wheel radius, with the zero value of ψ corresponding to the zero value of ξ; $\psi = -\frac{\xi}{r}$;

m is the mass lumped at the end of the pendulum;

M, J, and r are the mass, moment of inertia, and radius of the wheel, respectively;

$\theta = \psi\frac{r}{l}$ is the reduced wheel rotation angle;

U is the torque developed by the drive and applied between the pendulum and the wheel;

$u = \frac{U}{mgl}$, where g is the acceleration of gravity;

t is the time variable, and $\tau = t\sqrt{g/l}$ is the new dimensionless independent variable;

$'$ is the derivative with respect to the variable τ.

Also, we denote the angular velocities by $\omega = \varphi'$, $\delta = \theta'$, and $x = (\varphi, \omega, \theta, \delta)^{\mathsf{T}}$. Applying the Lagrangian formalism and the independent variable τ yields the motion equations

$$\begin{aligned} \varphi' &= \omega, \\ \omega' &= f_1(x) + h_1(x)u, \\ \theta' &= \delta, \\ \delta' &= f_2(x) + h_2(x)u, \end{aligned} \tag{1}$$

where

$$\begin{aligned} f_1(x) &= \frac{\sin\varphi}{d}[-\omega^2\cos\varphi + (1+\beta)], \\ f_2(x) &= \frac{\sin\varphi}{d}(\omega^2 - \cos\varphi), \\ h_1(x) &= \frac{1}{d}(\cos\varphi + 1 + \beta), \\ h_2(x) &= \frac{1}{d}(-\cos\varphi - 1), \\ \beta &= \frac{M + J/r^2}{m}, \\ d &= \beta + \sin^2\varphi. \end{aligned} \tag{2}$$

(For details, see [1].)

Denoting $f = (\omega, f_1, \delta, f_2)^{\mathsf{T}}$ and $h = (0, h_1, 0, h_2)^{\mathsf{T}}$, we write system (1) as

$$x' = f + hu. \tag{3}$$

(For the sake of simplicity, the dependence on x is omitted here.)

3. CONTROL DESIGN TO STABILIZE THE TRIVIAL EQUILIBRIUM OF SYSTEM (1)

We choose the system output

$$y = \varphi + \theta \tag{4}$$

and design a control law for system (1) that ensures the asymptotic stability with respect to this output. The corresponding control design method was described, e.g., in [4, Chapter 12]. The control law is given by

$$\begin{aligned} u^*(x) &= -\frac{\lambda^2 y + 2\lambda y' + F}{H} \\ &= -\frac{d\lambda^2(\varphi+\theta) + 2d\lambda(\omega+\delta) + \sin\varphi\left[(1-\cos\varphi)(\omega^2+1) + \beta\ \right]}{\beta}, \end{aligned} \tag{5}$$

$$F = \frac{\sin\varphi}{d}\left[(1-\cos\varphi)(\omega^2+1) + \beta\ \right],\ \ H = \frac{\beta}{d}. \tag{6}$$

In (5), the parameter λ is the desired exponential decay rate for the closed-loop system output. This output will satisfy a linear second-order differential equation with the characteristic polynomial root $-\lambda$ of multiplicity 2.

System (1) closed by the control law (5) takes the form

$$x' = f(x) + h(x)u^*(x), \tag{7}$$

where

$$f(0) = 0, \;\; u^*(0) = 0, \;\; \left.\frac{\partial h(x)}{\partial x}\right|_{x=0} = 0.$$

Applying the linearization procedure to system (7) in a neighborhood of zero yields

$$x' = \Phi x, \;\; \Phi = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} + h(0) \left.\frac{\partial u^*(x)}{\partial x}^{\mathsf{T}}\right|_{x=0}. \tag{8}$$

We have

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(\beta+2)\lambda^2+1}{\beta} & -2\lambda\left(\dfrac{\beta+2}{\beta}\right) & -\lambda^2\left(\dfrac{\beta+2}{\beta}\right) & -2\lambda\left(\dfrac{\beta+2}{\beta}\right) \\ 0 & 0 & 0 & 1 \\ \dfrac{2\lambda^2+1}{\beta} & 4\dfrac{\lambda}{\beta} & 2\dfrac{\lambda^2}{\beta} & 4\dfrac{\lambda}{\beta} \end{bmatrix}. \tag{9}$$

Let us make the linear change of variables $\zeta = Sx$, where

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{10}$$

In other words,

$$\zeta_1 = \varphi, \;\; \zeta_2 = \omega, \;\; \zeta_3 = y, \;\; \zeta_4 = y'.$$

In the new variables, matrix (9) takes the form

$$\Phi_\zeta = S\Phi S^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{1}{\beta} & 0 & -\lambda^2\left(\dfrac{\beta+2}{\beta}\right) & -2\lambda\left(\dfrac{\beta+2}{\beta}\right) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda^2 & -2\lambda \end{bmatrix}. \tag{11}$$

The characteristic polynomials of the matrices Φ and Φ_ζ coincide. Due to the block-triangular form of matrix (11), its spectrum consists of those of the two diagonal blocks of dimensions 2×2. Thus, the eigenvalues of matrix (11) are

$$\left\{ -\frac{i}{\sqrt{\beta}}, \frac{i}{\sqrt{\beta}}, -\lambda, -\lambda \right\}, \tag{12}$$

where i indicates the imaginary unit. Like Φ, the matrix Φ_ζ has a pair of pure imaginary roots and a multiple negative root. Under this distribution of the roots, the closed loop system is asymptotically stable with respect to the output and has zero dynamics that are not asymptotically stable.

Let us design a control law ensuring the asymptotic stability of system (1). In the previous studies, the dissipative term

$$u^{**}(x) = u^*(x) - k(\omega - \delta), \quad k > 0, \tag{13}$$

was added to the control law (5), interpreted as viscous friction at the junction point of the pendulum and the wheel [1].

This paper considers a more general parametric expansion of the control law (5). In addition to the gain k, which can be of arbitrary sign, we introduce a factor s. The new control law is given by

$$u^{***}(x) = su^*(x) - k(\omega - \delta). \tag{14}$$

In the new variables, the matrix of system (1) closed by the control law (14) and linearized in a neighborhood of zero has the form

$$\Phi_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(\beta+1)(1-s)}{\beta} - \dfrac{s}{\beta} & -2k\dfrac{\beta+2}{\beta} & -\dfrac{(\beta+2)\lambda^2}{\beta}s & (k-2\lambda s)\dfrac{(\beta+2)}{\beta} \\ 0 & 0 & 0 & 1 \\ 1-s & -2k & -\lambda^2 s & k - 2\lambda s \end{bmatrix}. \tag{15}$$

The characteristic polynomial of this matrix is

$$\begin{gathered} N(\mu, s) = \det(\mu I - \Phi_s) \\ = \mu^4 + \left(2\lambda s + k\frac{4+\beta}{\beta}\right)\mu^3 + \left(\lambda^2 s + \frac{\beta+2}{\beta}(s-1) + \frac{1}{\beta}\right)\mu^2 + \frac{2\lambda s - k}{\beta}\mu + \frac{\lambda^2}{\beta}s. \end{gathered} \tag{16}$$

Under which values of the parameters s and k does the polynomial (16) have roots with negative real parts? To answer this question, we apply the Liénard–Chipart criterion; for example, see [14, Section 3.5]. We compile the Hurwitz matrix

$$\begin{pmatrix} 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} & 0 & 0 \\ 1 & \lambda^2 s + \dfrac{\beta+2}{\beta}(s-1) + \dfrac{1}{\beta} & \dfrac{\lambda^2}{\beta}s & 0 \\ 0 & 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} & 0 \\ 0 & 1 & \lambda^2 s + \dfrac{\beta+2}{\beta}(s-1) + \dfrac{1}{\beta} & \dfrac{\lambda^2}{\beta}s \end{pmatrix}. \tag{17}$$

The polynomial (16) is Hurwitz if and only if its coefficients are positive and the third principal minor of matrix (17) is positive. Due to the positivity of λ and β, the coefficients of this polynomial are positive under the conditions

$$-2\lambda s\frac{\beta}{\beta+4} < k < 2\lambda s \tag{18}$$

and

$$s > \frac{\beta+1}{\lambda^2\beta + \beta + 2} \doteq \bar{s}. \tag{19}$$

For the third principal minor of matrix (17), the positivity condition takes the form

$$\det\begin{pmatrix} 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} & 0 \\ 1 & \lambda^2 s + \dfrac{\beta+2}{\beta}(s-1) + \dfrac{1}{\beta} & \dfrac{\lambda^2}{\beta}s \\ 0 & 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} \end{pmatrix} > 0. \tag{20}$$

After necessary transformations, we obtain

$$\frac{\beta+2}{\beta^3}\Big\{k^2\Big[\beta+2-(\beta+4)(1+2\lambda^2)s\Big] - 4\lambda sk + 4\lambda s^2\Big[\lambda\beta(s-1)+k(2-\lambda^2\beta)\Big]\Big\} > 0.$$

(Hereinafter, the computer algebra system Maxima is used [13].) Reducing by the positive factor $(\beta+2)/\beta^3$, we finally write this condition as

$$k^2\Big[\beta+2-(\beta+4)(1+2\lambda^2)s\Big] + 4k\lambda s\Big[s(2-\lambda^2\beta)-1\Big] + 4\lambda^2 s^2\beta(s-1) > 0. \tag{21}$$

The closed loop linearized system can be asymptotically stable even without the dissipative term in the control law (14) (i.e., for $k = 0$.) In particular, the following result is valid.

Lemma 1. *For $k = 0$, the polynomial* (16) *is Hurwitz for any*

$$s > 1. \tag{22}$$

The proof of Lemma 1 is provided in the Appendix. Thus, asymptotic stability with respect to the output y turns into asymptotic stability with respect to the state under the control law (5) with a gain exceeding 1.

The terms in (21) are grouped as a quadratic polynomial in k. We denote its coefficients by

$$\begin{gathered} c_0 = \beta+2-(\beta+4)(1+2\lambda^2)s, \quad c_1 = 4\lambda s\Big[s(2-\lambda^2\beta)-1\Big], \\ c_2 = 4\lambda^2 s^2\beta(s-1). \end{gathered} \tag{23}$$

For the sake of brevity, the dependence of the coefficients on s is omitted. We denote by s_0 the value for which c_0 vanishes:

$$s_0 = \frac{\beta+2}{(\beta+4)(1+2\lambda^2)}. \tag{24}$$

Note that

$$c_0 < 0 \quad \text{for} \quad s > s_0. \tag{25}$$

Then we have the following result.

Lemma 2. *For any λ and $\beta > 0$,*

$$s_0 < \bar{s} < 1. \tag{26}$$

In other words, condition (19) *implies the inequality $c_0 < 0$.*

The proof is postponed to the Appendix.

Now let us investigate the asymptotic stability domain of system (15) in the parameter space (s, k). We denote this domain by Ω. Note that the boundary point (1, 0) does not belong to the domain Ω. By Lemma 1, the segment $s > 1$, $k = 0$ belongs to Ω. How far can this domain be extended by choosing $k \neq 0$ under different s satisfying condition (19)? The answer to this question is given below.

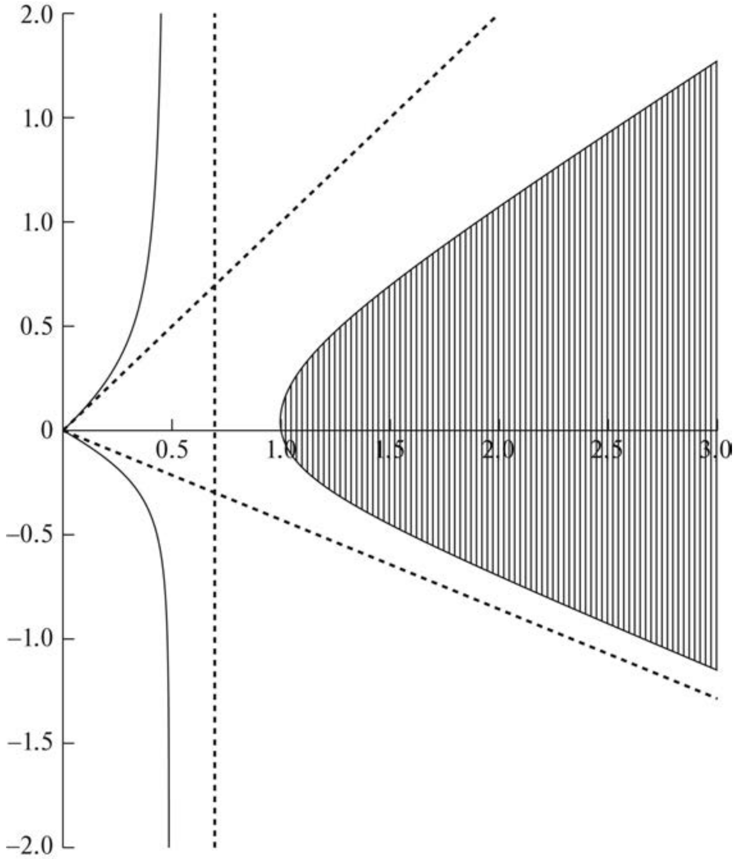

Fig. 2. The domain Ω corresponding to $\beta = 3$ and $\lambda = 0.5$.

Theorem 1. *A point* (s, k) *belongs to the domain* Ω *if and only if* (25) *holds jointly with conditions* (18), (19),

$$c_1^2 - 4c_0c_2 > 0, \tag{27}$$

and

$$\frac{-c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2c_0} < k < \frac{-c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2c_0}. \tag{28}$$

For $s = 1$, *a point* $(1, k)$ *belongs to the domain* Ω *if and only if*

$$0 < k < \frac{2\lambda(1 - \lambda^2\beta)}{1 + \lambda^2(\beta + 4)}. \tag{29}$$

This theorem is proved in the Appendix.

Inequality (29) coincides with the previous estimate of the gain k ensuring the asymptotic stability of the linearized system closed by the control law (13); see [1].

Figure 2 shows an example of the domain Ω. Here, the slanted dashed lines represent the boundary of the domain (18) whereas the vertical dashed line the boundary of the domain (19). The solid thin line indicates the boundary of the definitional domain of inequality (21). According to the figure, this domain has three parts with curvilinear boundaries. Only one of them satisfies the conditions of Theorem 1, thus being the domain Ω (see shading with vertical lines).

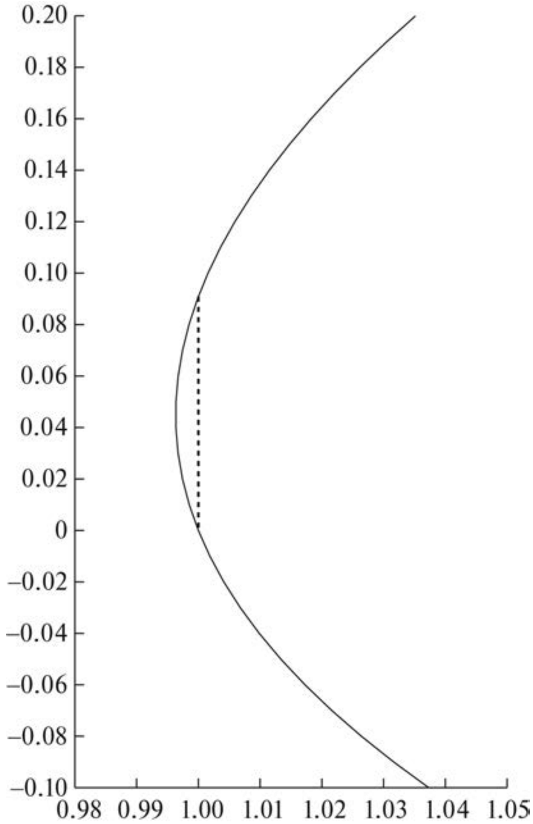

Fig. 3. Comparing the domain Ω and its part shown by dashed line. This part corresponds to $s = 1$ and is defined by inequality (29).

For comparison, Fig. 3 presents an enlarged fragment of the domain Ω and the stability domain defined by inequality (29) for $s = 1$ and the same values of the parameters β and λ. According to the figure, introducing the parameter s into the control law (14) allows extending the stability domain significantly. Moreover, this control law can ensure asymptotic stability for those values of the parameters β and λ under which stabilization by the control law (13) is impossible.

Now we consider the behavior of system (1) closed by the control law (14) in the entire space instead of a neighborhood of zero. After passing to the variables φ, ω, y, and y', the closed loop system takes the form

$$\begin{aligned}
\varphi' &= \omega, \qquad (30)\\
\omega' &= -s\frac{\beta+\cos\varphi+1}{\beta}\left(\lambda^2 y + 2\lambda z\right) - s\frac{1+\omega^2}{\beta}\sin\varphi\\
&\quad - k\frac{\beta+\cos\varphi+1}{\sin^2\varphi+\beta}(2\omega - z) - (s-1)\frac{(\beta+1-\omega^2\cos\varphi)}{\sin^2\varphi+\beta}\sin\varphi,\\
y' &= z,\\
z' &= -s\left(\lambda^2 y + 2\lambda z\right) - k\frac{\beta}{\sin^2\varphi+\beta}(2\omega - z)\\
&\quad - (s-1)\frac{(\omega^2+1)(1-\cos\varphi)+\beta}{\sin^2\varphi+\beta}\sin\varphi.
\end{aligned}$$

System (30) can be written as

$$\zeta' = \Psi(\gamma)\zeta,$$
$$\Psi(\gamma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma_4 - 1)}{\beta}s - \gamma_5(s-1) & -2\gamma_3 k & -\gamma_2\lambda^2 s & \gamma_3 k - 2\gamma_2\lambda s \\ 0 & 0 & 0 & 1 \\ -\gamma_6(s-1) & -2\gamma_1 k & -\lambda^2 s & \gamma_1 k - 2\lambda s \end{bmatrix}, \tag{31}$$

where, due to $d = \beta + \sin^2\varphi$,

$$\begin{aligned} \gamma_1 &= \frac{\beta}{d}, \\ \gamma_2 &= \frac{1+\beta+\cos\varphi}{\beta}, \\ \gamma_3 &= \frac{1+\beta+\cos\varphi}{d}, \\ \gamma_4 &= 1 - \frac{\sin\varphi}{\varphi}(1+\omega^2), \\ \gamma_5 &= \frac{1+\beta-\omega^2\cos\varphi}{d}\left(\frac{\sin\varphi}{\varphi}\right), \\ \gamma_6 &= \frac{(1+\omega^2)(1-\cos\varphi)+\beta}{d}\left(\frac{\sin\varphi}{\varphi}\right). \end{aligned} \tag{32}$$

Regarding the variation of the angle φ, we suppose the following.

Assumption 1. On the trajectories of the controlled system (1),

$$|\varphi| \leqslant \varphi_0 < \frac{\pi}{2}, \quad |\omega| \leqslant \omega_0, \tag{33}$$

where φ_0 and ω_0 are some positive constants.

System (31), equivalent to (30), is nonlinear. Along with this system, we consider the linear time-varying system

$$\zeta' = \Psi(\gamma(\tau))\zeta, \tag{34}$$

where $\gamma_l(\tau)$, $l = 1, \cdots, 6$, represent arbitrary measurable functions of the time variable τ that are subjected only to the two-sided constraints following from the expressions (32) and Assumption 1 :

$$\begin{aligned} \gamma_1(\tau) &\in \left[\frac{\beta}{d_0}, 1\right], \\ \gamma_2(\tau) &\in \left[\frac{1+\beta+\cos\varphi_0}{\beta}, \frac{2+\beta}{\beta}\right], \\ \gamma_3(\tau) &\in \left[\frac{1+\beta+\cos\varphi_0}{d_0}, \frac{2+\beta}{\beta}\right], \\ \gamma_4(\tau) &\in \left[-\omega_0^2, 1-r_0\right], \\ \gamma_5(\tau) &\in \left[\min\left\{\frac{1+\beta-\omega_0^2}{\beta}, \frac{1+\beta-\omega_0^2\cos\varphi_0}{d_0}r_0\right\}, \frac{\beta+1}{\beta}\right], \\ \gamma_6(\tau) &\in \left[\frac{1-\cos\varphi_0+\beta}{d_0}r_0, \max\left\{1, \frac{(1-\cos\varphi_0)(1+\omega_0^2)+\beta}{d_0}r_0\right\}\right], \end{aligned} \tag{35}$$

with $d_0 = \beta + \sin^2\varphi_0$ and $r_0 = \frac{\sin\varphi_0}{\varphi_0}$. For all possible values of the functions $\gamma_l(\tau)$, the solution set of system (34) is wider than that of the nonlinear system (30). Therefore, the absolute stability requirement for the zero (trivial) solution of system (34) in the class of functions $\gamma_l(\tau)$ subjected to the constraints (35) will also ensure the stability of the trivial solution of system (30). Immersion in a wider class of systems (in the sense of the solution set) yields sufficient conditions for the stability of the trivial equilibrium of system (30). To derive such conditions, we choose a Lyapunov function having a negative derivative for all systems (34), (35) simultaneously.

As a candidate Lyapunov function we choose

$$V(\zeta) = \frac{1}{2}\zeta^{\mathsf{T}} P\zeta + \alpha\left[1 - \cos\varphi + \frac{\beta}{2}\ln\left(1+\omega^2\right) - \frac{\varphi^2}{2} - \frac{\omega^2\beta}{2}\right], \tag{36}$$

which is parameterized by a positive definite matrix $P \succ 0$ and a nonnegative number $\alpha \geqslant 0$. (The signs $\succ$, $\prec$, $\succeq$, and $\preceq$ mean positive and negative definiteness and positive and negative semidefiniteness, respectively.) In the expression (36), the Taylor expansion of the nonlinear term with the factor α starts from the third-order terms and corrects the quadratic form for better consideration of the nonlinear properties of system (34). The expression $1 - \cos\varphi + \frac{\beta}{2}\ln\left(1+\omega^2\right)$ is the first integral of the limit system introduced in [1]. The requirement that the derivative of the function (36) along the trajectories of system (34) be negative definite for all possible values of the functions $\gamma_l(\tau)$, $l = 1, \cdots, 6$, from the intervals (35) will be written as a system of LMIs.

The derivative of the function (36) along the trajectories of system (34) has the form

$$\begin{aligned}
V' &= \zeta^{\mathsf{T}} P\Psi(\gamma)\zeta + \alpha\Big[\sin\zeta_1\zeta_2 - \zeta_1\zeta_2 \\
&\quad + \frac{\beta\zeta_2}{1+\zeta_2^2}\left(\gamma_5(1-s)\zeta_1 - (1+\zeta_2^2)\frac{s}{\beta}\sin\zeta_1 - 2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right) \\
&\quad - \beta\zeta_2\left(\gamma_5(1-s)\zeta_1 - (1+\zeta_2^2)\frac{s}{\beta}\sin\zeta_1 - 2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right)\Big] \\
&= \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta \\
&\quad + \alpha\Big[\zeta_2\left(-2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right)\left(\frac{1}{1+\zeta_2^2} - 1\right)\beta - \zeta_1\zeta_2\gamma_4 \\
&\qquad + (s-1)\zeta_1\zeta_2\left(1 - \beta\gamma_5\left(\frac{1}{1+\zeta_2^2} - 1\right) - \gamma_4 - \frac{\sin\zeta_1}{\zeta_1}\right)\Big] \\
&= \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta \\
&\quad + \alpha\Big[\zeta_2\left(-2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right)\left(\frac{1}{1+\zeta_2^2} - 1\right)\beta - \zeta_1\zeta_2\gamma_4 \\
&\qquad + (s-1)\zeta_1\zeta_2\left(\zeta_2^2\frac{\sin\zeta_1}{\zeta_1} - \beta\gamma_5\left(\frac{1}{1+\zeta_2^2} - 1\right)\right)\Big] \\
&= \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta \\
&\quad + \alpha\left[\zeta_2\left(-2k\gamma_9\zeta_2 - \lambda^2\gamma_8 s\zeta_3 + (k\gamma_9 - 2\lambda\gamma_8 s)\zeta_4\right)\beta - \gamma_4\zeta_1\zeta_2 + \gamma_7(s-1)\zeta_1\zeta_2\right],
\end{aligned} \tag{37}$$

where

$$\gamma_0 = -\frac{\zeta_2^2}{1+\zeta_2^2},$$
$$\gamma_7 = \frac{\omega^2 \sin\varphi}{\varphi}\left(1 + \frac{\beta}{(1+\omega^2)}\frac{(\beta+1-\omega^2\cos\varphi)}{d}\right),$$
$$\gamma_8 = \gamma_0\gamma_2,$$
$$\gamma_9 = \gamma_0\gamma_3.$$

For the values defined above, we have the two-sided estimates

$$\begin{aligned}
\gamma_0 &\in \left[-\frac{\omega_0^2}{1+\omega_0^2}, 0\right],\\
\gamma_7 &\in \left[0, \max\left\{\omega_0^2 r_0\left(1+\frac{\beta\,(\beta+1-\omega_0^2\cos\varphi_0)}{(1+\omega_0^2)(\beta+\sin^2\varphi_0)}\right), \frac{\omega_0^2(2+\beta)}{(1+\omega_0^2)}\right\}\right],\\
\gamma_8 &\in \left[-\frac{\omega_0^2(2+\beta)}{\beta(1+\omega_0^2)}, 0\right],\\
\gamma_9 &\in \left[-\frac{\omega_0^2(2+\beta)}{\beta(1+\omega_0^2)}, 0\right].
\end{aligned} \tag{38}$$

Due to the expressions (35), the parameters γ_2 and γ_3 achieve their maximum values simultaneously. Therefore, the parameters γ_8 and γ_9 achieve the minimum value $-\frac{\omega_0^2(2+\beta)}{\beta(1+\omega_0^2)}$ and the maximum value 0 simultaneously. In this case, it suffices to keep the single parameter γ_8 in the expression (37) and write

$$V' = \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta + \alpha\left[\beta\gamma_8\zeta_2\left(-2k\zeta_2 - \lambda^2 s\zeta_3 + (k-2\lambda s)\zeta_4\right) + (\gamma_7(s-1)-\gamma_4)\,\zeta_1\zeta_2\right]. \tag{39}$$

In the next section, we represent the condition $V' < 0$ in terms of LMIs.

4. ESTIMATING THE ATTRACTION DOMAIN OF THE TRIVIAL EQUILIBRIUM

We write the Lyapunov function (36) as

$$V(\zeta) = \frac{1}{2}\zeta^{\mathsf{T}}Q(\alpha)\zeta + \alpha\left[1-\cos\varphi + \frac{\beta}{2}\ln\left(1+\omega^2\right)\right] \geqslant \frac{1}{2}\zeta^{\mathsf{T}}Q(\alpha)\zeta, \tag{40}$$

where $Q(\alpha)$ denotes the matrix

$$Q(\alpha) = P - \alpha\begin{bmatrix}1&0&0&0\\0&\beta&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}. \tag{41}$$

By imposing the inequality

$$Q(\alpha) \succeq \varepsilon I,$$

where $\varepsilon > 0$ is sufficiently small, from (40) we conclude that the function $V(\zeta)$ is positive definite. Further, the expression (39) for V' affinely depends on the arbitrarily varying parameters γ_l,

$l = 1, \cdots, 8$, each ranging in a closed interval. Hence, the vector γ ranges in the Cartesian product of eight closed intervals. This set, denoted by $\Gamma \subset R^8$, is convex and has 256 extreme points obtained by equating the arbitrarily varying parameters γ_l, $l = 1, \cdots, 8$, to their minimum and maximum values on the closed intervals (35) and (38). The quadratic form (39), whose matrix affinely depends on the parameters γ, is negative definite for all $\gamma \in \Gamma$ if and only if it is negative definite at the extreme points of this set, i.e., at the vectors γ_i. Therefore, the condition $V' < 0$ is equivalent to a system of 256 LMIs, each corresponding to one of the vectors γ^i, $i = 1, \ldots, 256$. For all possible values γ^i, we obtain the system of LMIs (one LMI of high dimensions)

$$P\Psi(\gamma^i) + \Psi^T(\gamma^i)P - \alpha Y(\gamma^i) \preceq 0, \tag{42}$$

where

$$Y(\gamma^i) = \begin{bmatrix} 0 & \gamma_4^i - \gamma_7^i(s-1) & 0 & 0 \\ \gamma_4^i - \gamma_7^i(s-1) & 4\beta\gamma_8^i k & \beta\gamma_8^i\lambda^2 s & \beta\gamma_8^i(2\lambda s - k) \\ 0 & \beta\gamma_8^i\lambda^2 s & 0 & 0 \\ 0 & \beta\gamma_8^i(2\lambda s - k) & 0 & 0 \end{bmatrix}.$$

The resulting system of LMIs may be infeasible for given values φ_0 and ω_0. We introduce a parameter $a \in [0, 1]$ and choose

$$\varphi_0(a) = a\frac{\pi}{2}, \;\; \omega_0(a) = a\bar{\omega}$$

for φ_0 and ω_0, where $\bar{\omega} = \sqrt{e^{\frac{4}{\beta}} - 1}$. Each value of $\varphi_0(a)$ and $\omega_0(a)$ is associated with the corresponding limits of the intervals (35) and (38) and, consequently, with the 256 vectors $\gamma^i(a)$. By the expressions for these limits, the vectors $\gamma^i(a)$ continuously depend on a. For $a = 0$, the lower and upper limits of the intervals coincide. Therefore, for $a = 0$ we have

$$\gamma^i(0) \doteq \gamma^0 = \left[1, \frac{2+\beta}{\beta}, \frac{2+\beta}{\beta}, 0, \frac{1+\beta}{\beta}, 1, 0, 0\right]^T.$$

Further, $\Psi(\gamma^0) = \Phi_s$ due to the matrix formulas (15) and (31). According to Theorem 1, the system of LMIs (42) is feasible for sufficiently small $a > 0$.

Let a^* be the supremum of those a for which the system of LMIs

$$\begin{aligned} P\Psi(\gamma^i(a)) + \Psi^T(\gamma^i(a))P - \alpha Y(\gamma^i(a)) &\preceq 0, \\ Q(\alpha) &\succeq \varepsilon I, \\ \mathrm{tr(Q(\alpha))} &= 1 \end{aligned} \tag{43}$$

is feasible with respect to the variables P and α. The linear equation $\mathrm{tr(Q(\alpha))} = 1$ (the unit trace of the matrix $Q(\alpha)$) has been added to normalize the solution: otherwise, the feasibility set of the LMIs would be a cone and, together with any solution P and α, σP and $\sigma\alpha$ would be another solution for any $\sigma > 0$, including both arbitrarily large and arbitrarily small values.

The value a^* is obtained by successively checking the feasibility of (43) for an increasing numerical sequence $\{a\}$.

Thus, under Assumption 1, where $\varphi_0 = a^*\frac{\pi}{2}$ and $\omega_0 = a^*\bar{\omega}$, the Lyapunov function (36) has a negative definite derivative along the trajectories of system (31). If there exists a constant $c > 0$ such that the set

$$\Omega_c = \{\zeta : V(\zeta) \leqslant c\} \tag{44}$$

is inscribed in the set

$$\Pi_0 = \{\zeta : |\varphi| \leqslant \varphi_0,\ |\omega| \leqslant \omega_0\}, \tag{45}$$

then any trajectory of the closed loop system (31) evolving from the interior of the set Ω_c remains inside it at any time due to the negativity of the derivative V'. As a result, if

$$\Omega_c \subset \Pi_0, \tag{46}$$

Assumption 1 will hold along the entire trajectory of the closed loop system (31) evolving from the interior of the set Ω_c. In other words, we have established the following result.

Theorem 2. *Let the conditions of Theorem 1 be satisfied and the value a^* be chosen as the supremum of those a for which the LMI* (43) *is feasible. If the constant c is such that condition* (46) *holds, then the set Ω_c is the asymptotic stability domain of system* (1) *closed by the control law* (14).

A method for finding a constant c that ensures condition (46) was described in [1].

5. AN EXAMPLE OF CONSTRUCTING THE ATTRACTION DOMAIN

Consider an example corresponding to the value $\beta = 3$. Let the parameters of the control law (14) be $\lambda = 0.578$, $k = 0$, and $s = 1.5$ to ensure the conditions of Theorem 1. This example is interesting because for the chosen values of the parameters β and λ, condition (29) fails, i.e., the mechanical system under consideration cannot be stabilized by the control law (13) under any value of the parameter k. However, this system can be stabilized by the control law (14) with $s > 1$ and $k = 0$. Checking the feasibility of the LMI (43) for the maximum possible value a^* yields the following parameters of the Lyapunov function:

$$P = \begin{bmatrix} 0.068 & -0.040 & 0.037 & 0.167 \\ -0.040 & 0.128 & -0.070 & -0.247 \\ 0.037 & -0.070 & 0.070 & 0.175 \\ 0.167 & -0.247 & 0.175 & 0.734 \end{bmatrix}, \quad \alpha = 0.000017,$$

achieved at $a^* = 0.349$.

Using Theorem 2, the invariant attraction domain of the closed loop system is constructed in the same way as described in [1]. Omitting the details, we present the final value: $c = 0.0036466$. The feasibility of LMIs was checked in Scilab [15].

Figure 4 shows the trajectories of the closed loop system in the coordinates θ (the abscissa axis) and δ (the ordinate axis). The combined control law was applied. If the state of the system does not fall into the attraction domain Ω_c, then the control law described in [1, Section 5] is applied. The condition $V(\zeta^*) = c$ is the criterion for crossing the boundary of the domain Ω_c. Switching to the control law (14) occurs upon reaching the domain Ω_c. In the figures, the angular variables are measured in degrees whereas the angular velocity in degrees per unit of the dimensionless time τ.

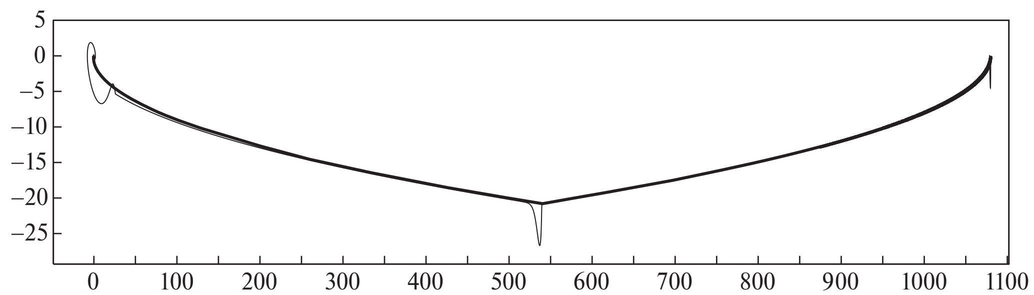

Fig. 4. Trajectories of the closed loop system in the coordinates θ (the abscissa axis) and δ (the ordinate axis).

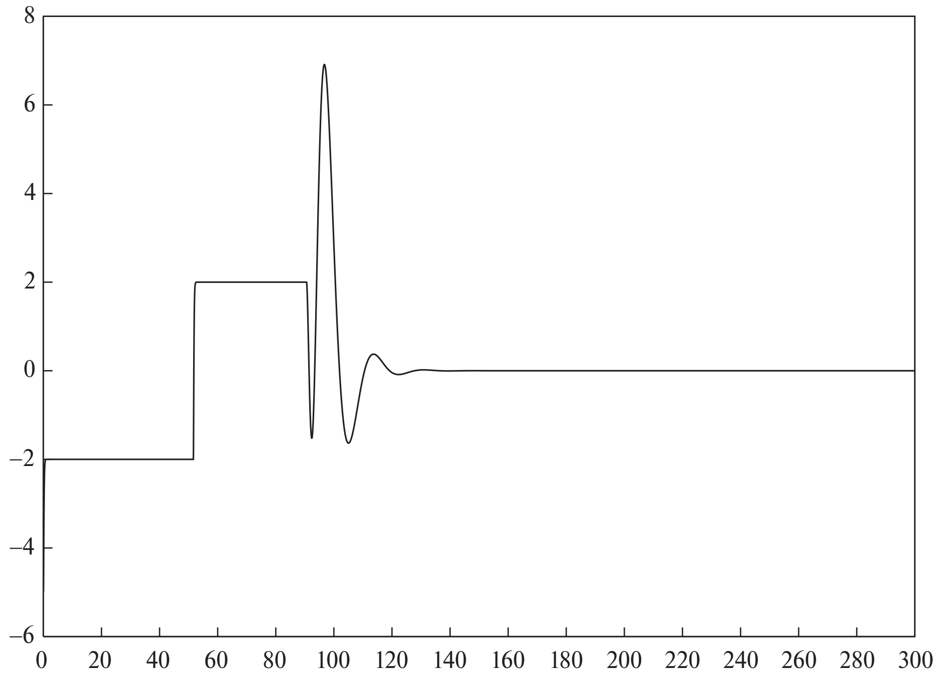

Fig. 5. Graph of the variable φ.

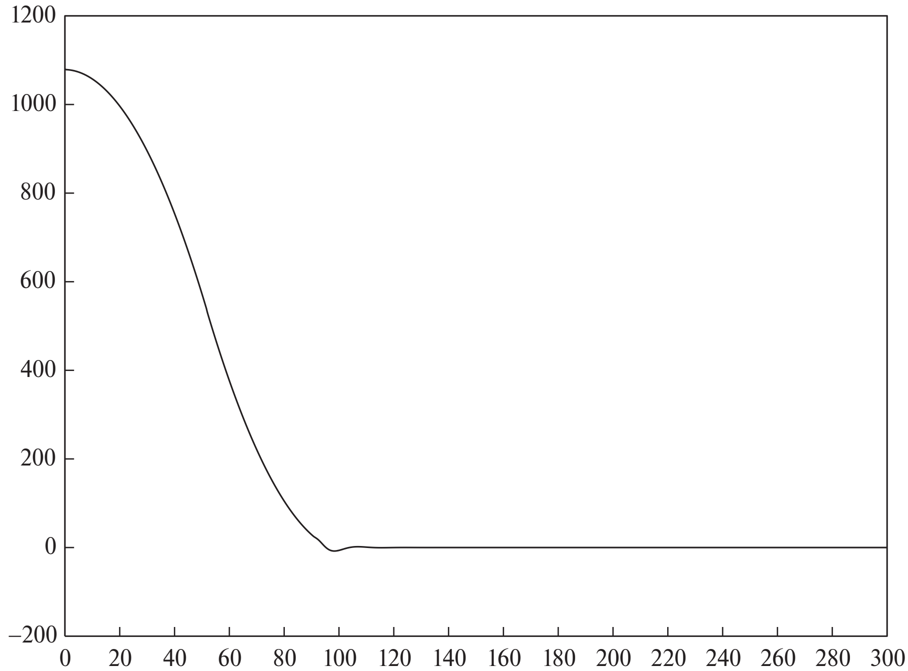

Fig. 6. Graph of the variable θ.

The thin line shows the trajectory of system (1) closed by the combined control law for $\bar{\varphi} = 2°$ (see formula (5.2) in [1]) and the initial conditions $\varphi(0) = -5°$, $\omega(0) = 0$, $\theta(0) = 1080°$, and $\delta(0) = 0$. The thick line shows the optimal trajectory of system (5.5) under the control law (5.6) of [1].

The graphs of the angles φ and θ are demonstrated in Figs. 5 and 6, respectively, where the abscissa axis is the dimensionless time τ. According to Fig. 5, at the initial stage with the control law (5.2) of [1], the variable φ stabilizes first at $-2°$ and then at $2°$; subsequently, at an approximate time of $\tau \sim 90$, switching to the control law (14) occurs, and the trivial equilibrium is asymptotically stabilized.

6. CONCLUSIONS

This paper has considered the problem of stabilizing the vertical position of an inverted pendulum on a wheel. A two-parameter extension has been proposed for the control law simultaneously stabilizing the pendulum angle (its deviation from the vertical line) and the wheel rotation angle. The stabilization problem has been solved by the output feedback linearization method, with the sum of the pendulum angle and the wheel rotation angle taken as the output. In contrast to the previous studies [1], an additional factor has been introduced along with a dissipative term. A numerical example has been provided to show that the new control law allows stabilizing the system in the case where it cannot be stabilized by the previous control law. The attraction domain of the trivial equilibrium has been estimated. This estimate has been constructed by solving a system of linear matrix inequalities.

APPENDIX

Proof of Lemma 1. For $k = 0$, condition (18) takes the form $s > 0$ whereas condition (21) the form

$$4\lambda^2 s^2 \beta(s-1) > 0.$$

Due to the first inequality and the positivity of β, it follows that $s > 1$. The proof of Lemma 1 is complete.

Proof of Lemma 2. By definition (19) of the value $\bar{s}$ and $\beta > 0$, we directly establish that $\bar{s} < 1$. Straightforward algebraic transformations in Maxima [13] yield

$$\bar{s} - s_0 = \frac{\lambda^2\beta^2 + 8\lambda^2\beta + 8\lambda^2 + \beta}{(\beta+4)(1+2\lambda^2)(\lambda^2\beta + \beta + 2)},$$

and the desired conclusion is obvious.

Proof of Theorem 1. Note that the lines $k = 2\lambda s$ and $k = -2\lambda s \frac{\beta}{\beta+4}$, which determine the boundary of the definitional domain of inequality (18), have no intersection with the boundary of the (possibly non-simply-connected) domain given by (20). Indeed, substituting $k = 2\lambda s$ into the left-hand side of inequality (20) leads to the contradiction

$$-8\lambda^4 s^3(\beta+2) > 0.$$

By analogy, we arrive at a contradiction when substituting the equality $k = -2\lambda s \frac{\beta}{\beta+4}$ into (20):

$$-\frac{8\beta\lambda^2 s^2(\beta+2)}{(\beta+4)^2} > 0.$$

Thus, the boundary of the domain Ω is formed by the points (s, k) satisfying condition (19) and turning the left-hand side of inequality (20) to zero. According to Lemma 2, for these values of s the coefficient c_0 in (23) takes negative values, i.e., the dependence of the left-hand side of (21) on the variable k is an inverted parabola. Then the closed interval of values k for which inequality (21) holds is of the form (28) provided the positive determinant of the quadratic inequality (21).

For $s = 1$, we have $c_2 = 0$, and inequality (21) takes the form

$$-k^2[2 + 2\lambda^2(\beta+4)] + k(4\lambda - 4\lambda^3\beta) > 0,$$

which implies inequality (29). The proof of Theorem 1 is complete.

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NONLINEAR SYSTEMS

On Forced Oscillations in a Relay System with Hysteresis

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Abstract—This paper discusses the phenomenon associated with the forced synchronization (“entrainment of a self-sustained oscillator by an external force”) in a relay system with hysteresis, which manifests itself in the occurrence of periodic motions close to the rhythmic activity of neurons, when packets of fast oscillations are interspersed with intervals of the slow dynamics.

To study this phenomenon, we introduce a circle mapping, which, depending on the parameters, can be a circle diffeomorphism or discontinuous map (“gap map”). In both cases, this mapping demonstrates the so-called period-adding bifurcation structure.

It is demonstrated that packets number of fast oscillations in the period of periodic motion is determined by the rotation number, and the length of the intervals between the packets may be found of the boundaries of the absorbing interval. The change in the number of pulses in the packet occurs through the border-collision bifurcation.

Keywords: forced relay system with hysteresis, system with two time scales, discontinuous circle mapping, border-collision and period adding bifurcations

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1. INTRODUCTION

Let us consider the relay system with hysteresis [1–3] and external periodic action. The behavior of this system is described by a differential equation

$$\dot{x} = f(t, x), \quad f(t, x) = \lambda \left(x - S(x, \eta) + \sigma(t)\right), \quad \sigma(t + \pi) = -\sigma(t). \tag{1}$$

Here t is the time variable, x is an unknown function of t and f is a given function of t and x $(t, x, f \in \mathbb{R})$. $\dot{x}$ is the derivative of x with respect to t and S is the output (switching function) signal of the relay element. The absolute value of λ is proportional to the time constant of the plant, and $\sigma(t) = \mu_0 + \mu_m \cos t$ is the forced signal, were μ_0, μ_m are the constant component and the amplitude of the variable component $\sigma(t)$, respectively.

The output signal S of the relay element is defined as

$$S(x, \eta) = \begin{cases} 1, & x < q - \chi, \qquad q - \chi < x < q + \chi, \qquad \eta = 1; \\ 0, & x > q + \chi, \qquad q - \chi < x < q + \chi, \qquad \eta = 0. \end{cases}$$

Here q is the setting signal; χ is the hysteresis of a relay element; $\eta = 0, +1$ are the values of the function S after the previous switching of the relay element.

As can be seen from equation (1), the right part depends on both t and x. The function f periodically changes by t with a period of 2π. In addition, the function $x(t)$ satisfying (1) is invariant with respect to the shift of the origin of t by 2π [4].

Parameters: $\lambda = -7.5/\pi$, $q = 4.0/\Gamma$; $\mu_0 = 1.5/\Gamma$; $\mu_m = 0.525/\Gamma$; $\chi = \chi_0/\Gamma$. In the following bifurcation analysis, we shall consider Γ and χ_0 as control parameters: $6.0 \leqslant \Gamma \leqslant 7.0$, $0.35 \leqslant \chi_0 \leqslant 0.65$.

Differential equations of the form (1) are used to study many problems of engineering, mechanics, physics [5–7], and biology.

Examples in biology include the analysis of one particular type of cardiac arrhythmia, atrioventricular conduction block [8–10], investigations of biological mechanisms for sleep-wake regulation [11–14], the study rhythmic activity of neurons [15–21].

In [22] it was shown that with accepted idealizations, a mathematical model of a vibration machine with unbalanced excitation of vibrations and relay control can be reduced to the equation (1).

The considered class of relay systems refers to systems with two time scales (for example see [23, 24]), the dynamics of which are determined by two frequencies: high frequency oscillations generated by fast switchings of the relay element modulated by the low-frequency reference signal.

Such systems demonstrate a phenomenon close to the rhythmic activity of neurons, when packets of fast oscillations are interspersed with intervals of the slow dynamics. A typical example of a model mapping describing this behavior of neurons is the discontinuous map of Rulkov [17] (see also the review [19]). The bifurcation mechanisms of the occurrence of such oscillations have been studied by many authors (for example see [18–21, 25]).

In this paper, we study the entrainment of a self oscillations of the relay system with hysteresis by an external periodic signal, which manifests itself in the occurrence of regular motions close to the rhythmic dynamics of neurons.

First, we reduce the differential equation (1) to circle mapping. It is shown that, depending on the parameters, such a mapping is a circle diffeomorphism or discontinuous ("gap map").

Next we obtain analytically the equation for the boundaries of the gap and boundaries of the absorbing interval in the phase space.

This allow us to obtain the boundary separating the domains of existence of a circle diffeomorphism and discontinuous map ("gap map") in the parameter plane.

We found, that in both regions, the mapping demonstrates the so-called period-adding bifurcation structure [26, 27]. In [22], it was shown both numerically and experimentally that the intervals of low-frequency oscillations, are interrupted by bursts of fast oscillations.

In the present study is demonstrated that packets number of fast oscillations in the period of periodic motion is determined by the rotation number, and the length of the intervals between the packets may be found of the boundaries of the absorbing interval.

The change in the number of pulses in the packet occurs through the border-collision bifurcation when one of the points of the periodic orbit collides with the switching manifold. This corresponds to a collision solution of the equation (1) for $S = 1$ with the upper switching threshold of the relay element. Such a bifurcation in nonsmooth differential equations is called "grazing bifurcation" [28–31].

2. MATHEMATICAL MODEL IN DISCRETE TIME

2.1. Return Map

Due to the periodicity of f by t with a period of 2π, the phase plane (1) is a rectangle with a width of 2π with identified points N', N (Fig. 1a). The equation (1) reduces to a system of autonomous differential equations

$$\dot{\theta} = 1, \ \dot{x} = f(\theta, x), \tag{2}$$

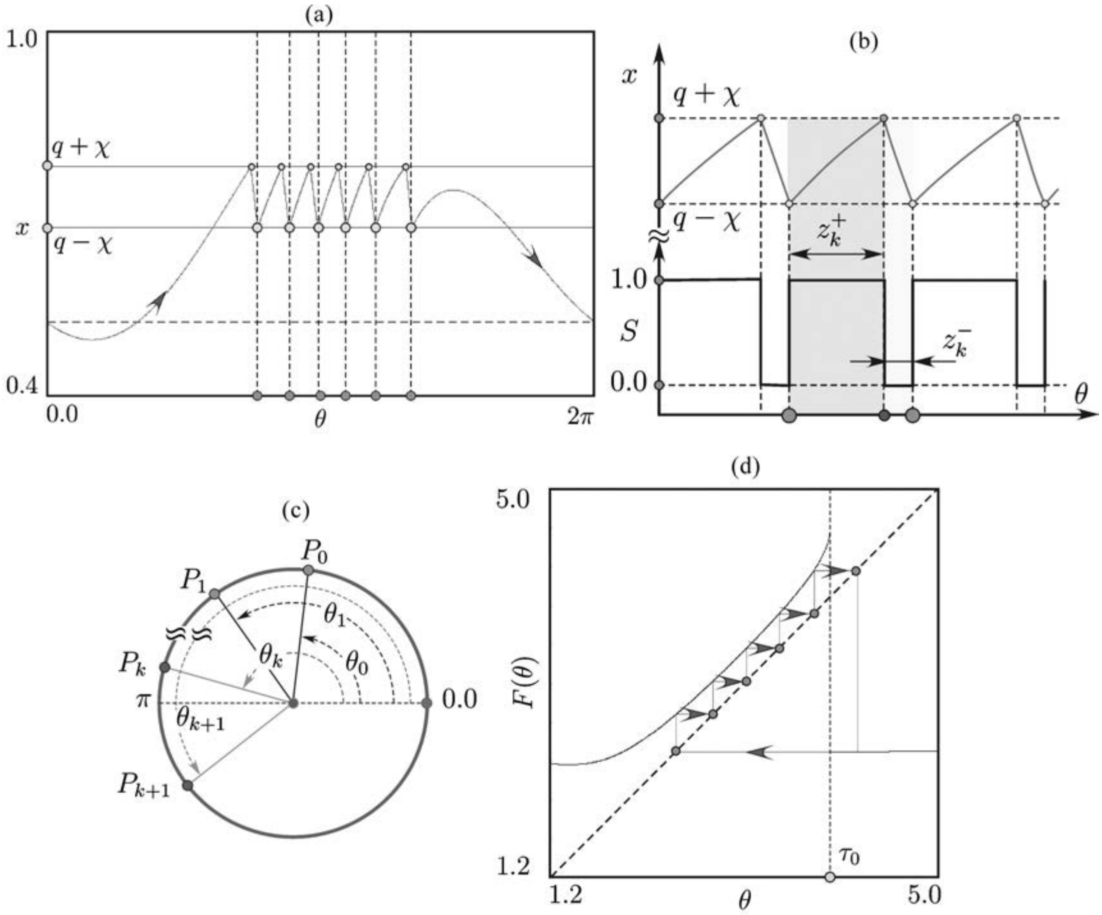

Fig. 1. (a) Periodic solution of a non-autonomous equation (1). (b) Magnified part of the periodic solution in (a), illustrating the technique of obtaining a mapping (3). (c) The mapping defined on the circle. (d) The periodic orbit of the mapping (3) corresponding to the periodic solution of the equation (1).

the phase variables of which are $\theta = t - 2\pi\lfloor t/(2\pi)\rfloor$ and x. Here $\lfloor\cdot\rfloor$ is the largest integer number not greater than argument (i.e., the integer part, or floor, of argument).

The solution to equation (2)) starting at the point of the lower boundary $q-\chi$ of the hysteresis zone maps to the point located on the upper boundary $q+\chi$ then, after switching the relay element $S = 1 \to S = 0$, this point maps back to the lower boundary one (see Fig. 1b).

Let's connect the ends of the segment $[0;\ 2\pi]$ and form a circle of unit radius [1] (Fig. 1c). Then one can introduce a function F, which maps each point P_k on the unit circle into a point P_{k+1}, to which it will map P_k when rotated by an angle θ_{k+1} according to the differential equations (2) after a time $z_k^+ + z_k^-$, where $z_k^\pm$ is the pulse width at $S = 1$ and $S = 0$, respectively (Figs. 1b and 1c).

Then the change of the angle between consecutive points is $P_k = P(1, \theta_k)$, $k = 0, 1, 2, \ldots$ on the unit circle is described by

$$\theta \mapsto F(\theta) \bmod 2\pi, \quad F(\theta) = \theta + z^+(\theta) + z^-(\theta), \quad 0{,}0 \leqslant \theta \leqslant 2\pi. \tag{3}$$

Here z^+ is the smallest non-negative solution of the equation

$$q + \chi = \mathrm{e}^{\lambda z^+}(q - \chi - 1 + \mu_0) + 1 - \mu_0 + A_m\ (\sin(\theta + z^+) - \lambda\cos(\theta + z^+))$$
$$-A_m\,\mathrm{e}^{\lambda z^+}(\sin\,\theta - \lambda\cos\,\theta)\,,$$

and z^-:

$$q - \chi = \mathrm{e}^{\lambda z^-}(q + \chi + \mu_0) - \mu_0 + A_m \left(\sin(\theta' + z^-) - \lambda \cos(\theta' + z^-)\right)$$
$$-A_m \mathrm{e}^{\lambda z^-}\left(\sin\theta' - \lambda\cos\theta'\right), \quad A_m = \frac{\lambda\mu_m}{1+\lambda^2},\ \theta' = \theta + z^+.$$

Hence the orbit of the point θ_0

$$\theta_1 = F(\theta_0),\ \theta_2 = F(\theta_1) = F^2(\theta_0),\ \ldots, \theta_k = F^k(\theta_0),\ \ldots.$$

The nature of the dynamics on a circle is determined by the rotation number

$$r = \lim_{k\to\infty} \frac{F^k(\theta) - \theta}{2\pi k}.$$

If the rotation number is rational $r = \frac{n}{m}$, were n, m $(m \neq 0)$ are integers, then there exists θ_0 such that

$$F^m(\theta_0) = \theta_0 \bmod 2\pi$$

and the orbit on the circle is m-periodic.

If r is an irrational number, then, in the case of diffeomorphism, the dynamics is quasiperiodic: each iteration of the mapping gives a new point on the unit circle and none of these points repeats. Then P_k forms a dense set on the circle.

This was related to the properties of diffeomorphisms. If the mapping is noninvertible or discontinuous, then the specified properties are preserved, but there are differences. A detailed discussion of these differences can be found in [26].

To describe the orbit of the system (3) starting at the point θ_0, one can use two characters $\mathcal{L}$ and $\mathcal{R}$ (“left,” “right”) [26]. Then the orbit $\mathcal{O}(\theta_0) = \{\theta_i = F^i(\theta_0),\ i = 0, 1, 2, \ldots\}$ is described by the sequence:

$$\bar\sigma_0\bar\sigma_1\bar\sigma_2\ \cdots, \tag{4}$$

where the symbol $\bar\sigma_i$, $i = 0, 1, 2, \ldots$ in this sequence for each $i \geqslant 0$ is defined as

$$\bar\sigma_i = \begin{cases} \mathcal{L}, & \theta_i < c_0; \\ \mathcal{R}, & \theta_i > c_0. \end{cases}$$

For one-dimensional maps with a single point of discontinuity, the rotation number r of the periodic orbit of the period m

$$\mathcal{O}_m(\theta) = \{\theta \in I : \theta,\ F^k(\theta),\ k = 1, \ldots, m-1\},$$
$$F^m(\theta) = \theta,\ F^k(\theta) \neq \theta$$

is defined as [26]

$$r = \frac{N_{\mathcal{R}}(\mathcal{O}_m)}{N_{\mathcal{L}}(\mathcal{O}_m) + N_{\mathcal{R}}(\mathcal{O}_m)},$$

where $N_{\mathcal{L}}(\mathcal{O}_m)$, $N_{\mathcal{R}}(\mathcal{O}_m)$ is a number of symbols $\mathcal{L}$ and $\mathcal{R}$ in equation (4).

2.2. Properties of the Mapping

- Let's rewrite the map (3) as

$$\theta_{k+1} = F(\theta_k),$$
$$F(\theta) = \begin{cases} F_{\mathcal{L}}(\theta), & \theta < c_0; \\ F_{\mathcal{R}}(\theta), & \theta > c_0. \end{cases}$$

  The functions $F_{\mathcal{L}}$, $F_{\mathcal{R}}$ are continuous and strictly increase on the segments $[F_{\mathcal{R}}(c_0); c_0]$ and $[c_0; F_{\mathcal{L}}(c_0)]$, respectively. The F mapping has no fixed points in $(F_{\mathcal{R}}(c_0); F_{\mathcal{L}}(c_0))$.
- If

$$F_{\mathcal{R}} \circ F_{\mathcal{L}}(c_0) < F_{\mathcal{L}} \circ F_{\mathcal{R}}(c_0),$$

  then each point $\theta \in J$ has either a single preimage, or has no preimages inside J.
  If there is a non-empty subinterval $(F_{\mathcal{R}} \circ F_{\mathcal{L}}(c_0); F_{\mathcal{L}} \circ F_{\mathcal{R}}(c_0))$ consisting of points that have no preimages in J, then in this case it is said that F is discontinuous ("gap map").
- Figure 2a shows the case when the F function is continuous and monotonically increasing. Then F contains one fictitious point of discontinuity such that

$$F_{\mathcal{R}}(c_0) = F_{\mathcal{L}}(c_0)$$

  and $F(\theta + 2\pi) = F(\theta) + 2\pi$.
- Figure 2b shows the case when F contains a single point of discontinuity. How the gap occurs is explained below. Moreover, the points $c_{\mathcal{L}} = F_{\mathcal{R}}(c_0)$ and $c_{\mathcal{R}} = F_{\mathcal{L}}(c_0)$, called critical points of rank one, define the boundaries of the invariant absorbing interval $J = [c_{\mathcal{L}}; c_{\mathcal{R}}]$. The function F may have local extremes, but they are outside of J. Therefore, inside J, the function F is piecewise increasing.
  Moreover, in this case $F_{\mathcal{L}}(c_0) < F_{\mathcal{R}}(c_0)$, so that in the absorbing interval J the mapping F is discontinuous ("gap map")(see Fig. 2c).
- The condition when the mapping F becomes discontinuous is formulated by the following Statement 1.

Statement 1. *A discontinuity F occurs when the solution $x(t)$ of the equation (1) for $S = 1$, starts at the point of the lower threshold $q - \chi$ switching of the relay element, tangents the upper threshold of switching $q + \chi$. In this case, the function F has a point of discontinuity c_0, which satisfies the equation*

$$F(c_0) = c_*, \tag{5}$$

where $F(y) = y + z^+(y)$ [22, 27].

It is easy to see that the function Q on the right side of the equation (5) is given implicitly. Our goal is to obtain the equation with respect to the point of discontinuity c_0 explicitly.

The condition of tangency $x(t)$ to the upper switching threshold of the relay element is written as

$$\varphi_t + \varphi_x \, \dot{x}(t)|_{t=c_*} = 0, \quad \varphi = q + \chi - x. \tag{6}$$

Since $\varphi_t = 0$ and $\varphi_x = -1$, then (6) is equivalent to

$$\dot{x}(t)|_{t=c_*} = 0, \quad \dot{x} = \lambda(x - 1 + \mu_0 + \mu_m \cos t).$$

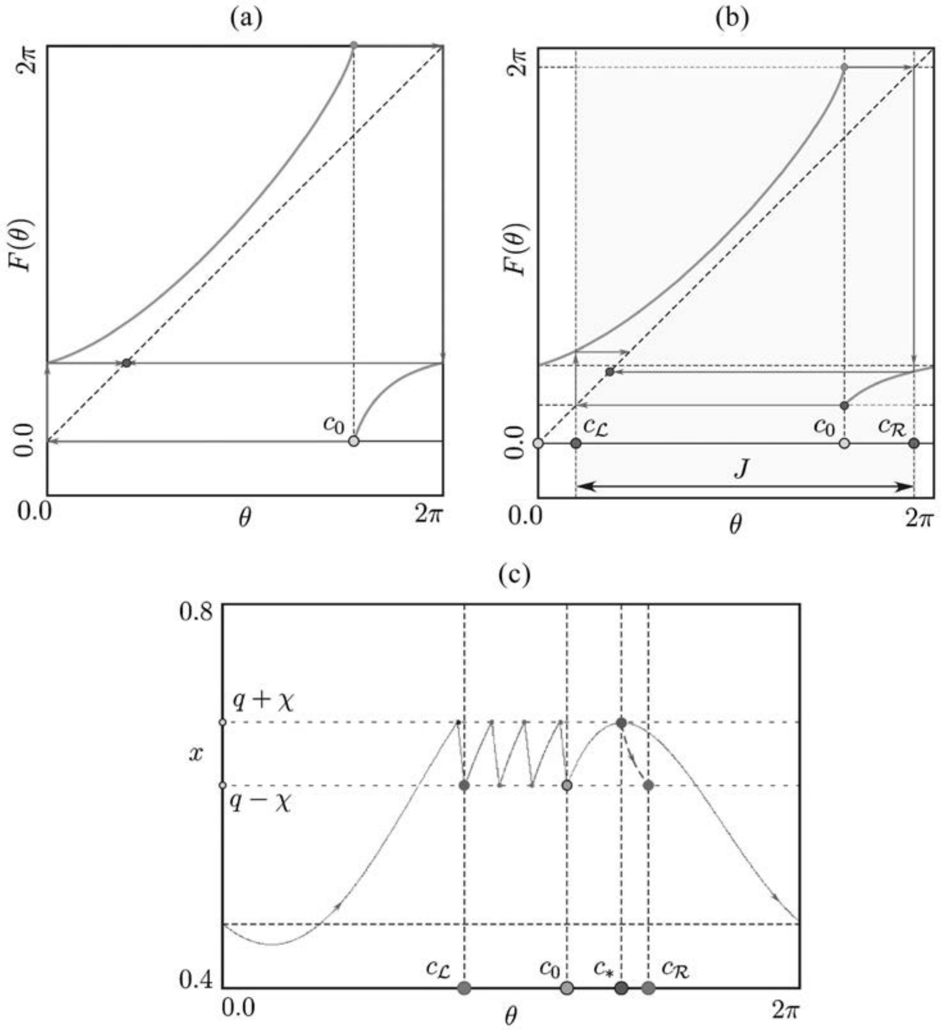

Fig. 2. (a) Circle diffeomorphism. (b) Discontinuous map (“gap map“). (c) Determination of the discontinuity point F and boundaries of the absorbing interval J.

Hence

$$x(c_*) - 1 + \mu_0 + \mu_m \cos c_* = 0.$$

Taking into account that $x(c_*) = q + \chi$, we get

$$q + \chi - 1 + \mu_0 + \mu_m \cos c_* = 0.$$

By solving this equation with respect to c_*, we have

$$c_* = 2\pi - \arccos\left(\frac{1 - q - \chi - \mu_0}{\mu_m}\right),$$

if $-1 \leqslant \frac{1-q-\chi-\mu_0}{\mu_m} \leqslant +1$.

From the condition $-1 \leqslant \frac{1-q-\chi-\mu_0}{\mu_m} \leqslant +1$, we find the boundary L separating the regions of existence of a discontinuous map and a circle diffeomorphism in the parameter space. Let's denote these domains as Π_{span} and Π_{circle}. The boundary between Π_{gap} and Π_{circle} on the parameter plane (χ_0, Γ) is

$$L = \{(\chi_0, \Gamma) : \Gamma = 6.025 + \chi_0\}. \tag{7}$$

After c_* is found, we determine the point of discontinuity c_0. To obtain the equation with respect to c_0 explicitly, we use the tangency condition

$$x(c_*) - 1 + \mu_0 + \mu_m \cos c_* = 0. \tag{8}$$

Since

$$\begin{gathered} x(c_*) = \mathrm{e}^{\lambda(c_* - c_0)}(q - \chi - 1 + \mu_0) + 1 - \mu_0 \\ + A_m\, t(\sin\, c_* - \lambda\, \cos\, c_*) - A_m\, \mathrm{e}^{\lambda(c_* - c_0)}\, (\sin\, c_0 - \lambda\, \cos\, c_0), \end{gathered} \tag{9}$$

then, substituting (9) into (8), we get:

$$\begin{gathered} \mathrm{e}^{\lambda(c_* - c_0)}(q - \chi - 1 + \mu_0) + A_m(\sin c_* - \lambda \cos c_*) \\ -A_m \mathrm{e}^{\lambda(c_* - c_0)}(\sin c_0 - \lambda \cos c_0) + \mu_m \cos c_* = 0. \end{gathered}$$

This equation is solved numerically.

3. BIFURCATION ANALYSIS

Figure 3a presents the two-dimensional bifurcation diagram in the (χ_0, Γ) parameter plane. Boundary L (7) separates the domains Π_{circle} and Π_{gap}.

Below the boundary of L (within the domain of Π_{circle}), the mapping (2) is a circle diffeomorphism, and the region Π_{gap} corresponds to the region of existence of a discontinuous mapping ("gap map").

Figure 3b shows a bifurcation diagram obtained for the scan A in Fig. 3a: $0.4 \leqslant \chi_0 \leqslant 0.61$, $\Gamma = 6.6$ and Fig. 3c is a magnification of that part of this diagram, illustrating the transition to and from the synchronization $1:5$ through the saddle-node bifurcation in Π_{circle}. Here dotted lines correspond to the unstable 5-cycle and solid lines to the stable one. At the boundaries of the resonant tongue with $1:5$ stable and saddle 5-cycles merge and disappear.

Resonance domains have a classical structure, the so-called Arnold tongues. Between each two quasiperiodic regimes with different rotation numbers there exists a region of resonance dynamics.

Figure 3d presents a magnified part of the diagram in Fig. 3b illustrating the transition when we cross the boundary between the regions Π_{circle} and Π_{gap}. As you can see from this diagram, entering synchronization mode with $2:9$ occurs in Π_{circle}, and the transition from the synchronization mode $2:9$ occurs in the domain Π_{gap}.

Studies have shown that transitions at the left boundary of the tongue with $2:9$ occur through the classical saddle-node bifurcation. In [22] it was shown that in the domains Π_{gap} such transitions are associated with border-collision fold bifurcation.

Figure 3e illustrates the so-called "Devil's staircase: the dependence of the rotation number r on χ_0. In this figure, the plateaux correspond to the resonance tongues with $1:6$, $1:5$, $2:9$ $1:4$. In Fig. 3f displays the bifurcation diagram for Π_{gap}, which demonstrates a sequence of period-adding bifurcations.

Finally, Fig. 4a shows periodic motion demonstrating the bursting dynamics in the Π_{gap} tongue with $1:6$. As you can see from this figure, the number of packets with fast oscillations is determined by the numerator of the rotation number. The interval between the packets is determined by the boundaries of the absorbing interval. The change in the number of pulses in the packets occurs through the border-collision bifurcations, when one of the periodic points collides with the boundary of the gap F.

Periodic solution in the domain Π_{circle} is depicted in Fig. 4b, where this attractor is localized in the zone hysteresis of the relay element. The question is how to determine the characteristics of such behavior for Π_{circle}, remains open.

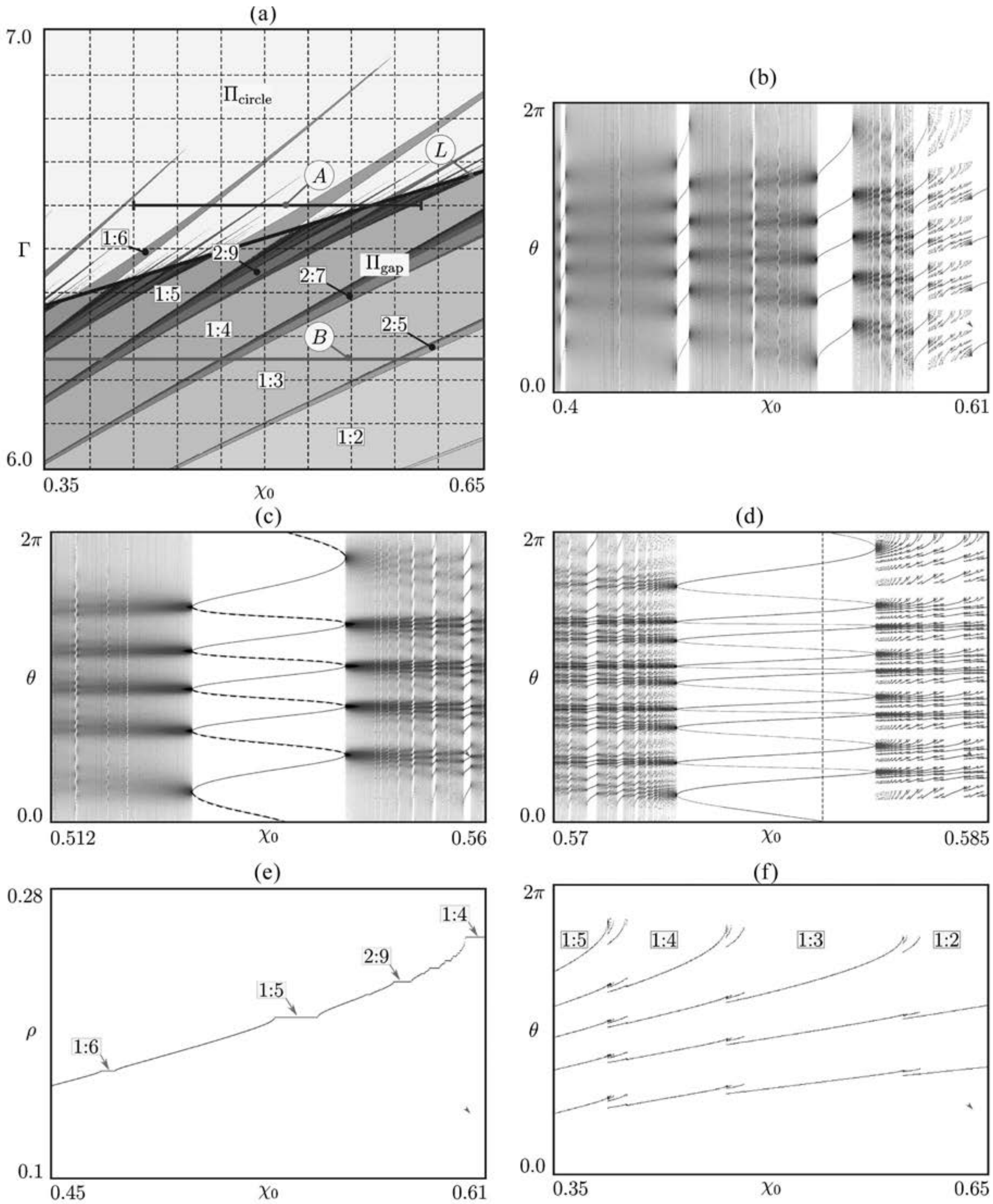

Fig. 3. (a) Bifurcation structure of the parameter plane (χ_0, Γ) of the map (3). The boundary L (7) separates the domains Π_{circle} and Π_{gap} in the parameter plane (χ_0, Γ). (b) Bifurcation diagram for the scan A in (a) $(0.4 \leqslant \chi_0 \leqslant 0.61,\ \Gamma = 6.6)$. (c) Magnified part of the bifurcation diagram in (b) near the tongue with 1:5 for Π_{circle}: $(0.512 \leqslant \chi_0 \leqslant 0.56,\ \Gamma = 6.6)$. Here the dotted lines correspond to the unstable 5-cycle and solid lines to the stable one. (d) Magnified part of the bifurcation diagram in (b) for the neighborhood of the resonance 2:9, part of which is located in the region Π_{circle}, and part—in Π_{gap}. (scan A): $0.57 \leqslant \chi_0 \leqslant 0.585$ $\ \Gamma = 6.6$. (e) Devils staircase: dependence of the rotation number r on the parameter χ_0, $0.45 \leqslant \chi_0 \leqslant 0.61$, $\Gamma = 6.6$. (f) Diagram for Π_{gap}, illustrating a sequence of the period adding bifurcations: $0.35 \leqslant \chi_0 \leqslant 0.65$, $\Gamma = 6.25$, (scan B).

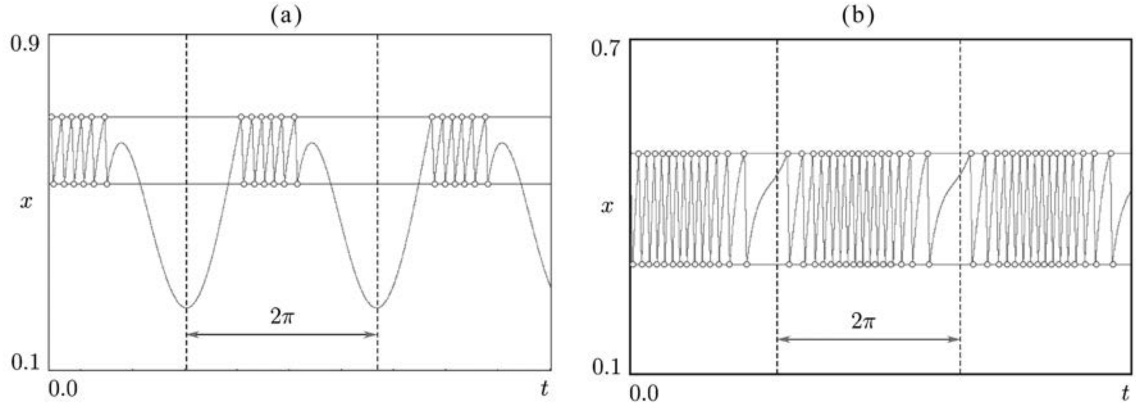

Fig. 4. (a) Periodic solution for Π_{gap}. (b) Periodic solution for Π_{circle}.

4. CONCLUSIONS

The paper was devoted to the discussion of an unusual phenomenon in a relay system with hysteresis and external periodic excitation. This phenomenon is associated with forced synchronization of relay systems and manifests itself in the occurrence of periodic dynamics close to the rhythmic activity of neurons, when packets of fast oscillations are interspersed with intervals of the slow dynamics.

To study this phenomenon, a circle mapping was introduced. Depending on the parameters, the mapping can be a circle diffeomorphizable or discontinuous map ("gap map"). It was shown that in both cases the mapping demonstrates the so-called period-adding bifurcation structure.

We found that in the Π_{gap} region, the number of packets with fast oscillations is determine in the period of periodic motion is determined by its rotation number, and the length of the intervals between packets is determined by the boundaries of the absorbing interval. The change in the number of pulses in a packet occurs through border-collision bifurcations, which in nonsmooth differential equations is called grazing bifurcations.

Due to the fact that the function F is piecewise monotonically increasing with one point of discontinuity c_0 in J (see Figs. 2a and 2b, the proof is given in [22]), Arnold tongues with rotation numbers $1:m$, $m = 2, 3, \ldots$ (m is the cycle period) do not intersect on the parameter plane [7].

The regions of periodicity corresponding to stable m cycles of higher levels of complexity located between these tongues also do not overlap. The map (3) demonstrates the typical period adding phenomenon, and there can be no bistable behavior in one-dimensional maps of the class under consideration [7, 26, 32–34].

However, it is well known that if, for example, the function F increases to the left of the point of discontinuity, and decreases to the right one, then the bistability is possible [26]. In overlapped maps ("overlapping maps" [26]) the dynamics may be more complicated associated with multistability, as in multidimensional systems (see, for example, [35–38] and the the list of references there).

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STOCHASTIC SYSTEMS

On the Transformation of a Stationary Fuzzy Random Process by a Linear Dynamic System

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Abstract—In this paper, stationary random processes with fuzzy states are studied. The properties of their numerical characteristics—fuzzy expectations, expectations, and covariance functions—are established. The spectral representation of the covariance function, the generalized Wiener–Khinchin theorem, is proved. The main attention is paid to the problem of transforming a stationary fuzzy random process (signal) by a linear dynamic system. Explicit-form relationships are obtained for the fuzzy expectations (and expectations) of input and output stationary fuzzy random processes. An algorithm is developed and justified to calculate the covariance function of a stationary fuzzy random process at the output of a linear dynamic system from the covariance function of a stationary input fuzzy random process. The results rest on the properties of fuzzy random variables and numerical random processes. Triangular fuzzy random processes are considered as examples.

Keywords: stationary random processes, fuzzy states, fuzzy expectations, covariance functions, the transformation of a fuzzy random process by a linear dynamic system

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1. INTRODUCTION

In this paper, continuous random processes with fuzzy states (fuzzy random processes) are studied. Specifically, the time variable and the set of possible fuzzy states are considered to be continuous. The cross-section of a continuous fuzzy random process at any time instant is a fuzzy random variable. We apply the well-known results of fuzzy modeling [1, 2] and the theory of fuzzy random variables [3–5] and classical results of the theory of real random processes [6, 7].

This work continues the previous research on the theory of continuous random processes with fuzzy states [8]. In particular, the properties of the fuzzy expectations, expectations, and covariance functions of continuous fuzzy random processes were studied, and the relationship between the characteristics of fuzzy random signals at the input and output of a linear dynamic system was investigated using the Green function method.

Below, we introduce and analyze stationary fuzzy random processes, proving the spectral representation of the covariance function (the generalized Wiener–Khinchin theorem). Based on this theorem, we propose and justify an algorithm for calculating the characteristics of a stationary fuzzy random process (signal) at the output of a linear dynamic system, namely, its fuzzy expectation, expectation, and covariance functions, from the corresponding characteristics of the input fuzzy random process (signal). The results obtained in this area develop to the fuzzy case the well-known ones for real continuous random processes; for example, see [6, Chapter 7; 7, Chapter VII].

Let us emphasize the difference between the approach and results of this paper and the studies on continuous-time random processes with discrete fuzzy states. For example, fuzzy queueing systems were discussed in [9–12]; stochastic fuzzy dynamic automatic control systems were considered in [13, 14]. At the same time, stationary fuzzy random processes and their covariance functions were not addressed in the publications cited above.

In what follows, a fuzzy number $\tilde{z}$ defined on the universal space R of real numbers is understood as a set of ordered pairs $(x, \mu_{\tilde{z}}(x))$, where the membership function $\mu_{\tilde{z}} : R \to [0,1]$ determines the degree (grade) of membership $\forall\, x \in R$ to the set $\tilde{z}$ [1, Chapter 5]. In this paper, the interval representation of fuzzy numbers is used [1, Chapter 5]. In this case, the α-level set of a fuzzy number $\tilde{z}$ with a membership function $\mu_{\tilde{z}}(x)$ is defined as $Z_\alpha = \{x | \mu_{\tilde{z}}(x) \geqslant \alpha\}$ $(\alpha \in (0,1])$, $Z_0 = cl\{x | \mu_{\tilde{z}}(x) > 0\}$, where cl indicates the closure of an appropriate set.

Assume that all α-levels of a fuzzy number are closed and bounded intervals of the real axis. Thus, $Z_\alpha = [z^-(\alpha), z^+(\alpha)]$, where $z^-(\alpha)$ and $z^+(\alpha)$ are the left and right α-indices of a fuzzy number, respectively.

We will consider the set J of fuzzy numbers for which the indices $z^\pm(\alpha)$ satisfy the following standard conditions:

1. $z^-(\alpha) \leqslant z^+(\alpha)$, $\forall\, \alpha \in [0,1]$.

2. The function $z^-(\alpha)$ is bounded, nondecreasing, left-continuous on the interval $(0,1]$, and right-continuous at the point 0.

3. The function $z^+(\alpha)$ is bounded, nonincreasing, left-continuous on the interval $(0,1]$, and right-continuous at the point 0.

The sum of fuzzy numbers is a fuzzy number whose indices represent the sums of the corresponding indices of the summands. Multiplication of a fuzzy number by a positive real number means multiplying its indices by the latter number. Multiplication of a fuzzy number by a negative real number means multiplying its indices by the latter number and reversing them. Equality of fuzzy numbers is understood as equality of the corresponding α-indices $(\forall\, \alpha \in [0,1])$.

A real number r is associated with a fuzzy number whose left and right α-indices coincide with r $\forall\, \alpha \in [0,1]$.

2. THE FUZZY EXPECTATIONS, EXPECTATIONS, AND COVARIANCES OF FUZZY RANDOM VARIABLES

Let (Ω, Σ, P) be a probability space, where Ω denotes the set of elementary events, Σ is a σ-algebra consisting of all subsets of the set Ω, and P is a probability measure. Consider a mapping $\tilde{X} : \Omega \to J$. For a fixed $\omega \in \Omega$, its α-level intervals $X_\alpha(\omega)$ are given by $X_\alpha(\omega) = \{r \in R : \mu_{\tilde{X}(\omega)}(r) \geqslant \alpha\}$ $\alpha \in (0,1]$, $X_0(\omega) = cl\{\mu_{\tilde{X}(\omega)}(r) > 0\}$, where $\mu_{\tilde{X}(\omega)}(r)$ means the membership function of the fuzzy number $\tilde{X}(\omega)$. An interval $X_\alpha(\omega)$ can be represented as $X_\alpha(\omega) = [X^-(\omega,\alpha), X^+(\omega,\alpha)]$, and the bounds $X^-(\omega,\alpha)$ and $X^+(\omega,\alpha)$ are called the left and right α-indices of $\tilde{X}(\omega)$, respectively.

A mapping $\tilde{X} : \Omega \to J$ is called a fuzzy random variable (FRV) if the real-valued functions $X^\pm(\omega,\alpha)$ are measurable in ω $\forall\, \alpha \in [0,1]$; for example, see [3, 4]. In this case, α-indices are real random variables $\forall\, \alpha \in [0,1]$.

We will consider the class $\mathcal{X}$ of FRVs for which the indices $X^-(\omega,\alpha)$ and $X^+(\omega,\alpha)$ are square summable on $\Omega \times [0,1]$. Let

$$x^-(\alpha) = EX^-(\omega,\alpha), \qquad x^+(\alpha) = EX^+(\omega,\alpha). \tag{1}$$

Throughout this paper, E means the expectation of a random variable, i.e., $E\xi = \int_\Omega \xi(\omega)\, dP$ for a random variable $\xi(\omega)$.

A fuzzy number with the indices given by (1) is called the fuzzy expectation of an FRV $\tilde{X}$ and denoted by $M(\tilde{X})$; its indices are denoted by $[M(\tilde{X})]^{\pm}_{\alpha}$.

The expectation $m(\tilde{X})$ of an FRV $X \in \mathcal{X}$ is defined as the mean of a fuzzy number $M(\tilde{X})$ with the α-indices $M^{\pm}(\alpha)$ given by (1):

$$m(\tilde{X}) = \frac{1}{2}\int_0^1 \Big([M(\tilde{X})]^-(\alpha) + [M(\tilde{X})]^+(\alpha)\Big)\, d\alpha. \tag{2}$$

(For details, see [15].)

The covariance of FRVs $\tilde{X}$ and $\tilde{Y}$ is defined as

$$cov(\tilde{X},\tilde{Y}) = \frac{1}{2}\int_0^1 \left(cov(X^-_\alpha, Y^-_\alpha) + cov(X^+_\alpha, Y^+_\alpha)\right) d\alpha, \tag{3}$$

and the variance of an FRV $\tilde{X}$ is defined as $D(\tilde{X}) = cov(\tilde{X},\tilde{X})$ [4]. In the expression (3), the covariances of real random variables $X^{\pm}_\alpha$ and $Y^{\pm}_\alpha$ are given by the conventional formula $cov(X^{\pm}_\alpha, Y^{\pm}_\alpha)E(X^{\pm}_\alpha - E(X^{\pm}_\alpha))(Y^{\pm}_\alpha - E(Y^{\pm}_\alpha))$ [16, Chapter 14].

The properties of the fuzzy expectations, expectations, covariances, and variances of FRVs were discussed in [4, 5, 17, Chapter 6].

3. CONTINUOUS RANDOM PROCESSES WITH FUZZY STATES

In Sections 3 and 4, we will use the notion of the limit, continuity, and differentiability of real random processes in the mean-square (m.s.) sense. Consider the Hilbert space $\mathcal{H}$ of real random variables ξ defined on a probability space (Ω, Σ, P) that have a finite second moment, i.e., $E\xi^2 < \infty$. The scalar product and norm in $\mathcal{H}$ are given by $(\xi,\eta) = E\xi\eta$ and $\|\eta\| = (\xi,\xi)^{\frac{1}{2}}$, respectively. Let $\xi(t)$ be a real random process such that $\xi(t) \in \mathcal{H}\ \forall\, t \in [t_0, T]$. For this process, m.s. continuity and m.s. differentiability are defined as the corresponding notions for functions ranging in $\mathcal{H}$ (see [7, Chapter I]).

Let $[t_0, T]$ be an extended segment of the real axis. A continuous random process with fuzzy states (a fuzzy random process, FRP) $\tilde{X}(t)$ is a mapping $\tilde{X} : [t_0, T] \to \mathcal{X}$, i.e., a function $\tilde{X}(t) = \tilde{X}(\omega, t)$ whose values are FRVs from $\mathcal{X}\ \forall\, t \in [t_0, T]$.

We denote by $X^{\pm}_\alpha(\omega, t)$ the α-indices of an FRP $\tilde{X}(\omega, t)$. Further considerations will focus on the class of FRPs for which the real functions $X^{\pm}_\alpha(\omega, t)$ are jointly square summable (square summable in the aggregate of the variables) on $\Omega \times [0,1] \times [t_0, T]$.

Let the fuzzy expectation $M(\tilde{X}(t)) = M(\tilde{X}(\omega, t))$ of an FRP $\tilde{X}(\omega, t)\ \forall\, t \in [t_0, T]$ be defined as the fuzzy expectation (1) of the corresponding FRV with the α-indices equal to

$$[M(\tilde{X}(t))]^{\pm}_\alpha = EX^{\pm}_\alpha(\omega, t), \qquad (\forall\, \alpha \in [0,1]). \tag{4}$$

The properties of the fuzzy expectations of FRVs (see [5, 18]) imply the following result.

Proposition 1. *The fuzzy expectations of FRPs possess the following properties:*

1. For a nonrandom function $\tilde{z} : [t_0, T] \to J$, $M(\tilde{z}(t)) = \tilde{z}(t)$.

2. If $\varphi : [t_0, T] \to R$ is a nonrandom scalar factor and $\tilde{X}(t)$ is an FRP, then $M(\varphi(t)\tilde{X}(t)) = \varphi(t)M(\tilde{X}(t))$.

3. For FRPs $\tilde{X}(t)$ and $\tilde{Y}(t)$, $M(\tilde{X}(t) + \tilde{Y}(t)) = M(\tilde{X}(t)) + M(\tilde{Y}(t))$.

According to (2), the expectation of an FRP $\tilde{X}(t)\ \forall\, t \in [t_0, T]$ is given by

$$m(\tilde{X}(t)) = \frac{1}{2}\int_0^1 \Big([M(\tilde{X}(t))]^-_\alpha + [M(\tilde{X}(t))]^+_\alpha\Big)\, d\alpha.$$

Example 1. Let real random processes $\xi_i(\omega, t)$ ($i = 1, 2, 3$; $\omega \in \Omega$, $t \in [t_0, T]$) be square summable on $\Omega \times [t_0, T]$ and $\xi_1(\omega, t) < \xi_2(\omega, t) < \xi_3(\omega, t)$ for all $\omega \in \Omega$, $t \in [t_0, T]$.

Consider an FRP $\tilde{X}(t)$ for which the fuzzy number $\tilde{X}(\omega, t)$ has the triangular form ($\xi_1(\omega, t)$, $\xi_2(\omega, t)$, $\xi_3(\omega, t)$) for all $\omega \in \Omega$ and $t \in [t_0, T]$. In other words, the membership function of $\tilde{X}(\omega, t)$ $\forall \omega \in \Omega$, $t \in [t_0, T]$ is described by

$$\mu_{\omega,t}(x) = \begin{cases} \dfrac{x - \xi_1(\omega, t)}{\xi_2(\omega, t) - \xi_1(\omega, t)} & \text{if } x \in [\xi_1(\omega, t), \xi_2(\omega, t)]; \\ \dfrac{x - \xi_3(\omega, t)}{\xi_2(\omega, t) - \xi_3(\omega, t)} & \text{if } x \in [\xi_2(\omega, t), \xi_3(\omega, t)]; \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the α-indices of $\tilde{X}(t)$ are

$$X^-_\alpha(t) = (1 - \alpha)\xi_1(t) + \alpha\xi_2(t), \qquad X^+_\alpha(t) = (1 - \alpha)\xi_3(t) + \alpha\xi_2(t). \tag{5}$$

Due to (4) and (5), the fuzzy expectation $M(\tilde{X}(t))$ is given by the following formulas for the α-indices:

$$[M(\tilde{X}(t))]^-_\alpha = (1 - \alpha)E\xi_1(t) + \alpha E\xi_2(t)\ (\forall\, \alpha \in [0, 1]),$$
$$[M(\tilde{X}(t))]^+_\alpha = (1 - \alpha)E\xi_3(t) + \alpha E\xi_2(t)\ (\forall\, \alpha \in [0, 1]).$$

In addition, by (2), the FRP $\tilde{X}(t)$ has the expectation

$$m(\tilde{X}(t)) = \frac{1}{4}(E\xi_1(t) + 2E\xi_2(t) + E\xi_3(t)).$$

We proceed to the notion of the covariance function of an FRP and its properties. In accordance with (3), let the covariance function of an FRP $\tilde{X}(t)$ be defined as

$$K_{\tilde{X}}(t, s) = cov(\tilde{X}(t), \tilde{X}(s)) = \frac{1}{2}\int_0^1 \Big(K_{X^-_\alpha}(t, s) + K_{X^+_\alpha}(t, s)\Big)\, d\alpha. \tag{6}$$

Here, $K_{X^-_\alpha}(t, s)$ and $K_{X^+_\alpha}(t, s)$ represent the covariance functions of the real random processes $X^-_\alpha(t)$ and $X^+_\alpha(t)$, respectively, given by

$$K_{X^\pm_\alpha}(t, s) = E\left(X^\pm_\alpha(t) - E(X^\pm_\alpha(t))\right)\left(X^\pm_\alpha(s) - E(X^\pm_\alpha(s))\right).$$

The variance of an FRP $\tilde{X}(t)$ is $D_{\tilde{X}}(t) = K_{\tilde{X}}(t, t)$.

Example 2. Within Example 1, let the random processes $\xi_1(t)$ and $\xi_2(t)$, as well as $\xi_2(t)$ and $\xi_3(t)$, be pairwise uncorrelated. Then the covariance function $K_{\tilde{X}}(t_1, t_2)$ of the FRP $\tilde{X}(t)$ of the triangular form ($\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$) is expressed through the covariance functions $K_{\xi_1}(t_1, t_2)$, $K_{\xi_2}(t_1, t_2)$, and $K_{\xi_3}(t_1, t_2)$ of the random processes $\xi_1(t)$, $\xi_2(t)$, and $\xi_3(t)$, respectively, as follows:

$$K_{\tilde{X}}(t_1, t_2) = \frac{1}{6}\left\{K_{\xi_1}(t_1, t_2) + 2K_{\xi_2}(t_1, t_2) + K_{\xi_3}(t_1, t_2)\right\}.$$

Indeed, by formula (5), assuming the uncorrelatedness of $\xi_1(t)$ and $\xi_2(t)$, for the left index $X_\alpha^-(t)$ of the triangular FRP we obtain

$$K_{X_\alpha^-}(t_1,t_2) = (1-\alpha)^2 K_{\xi_1}(t_1,t_2) + \alpha^2 K_{\xi_2}(t_1,t_2).$$

Similarly, based on the uncorrelatedness of $\xi_3(t)$ and $\xi_2(t)$,

$$K_{X_\alpha^+}(t_1,t_2) = (1-\alpha)^2 K_{\xi_3}(t_1,t_2) + \alpha^2 K_{\xi_2}(t_1,t_2).$$

Then definition (6) of the covariance function of an FRP finally leads to the desired conclusion.

According to (6) and the properties of the covariances of real random processes (see [16, Chapter 23]), we arrive at the following result.

Proposition 2. *The covariance function of an FRP $\tilde{X}(t)$ possesses the following properties:*

1. $K_{\tilde{X}}(t_1,t_2) = K_{\tilde{X}}(t_2,t_1)\ \forall\, t_1,t_2 \in [t_0,T]$ (symmetry).

2. Let $\tilde{X}(t)$ be an FRP and $\varphi(t)$ be a nonrandom real function. For an FRP $\tilde{Y}(t) = \varphi(t)\tilde{X}(t)$, we have $K_{\tilde{Y}}(t_1,t_2) = \varphi(t_1)\varphi(t_2)K_{\tilde{X}}(t_1,t_2)$ if $\varphi(t_1)\varphi(t_2) \geqslant 0$.

3. If $\tilde{Y}(t) = \tilde{X}(t) + \varphi(t)$, then $K_{\tilde{Y}}(t_1,t_2) = K_{\tilde{X}}(t_1,t_2)$.

4. $|K_{\tilde{X}}(t_1,t_2)| \leqslant \sqrt{D_{\tilde{X}}(t_1)D_{\tilde{X}}(t_2)}$.

An FRP $\tilde{X}(\omega,t)$ with the α-intervals $[X_\alpha^-(\omega,t), X_\alpha^+(\omega,t)]$ is said to be continuous at a point t if all its α-indices $X_\alpha^\pm(\omega,t)$ are continuous in t as scalar random processes in the mean-square sense [8].

An FRP $\tilde{X}(\omega,t)$ with the α-indices $X_\alpha^-(\omega,t)$ and $X_\alpha^+(\omega,t)$ is said to be Seikkala differentiable at a point t if all its α-indices are differentiable with respect to t as scalar random processes in the mean-square sense and the derivatives $\frac{\partial}{\partial t}X_\alpha^-(\omega,t)$ and $\frac{\partial}{\partial t}X_\alpha^+(\omega,t)$ are the lower and upper α-indices of some FRV, respectively, which is called the derivative at the point t [8]. (Compare this definition with the one for a fuzzy-valued function [19].) In this case, the t-derivative of an FRP $\tilde{X}(t)$ will be denoted by $\tilde{X}'(t) = \frac{\partial}{\partial t}\tilde{X}(\omega,t)$.

The second and higher derivatives are sequentially defined in a conventional way.

Remark 1. By the definition of an FRP and the arithmetic properties of fuzzy numbers in the interval form, the differentiation of FRPs is a linear operation: the derivative of the sum (difference) of FRPs equals the sum (difference) of the corresponding derivatives, and a constant multiplier can be factored outside the derivative sign.

Remark 2. The derivative of an FRV (a constant) coincides with the fuzzy number whose left and right α-indices are zero $\forall\,\alpha \in [0,1]$.

The next theorem concerns the fuzzy expectation of the derivative of an FRP.

Theorem 1. *Let an FRP $\tilde{X}(t)$ be differentiable in a domain $t \in (t_0,T)$. Then the derivative of the fuzzy expectation of the FRP $\tilde{X}(t)$ is well defined, and the fuzzy expectation of its derivative equals the derivative of its fuzzy expectation:*

$$M\tilde{X}'(t) = (M\tilde{X}(t))'. \tag{7}$$

Proof. According to the definitions of the derivative of an FRP and the fuzzy expectation of an FRP, we write

$$[M(\tilde{X}'(t))]_\alpha^\pm = E[\tilde{X}'(t)]_\alpha^\pm = E(X_\alpha^\pm)'(t) = (EX_\alpha^\pm(t))'.$$

The latter equality follows from the corresponding property of real random processes [6, Chapter 6]. Then, using the definition of the Seikkala derivative of the fuzzy-valued function $M\tilde{X}(t)$ [19] and the interval sign of equality of fuzzy numbers, we obtain (7).

Let us emphasize that (7) is the equality of fuzzy-valued functions. Note that Theorem 1 somewhat strengthens the result from [8].

Corollary 1. *Under the hypotheses of Theorem 1, the expectations of an FRP and its derivative satisfy the relation*

$$m\tilde{X}'(t) = (m\tilde{X}(t))'. \tag{8}$$

In addition, the following result is valid for the covariance function of the derivative of an FRP.

Theorem 2 [8]. *Let the covariance functions of the α-indices $X_\alpha^\pm(t)$ of an FRP $\tilde{X}(t)$ have well-defined second derivatives $\frac{\partial^2 K_{X_\alpha^-}(t,s)}{\partial t \partial s}$ and $\frac{\partial^2 K_{X_\alpha^+}(t,s)}{\partial t \partial s}$ jointly continuous in the variables t, s, and α. Then the covariance function $K_{\tilde{X}'}(t,s)$ of the derivative $\tilde{X}'(t)$ of the FRP $\tilde{X}(t)$ is given by*

$$K_{\tilde{X}'}(t,s) = \frac{\partial^2 K_{\tilde{X}}(t,s)}{\partial t \partial s}. \tag{9}$$

4. STATIONARY FUZZY RANDOM PROCESSES

As is well known, a real random process $\xi(t)$ with $E|\xi(t)|^2 < \infty$, $t \in [0,\infty)$, is said to be stationary (in the wide sense) if it has a constant expectation $E\xi(t) = a$ and the covariance function $E[\xi(t) - a][\xi(s) - a] = K_\xi(t - s)$ that depends only on the difference of the arguments; for details, see [6, Chapter 7; 7, Chapter VII].

We say that an FRP $\tilde{X}(t)$, $t \in [0,\infty)$, is stationary if its α-indices $\forall\,\alpha \in [0,1]$ are real stationary random processes.

Example 3. Within Example 2, let all the random processes $\xi_j(t)$ $(j = 1,2,3;\ t \in [0,\infty))$ be stationary. Then the triangular FRP $\tilde{X}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ is stationary.

This fact follows from the expressions for the fuzzy expectations and covariance functions of the triangular FRP $\tilde{X}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ derived in Examples 1 and 2.

Theorem 3. *Let $\tilde{X}(t)$, $t \in [0,\infty)$, be a stationary FRP. Then its fuzzy expectation $M(\tilde{X}(t))$ and expectation $m(\tilde{X}(t))$ are constant, and the covariance function $K_{\tilde{X}}(t_1,t_2) = K_{\tilde{X}}(t_2 - t_1)$ depends on the difference $(t_2 - t_1) = \tau$ of the arguments.*

Proof. Assume that $\tilde{X}(t)$ is a stationary FRP. We denote by $m_\alpha^\pm$ the constant expectations of the α-indices $X_\alpha^\pm(t)$ of the FRP $\tilde{X}(t)$ $\forall\,\alpha \in [0,1]$. By definition (3), they are the α-indices of the fuzzy expectation $M_\alpha^\pm = m_\alpha^\pm$. Then the fuzzy expectation $M(\tilde{X}(t))$ is constant, and the expectation $m(\tilde{X}(t)) = \frac{1}{2}\int_0^1 (m_\alpha^+ + m_\alpha^-)d\alpha$ is constant as well.

We denote by $K_{X_\alpha^\pm}(t_1,t_2)$ the covariance functions of the α-indices, i.e., those of the real random processes $X_\alpha^\pm(t)$. According to the assumption, $X_\alpha^\pm(t)$ is a stationary random process; hence, $K_{X_\alpha^\pm}(t_1,t_2) = K_{X_\alpha^\pm}(t_2 - t_1)$. In this case, by definition (6), the covariance function $K_{\tilde{X}}(t_1,t_2)$ of the FRP $\tilde{X}(t)$ depends on the difference $(t_2 - t_1) = \tau$ of the arguments.

Theorem 4. *The covariance function $K_{\tilde{X}}(\tau)$ of a stationary FRP $\tilde{X}(t)$ possesses the following properties:*

1. $K_{\tilde{X}}(\tau) = K_{\tilde{X}}(-\tau)$ (evenness).

2. The variance of the stationary FRP $\tilde{X}(t)$ is constant and equals $D_{\tilde{X}} = K_{\tilde{X}}(0)$.

3. $|K_{\tilde{X}}(\tau)| \leqslant K_{\tilde{X}}(0)$ $(\forall\,\tau \in R)$.

Theorem 4 follows from the corresponding properties of the covariance functions $K_{X_\alpha^-}(\tau)$ and $K_{X_\alpha^+}(\tau)$ of the α-indices (see [16, Chapter 24]) and the representation (6).

For stationary FRPs, Theorem 2 can be refined as follows.

Theorem 5. *Under the conditions of Theorem 2, the covariance function of the derivative $\tilde{X}'(t)$ of a differentiable stationary FRP $\tilde{X}(t)$ equals the second derivative of its covariance function taken with the minus sign:* $K_{\tilde{X}'}(\tau) = -K''_{\tilde{X}}(\tau)$.

Proof. Due to formula (9), $\forall\, t_1, t_2 \in [0, \infty)$ we have

$$K_{\tilde{X}'}(t_1, t_2) = \frac{\partial^2 K_{\tilde{X}}(t_1, t_2)}{\partial t_1 \partial t_2}.$$

By the assumption, $\tilde{X}(t)$ is a stationary FRP. According to Theorem 3, its covariance function depends on the difference of the arguments: $K_{\tilde{X}}(t_1, t_2) = K_{\tilde{X}}(\tau)$, where $\tau = (t_2 - t_1)$. Consequently,

$$K_{\tilde{X}'}(t_1, t_2) = \frac{\partial^2 K_{\tilde{X}}(\tau)}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_1}\left(\frac{\partial K_{\tilde{X}}(\tau)}{\partial t_2}\right) = \frac{\partial}{\partial t_1}\left(\frac{\partial K_{\tilde{X}}(\tau)}{\partial \tau}\frac{\partial \tau}{\partial t_2}\right)$$
$$= \frac{d^2 K_{\tilde{X}}(\tau)}{d\tau^2}\frac{\partial \tau}{\partial t_1} = K''_{\tilde{X}}(\tau)(-1) = -K''_{\tilde{X}}(\tau).$$

In these formulas, we have utilized the equalities $\frac{\partial \tau}{\partial t_1} = -1$ and $\frac{\partial \tau}{\partial t_2} = 1$. Thus, the covariance function of the FRP $\tilde{X}'(t)$ depends only on the difference of the arguments, and the desired conclusion follows.

Proposition 3. *The derivative $\tilde{X}'(t)$ of a stationary differentiable FRP $\tilde{X}(t)$, $t \in [0, \infty)$, is a stationary FRP.*

Indeed, by the condition, the α-indices $X_\alpha^\pm(t)$, $t \in [0, \infty)$, of the FRP $\tilde{X}(t)$ are real stationary random processes $\forall\, \alpha \in [0, 1]$. Then, based on the well-known property of real stationary processes [6, Chapter 7], their derivatives $(X_\alpha^\pm)'(t)$ are such as well. Therefore, Proposition 3 is immediate from the definition of the differentiability of FRPs.

Proposition 4. *Let $\tilde{Y}(t)$, $t \in [0, \infty)$, be a stationary FRP k times differentiable on $(0, \infty)$ and at least one of given constants $b_s \geqslant 0$ $(s = 0, 1, \ldots, k)$ be nonzero. Then the linear combination of the derivatives, $\tilde{Z}(t) = \sum_{s=0}^{k} b_s \tilde{Y}^{(s)}(t)$, is a stationary FRP.*

Indeed, under the hypotheses of Proposition 4, the left and right indices $Z_\alpha^\pm(t)$ of the FRP $\tilde{Z}(t)$ $\forall\, \alpha \in [0, 1]$ have the form

$$Z_\alpha^\pm(t) = \left[\sum_{s=0}^{k} b_s (\tilde{Y})^{(s)}(t)\right]_\alpha^\pm = \sum_{s=0}^{k} b_s (Y_\alpha^\pm(t))^{(s)}.$$

Here, we have utilized the definition of the derivatives of FRPs and the properties of the arithmetical operations over fuzzy numbers in the interval form.

Since $Y_\alpha^\pm(t)$ are stationary real processes, by the well-known result for such processes [6, Chapter 7], $Z_\alpha^\pm(t)$ are stationary real random processes, and Proposition 4 is proved accordingly.

5. THE SPECTRAL DENSITY OF A STATIONARY FRP. THE GENERALIZED WIENER–KHINCHIN THEOREM

Consider the spectral representation problem of the covariance function of a stationary FRP.

The following result is well-known for a real stationary random process $\xi(t)$ defined on an infinite time interval $[0, \infty)$; for example, see [6, Chapter 7; 7, Chapter VII].

Lemma 1 (the Wiener–Khinchin theorem). *The covariance function $K_\xi(\tau)$ and spectral density $S_\xi(\tau)$ of a real stationary random process $\xi(t)$ are related by the self-reciprocal inverse Fourier* *cosine transforms:*

$$K_\xi(\tau) = \int_0^\infty S_\xi(\omega)\cos\omega\tau d\omega, \quad S_\xi(\omega) = \frac{2}{\pi}\int_0^\infty K_\xi(\tau)\cos\omega\tau d\tau. \tag{10}$$

Remark 3 [7, Chapter VII.] The existence of the spectral density $S_\xi(\omega)$ of a real stationary random process $\xi(t)$ and the relations (10) are ensured, e.g., by the continuity of the covariance function $K_\xi(\tau)$ of the process $\xi(t)$ and its summability on $(0,\infty)$ (i.e., $\int_0^\infty |K_\xi(\tau)|d\tau < \infty$).

Now we generalize Lemma 1 to the case of stationary FRPs. Let a stationary FRP $\tilde{X}(t)$ defined on $[0,\infty)$ have the α-indices $X_\alpha^\pm(t)$, and let $S_{X_\alpha^\pm}(\omega)$ be the spectral densities of the stationary random processes $X_\alpha^\pm(t)$ ($\forall\,\alpha \in [0,1]$) such that the functions $S_{X_\alpha^\pm}(\omega)$ are summable in α on $[0,1]$. We call the function

$$S_{\tilde{X}}(\omega) = \frac{1}{2}\int_0^1 \left(S_{X_\alpha^+}(\omega) + S_{X_\alpha^-}(\omega)\right) d\alpha \tag{11}$$

the spectral density of a stationary FRP $\tilde{X}(t)$.

Example 4. Within Example 3, we denote by $S_{\xi_i}(\omega)$ the spectral densities of the real stationary random processes $\xi_i(t)$ $(i = 1,2,3)$. Then the spectral density of the stationary FRP $\tilde{X}(t)$ of the triangular form $(\xi_1(t), \xi_2(t), \xi_3(t))$ is given by

$$S_{\tilde{X}}(\omega) = \frac{1}{6}\left(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)\right).$$

Indeed, let us denote by $K_{\xi_i}(\tau)$ the covariance functions of the real random processes $\xi_i(t)$. Due to (5) and the pairwise uncorrelatedness of the random processes $\xi_1(t)$ and $\xi_2(t)$, as well as of $\xi_2(t)$ and $\xi_3(t)$, for the covariance functions of the α-indices $X_\alpha^\pm(t)$ of the FRP $\tilde{X}(t)$ we write

$$K_{X_\alpha^-}(\tau) = (1-\alpha)^2 K_{\xi_1}(\tau) + \alpha^2 K_{\xi_2}(\tau),$$
$$K_{X_\alpha^+}(\tau) = (1-\alpha)^2 K_{\xi_3}(\tau) + \alpha^2 K_{\xi_2}(\tau).$$

Then, based on formulas (10) for the spectral densities of the real stationary random processes $X_\alpha^\pm$, it follows that

$$S_{X_\alpha^-}(\omega) = (1-\alpha)^2 S_{\xi_1}(\omega) + \alpha^2 S_{\xi_2}(\omega),$$
$$S_{X_\alpha^+}(\omega) = (1-\alpha)^2 S_{\xi_3}(\omega) + \alpha^2 S_{\xi_2}(\omega).$$

Therefore, according to (11), the spectral density of the FRP $\tilde{X}(t)$ of the triangular form $(\xi_1(t), \xi_2(t), \xi_3(t))$ is given by

$$S_{\tilde{X}}(\omega) = \frac{1}{2}\left(\int_0^1 (1-\alpha)^2 d\alpha S_{\xi_1}(\omega) + \int_0^1 \alpha^2 d\alpha S_{\xi_2}(\omega) + \int_0^1 (1-\alpha)^2 d\alpha S_{\xi_3}(\omega) + \int_0^1 \alpha^2 d\alpha S_{\xi_2}(\omega)\right)$$
$$= \frac{1}{6}\left(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)\right),$$

which finally implies Proposition 4.

The next result is valid by definition (11) and the properties of the spectral densities of real stationary random processes (see [16, Chapter 24]).

Proposition 5. *The spectral density of a stationary FRP possesses the following properties:*

1. The spectral density of a stationary FRP $\tilde{X}(t)$ is nonnegative, i.e., $S_{\tilde{X}}(\omega) \geqslant 0$.

2. Integrating the spectral density of a stationary FRP $\tilde{X}(t)$ over ω between zero and infinity gives the variance of the FRP $\tilde{X}(t)$, i.e., $\int_0^\infty S_{\tilde{X}}(\omega)d\omega = D_{\tilde{X}}$.

The appropriateness of the above definition (11) is confirmed by the generalized Wiener–Khinchin theorem established below.

Theorem 6. *Let $\tilde{X}(t)$, $t \in [0, \infty)$, be a stationary FRP such that the covariance function $K_{X_\alpha^\pm}(\tau)$ and spectral density $S_{X_\alpha^\pm}(\omega)$ are well-defined for its α-indices $X_\alpha^\pm(t)$ for any $\alpha \in [0,1]$ and, moreover, are jointly summable on $[0,\infty) \times [0,1]$. Then the covariance function and spectral density of the stationary FRP $\tilde{X}(t)$ are related by the self-reciprocal inverse Fourier cosine transforms:*

$$K_{\tilde{X}}(\tau) = \int_0^\infty S_{\tilde{X}}(\omega) \cos \omega\tau d\omega, \qquad S_{\tilde{X}}(\omega) = \frac{2}{\pi} \int_0^\infty K_{\tilde{X}}(\tau) \cos \omega\tau d\tau. \tag{12}$$

Proof. We begin with deriving the first formula in (12). For stationary real processes $X_\alpha^\pm(t)$, by Lemma 1, $K_{X_\alpha^-}(\tau) = \int_0^\infty S_{X_\alpha^-}(\omega) \cos \omega\tau d\omega$ and $K_{X_\alpha^+}(\tau) = \int_0^\infty S_{X_\alpha^+}(\omega) \cos \omega\tau d\omega$. Summing both sides of these equalities and integrating the resulting expression over α between 0 and 1 yield $\int_0^1 (K_{X_\alpha^-}(\tau) + K_{X_\alpha^+}(\tau))d\alpha = \int_0^1 \int_0^\infty (S_{X_\alpha^-}(\omega) + S_{X_\alpha^+}(\omega)) \cos \omega\tau d\omega d\alpha$. Changing the order of integration on the right-hand side (based on Fubini's theorem) and using (6) and (11), we finally arrive at the desired formula for $K_{\tilde{X}}(\tau)$.

In addition, according to Lemma 1, $S_{X_\alpha^\pm}(\omega) = \frac{2}{\pi} \int_0^\infty K_{X_\alpha^\pm}(\tau) \cos \omega\tau d\tau$. In view of (6) and (11), similar considerations as above lead to the second formula in (12).

Example 5. Within Example 4, by Theorem 6, the covariance function $K_{\tilde{X}}(t)$ of the triangular stationary fuzzy random process $\tilde{X}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ is given by

$$K_{\tilde{X}}(\tau) = \frac{1}{6} \int_0^\infty (S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)) \cos \omega\tau d\omega = \frac{1}{6} \left\{ K_{\xi_1}(\tau) + 2K_{\xi_2}(\tau) + K_{\xi_3}(\tau) \right\},$$

where $K_{\xi_i}(\tau)$ $(i = 1, 2, 3)$ denote the covariance functions of the real stationary random processes $\xi_i(t)$.

Note that this result agrees with Example 2.

6. ON THE TRANSFORMATION OF A STATIONARY FRP BY A LINEAR DYNAMIC TIME-INVARIANT SYSTEM

Suppose that a random signal $\xi(t)$ is supplied to the input of some device, and a random signal $\eta(t)$ is accordingly observed at its output. The device is called a linear dynamic time-invariant system of the nth order if the output $\eta(t)$ and input $\xi(t)$ are related by a linear differential equation of the nth order with constant coefficients of the form

$$\begin{aligned} & a_n \eta^{(n)}(t) + a_{n-1}\eta^{(n-1)}(t) + \cdots + a_1 \eta'(t) + a_0 \eta(t) \\ &= b_k \xi^{(k)}(t) + b_{k-1}\xi^{(k-1)}(t) + \cdots + b_1 \xi'(t) + b_0 \xi(t) \quad (t > 0). \end{aligned} \tag{13}$$

Here, a_j $(j = 0, 1, \ldots, n)$ and b_s $(s = 0, 1, \ldots, k)$ are real numbers, and $k < n$.

If the dynamic system (13) is asymptotically stable and a real stationary random process $\xi(t)$ is supplied to its input, then the output random process $\eta(t)$ can be considered stationary for sufficiently large values of t, i.e., after some transient period. The problem of establishing the relationship between the numerical characteristics (expectations and covariance functions) of the input and output real stationary random signals of the dynamical system (13) is widely known in the literature; for example, see [6, Chapter 7].

Suppose that a stationary FRP $\tilde{Y}(t)$ is supplied to the input of the dynamic system (13) and a stationary FRP $\tilde{X}(t)$ is accordingly observed at its output.

It is required to calculate the characteristics of the output stationary FRP $\tilde{X}(t)$ from the known characteristics of the input stationary FRP $\tilde{Y}(t)$ of the dynamic system (13).

Consider the problem of calculating the constant fuzzy expectation $M\tilde{X}$ (or the expectation $m\tilde{X}$) at the output of system (13) from the known constant fuzzy expectation $M\tilde{Y}$ (or the expectation $m\tilde{Y}$) at its input.

Proposition 6. *Let a stationary k times differentiable FRP $\tilde{Y}(t)$, $t \in [0, \infty)$, be supplied to the input of the dynamic system* (13) *and a stationary n times continuously differentiable FRP $\tilde{X}(t)$, $t \in [0, \infty)$, be accordingly observed at its output. Then*

$$M\tilde{X} = \frac{b_0}{a_0} M\tilde{Y}, \qquad m\tilde{X} = \frac{b_0}{a_0} m\tilde{Y}.$$

Indeed, by the condition,

$$\begin{aligned} &a_n\tilde{X}^{(n)}(t) + a_{n-1}\tilde{X}^{(n-1)}(t) + \dots + a_1\tilde{X}'(t) + a_0\tilde{X}(t) \\ &= b_k\tilde{Y}^{(k)}(t) + b_{k-1}\tilde{Y}^{(k-1)}(t) + \dots + b_1\tilde{Y}'(t) + b_0\tilde{Y}(t) \ (t > 0). \end{aligned} \tag{14}$$

We equate the fuzzy expectations of the left- and right-hand sides of (14). Based on the algebraic properties of fuzzy expectations (Proposition 1) and the properties of the derivative of fuzzy expectations (Theorem 1), it follows that

$$\begin{aligned} &a_n(M\tilde{X})^{(n)}(t) + a_{n-1}(M\tilde{X})^{(n-1)}(t) + \dots + a_1(M\tilde{X})'(t) + a_0M\tilde{X}(t) \\ &= b_k(M\tilde{Y})^{(k)}(t) + b_{k-1}(M\tilde{Y})^{(k-1)}(t) + \dots + b_1(M\tilde{Y})'(t) + b_0M\tilde{Y}(t). \end{aligned} \tag{15}$$

Since the fuzzy expectations of the stationary FRPs $\tilde{Y}(t)$ and $\tilde{X}(t)$ are constant, their derivatives of any order equal a fuzzy number whose left and right indices are all zero (see Remark 2). Then (15) entails the equality

$$a_0M\tilde{X} = b_0M\tilde{Y},$$

and, consequently, $M\tilde{X} = \frac{b_0}{a_0}M\tilde{Y}$. By analogy, we establish $m\tilde{X} = \frac{b_0}{a_0}m\tilde{Y}$ for their expectations.

Next, using the known covariance function $K_{\tilde{Y}}(\tau)$ of the input stationary FRP $\tilde{Y}(t)$ of the dynamic system (14), we find the covariance function $K_{\tilde{X}}(\tau)$ and variance $D_{\tilde{X}}$ of its output stationary FRP $\tilde{X}(t)$.

Theorem 7. *Let the coefficients of the dynamic system* (14) *be nonnegative, i.e., $a_j \geqslant 0$ ($j = 0, 1, \dots, n$), $b_s \geqslant 0$ ($s = 0, 1, \dots, k$). Let a k times continuously differentiable FRP $\tilde{Y}(t)$, $t \in [0, \infty)$, be supplied to the input of the dynamic system* (14), *and let the covariance functions $K_{Y_\alpha^\pm}(\tau)$ and spectral densities $S_{Y_\alpha^\pm}(\omega)$ be well defined for its α-indices $Y_\alpha^\pm(t)$ for any $\alpha \in [0, 1]$ and, moreover, be jointly summable on $[0, \infty) \times [0, 1]$.*

Let an n times continuously differentiable FRP $\tilde{X}(t)$, $t \in [0, \infty)$, be accordingly observed at the output of the dynamic system (14). *Then the algorithm for calculating (in real form) the covariance function $K_{\tilde{X}}(\tau)$ of the output stationary FRP $\tilde{X}(t)$ includes the following stages:*

1) Find the spectral density $S_{\tilde{Y}}(\omega)$ *of the input stationary FRP* $\tilde{Y}(t)$ *from its covariance function* $K_{\tilde{Y}}(\tau)$ *using the generalized Wiener–Khinchin formula* (12):

$$S_{\tilde{Y}}(\omega) = \frac{2}{\pi}\int\limits_0^\infty K_{\tilde{Y}}(\tau)\cos\omega\tau d\tau. \tag{16}$$

2) Find the frequency response $\Phi(i\omega)$ *for the differential equation* (14):

$$\Phi(i\omega) = \frac{b_k(i\omega)^k + \dots + b_1(i\omega) + b_0}{a_n(i\omega)^n + \dots + a_1(i\omega) + a_0}, \tag{17}$$

where i *indicates the imaginary unit.*

3) Find the spectral density $S_{\tilde{X}}(\omega)$ *of the output stationary FRP* $\tilde{X}(t)$ *from the spectral density* $S_{\tilde{Y}}(\omega)$ *of the input stationary FRP* $\tilde{Y}(t)$ *and the squared absolute value* $|\Phi(i\omega)|^2$ *of the frequency response* (17):

$$S_{\tilde{X}}(\omega) = |\Phi(i\omega)|^2 S_{\tilde{Y}}(\omega). \tag{18}$$

4) Find the covariance function $K_{\tilde{X}}(\tau)$ *and (or) variance* $D_{\tilde{X}}$ *of the output stationary FRP* $\tilde{X}(t)$ *from its spectral density* $S_{\tilde{X}}(\omega)$ (18) *using the generalized Wiener–Khinchine formula* (12):

$$K_{\tilde{X}}(\tau) = \int\limits_0^\infty S_{\tilde{X}}(\omega)\cos\omega\tau d\omega, \qquad D_{\tilde{X}} = \int\limits_0^\infty S_{\tilde{X}}(\omega)d\omega. \tag{19}$$

Proof. Due to equation (14), the definition of derivatives for FRPs, the assumption on the positive coefficients a_k and b_s, and the definition of summation for fuzzy numbers in the interval form, for the α-indices $X_\alpha^\pm(t)$ and $Y_\alpha^\pm(t)$ of the FRPs $\tilde{X}(t)$ and $\tilde{Y}(t)$, respectively, we have

$$\begin{aligned} &a_n(X_\alpha^\pm)^{(n)}(t) + a_{n-1}(X_\alpha^\pm)^{(n-1)}(t) + \dots + a_1(X_\alpha^\pm)'(t) + a_0X_\alpha^\pm(t) \\ &= b_k(Y_\alpha^\pm)^{(k)}(t) + b_{k-1}(Y_\alpha^\pm)^{(k-1)}(t) + \dots + b_1(Y_\alpha^\pm)'(t) + b_0Y_\alpha^\pm(t) \quad (t > 0). \end{aligned} \tag{20}$$

By the hypothesis of this theorem, the α-indices $Y_\alpha^\pm(t)$ and $X_\alpha^\pm(t)$ are real stationary random processes for $t \in [0,\infty)$. For each pair of real stationary processes $Y_\alpha^\pm(t)$ and $X_\alpha^\pm(t)$ related by the dynamic system (20), we apply the well-known algorithm consisting of Stages 1)–4) (see [6, Chapter 7]).

First, for any $\alpha \in [0,1]$, we find the spectral density $S_{Y_\alpha^\pm}(\omega)$ of the input random process $Y_\alpha^\pm(t)$ of the dynamic system (20) from its covariance function $K_{Y_\alpha^\pm}(\tau)$ using (10):

$$S_{Y_\alpha^\pm}(\omega) = \frac{2}{\pi}\int\limits_0^\infty K_{Y_\alpha^\pm}(\tau)\cos\omega\tau d\tau.$$

Next, using the frequency response $\Phi(i\omega)$, we calculate the spectral densities of the output stationary real random processes $X_\alpha^\pm(t)$ by the formulas

$$S_{X_\alpha^\pm}(\omega) = |\Phi(i\omega)|^2 S_{Y_\alpha^\pm}(\omega).$$

In view of (11), this result implies (18).

We proceed with determining the covariance functions

$$K_{X_\alpha^\pm}(\tau) = \int_0^\infty S_{X_\alpha^\pm}(\omega)\cos\omega\tau d\omega$$

of the output real stationary random processes $X_\alpha^\pm(t)$ of the dynamic system (20).

After that, we apply the above definitions of the covariance function (6) and spectral density (11) of the stationary FRPs:

$$K_{\tilde{X}}(\tau) = \frac{1}{2}\int_0^1 (K_{X_\alpha^-}(\tau) + K_{X_\alpha^+}(\tau))d\alpha = \frac{1}{2}\int_0^1 \left(\int_0^\infty (S_{X_\alpha^+}(\omega) + S_{X_\alpha^-}(\omega))\cos\omega\tau d\omega\right) d\alpha.$$

Changing the order of integration based on Fubini's theorem gives

$$K_{\tilde{X}}(\tau) = \int_0^\infty \left(\frac{1}{2}\int_0^1 (S_{X_\alpha^+}(\omega) + S_{X_\alpha^-}(\omega))d\alpha\right)\cos\omega\tau d\omega = \int_0^\infty S_{\tilde{X}}(\omega)\cos\omega\tau d\omega.$$

Thus, the first formula in (19) has been derived. The formula for the variance $D_{\tilde{X}}$ follows from the former one since $D_{\tilde{X}} = K_{\tilde{X}}(0)$.

Remark 4. Equation (20) can be interpreted as an equation in the Hilbert space $\mathcal{H}$ of random variables with a finite second moment. If the real parts of all roots of the characteristic equation $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0$ are negative, then equation (20) is asymptotically Lyapunov stable in the space $\mathcal{H}$ (see [21, Chapter II]).

Remark 5. Equation (15) is essentially a fuzzy differential equation. Such equations were considered in [19, 22]. Equation (14) is a fuzzy random equation; for details, see [23–25].

Example 6. Let an FRP of the triangular form $\tilde{Y}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$ be supplied to the input of the linear dynamic system (13), where the real random processes $\xi_i(t)$ $(i = 1, 2, 3)$ satisfy the conditions of Example 4. Then, according to Example 3, the FRP $\tilde{Y}(t)$ is stationary. Due to Example 2, its covariance function has the form

$$K_{\tilde{Y}}(\tau) = \frac{1}{6}\{K_{\xi_1}(\tau) + 2K_{\xi_2}(\tau) + K_{\xi_3}(\tau)\},$$

where $K_{\xi_i}(\tau)$ $(i = 1, 2, 3)$ are the covariance functions of the real random processes $\xi_i(t)$.

Moreover, according to Example 4, the spectral density of the input stationary triangular FRP $\tilde{Y}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))S_{\tilde{Y}}(\omega)$ of the dynamic system (13) is given by

$$S_{\tilde{Y}}(\omega) = \frac{1}{6}(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)),$$

where $S_{\xi_i}(\omega)$ $(i = 1, 2, 3)$ are the spectral densities of the real random processes $\xi_i(t)$.

Finally, using Theorem 7, we find the spectral density of the output stationary FRP $\tilde{X}(t)$ of the dynamic system (13):

$$S_{\tilde{X}}(\omega) = |\Phi(i\omega)|^2 S_{\tilde{Y}}(\omega) = \frac{1}{6}|\Phi(i\omega)|^2\{S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega)\},$$

where $\Phi(i\omega)$ is the frequency response of system (13). By Theorem 7, the covariance function $K_{\tilde{X}}(\tau)$ of the output FRP $\tilde{X}(t)$ of the dynamic system has the form

$$K_{\tilde{X}}(\tau) = \frac{1}{6}\int_0^\infty |\Phi(i\omega)|^2(S_{\xi_1}(\omega) + 2S_{\xi_2}(\omega) + S_{\xi_3}(\omega))\cos\omega\tau d\omega.$$

7. CONCLUSIONS

Theorems 3–7, as well as Propositions 3–6, form the essential content and scientific contribution of this paper. We emphasize the significance of the concept of the spectral density of a stationary FRP (Theorem 6) and the algorithm for calculating the covariance function of a stationary FRP at the output of a dynamic system from the covariance function of a stationary FRP at its input (Theorem 7). Examples 1–6 have illustrated the applicability of the novel theory to triangular FRPs.

The results of this work can be developed in the following directions:

1. They will remain valid if the covariance of fuzzy random variables from [5] is used instead of definition (3).

2. As is known [7], the Wiener–Khinchin theorem for real random processes (Lemma 1) can be written in a more general form: for this purpose, the spectral distribution function and the Riemann–Stiltjes integral should be considered instead of the spectral density and the Riemann integral.

The generalized Wiener–Khinchin theorem (Theorem 6) can be developed in this direction for stationary fuzzy random processes.

3. It is possible to extend some results of this paper to the case of generalized fuzzy numbers (see [26]).

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Self-Regulating Dynamic Measurement Method

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Abstract—A dynamic measuring system is considered. A new model of the measuring system, a method for processing of measurement results and a method of restoring the input signal of the system by the noisy output signal are proposed. The estimation of the accuracy of the method and a computational experiment demonstrating the efficiency of the signal recovery method are presented.

Keywords: measurement results processing, dynamic measurements, self-regulation, signal recovery

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1. INTRODUCTION

The efficiency of technological processes is directly related to ensuring the optimal parameters of process control systems. The accuracy of compliance with and optimization of the parameters depend on the accuracy of processing of data on the system status. In case of energy-intensive and high-speed technological processes, the system status changes within a short period of time. Due to the output signal's noise pollution and the time lag in the measuring system, we need information about the input signal in order to learn the actual system status. Another factor, which has a significant influence on the accuracy of processing the system status data, is the output signal's noise pollution. The noise problem, combined with the time lag in the measuring system, becomes especially serious during processing the dynamic measuring performed within a short period of time, when even slight noise in the input data can lead to severe misrepresentation of the data processing results. Many researchers have written their works on the problem of processing noise-contaminated dynamic signals. Among the works on this research topic we should highlight an approach based on using the control theory [1]. In this work, a model of a measuring system with modal control of the dynamic characteristics is proposed; and in work [2], a number of methods for dynamic error correction, based on the application of the automatic control theory, is proposed. Another approach to processing of noise-contaminated dynamic measurements implies the creation of engineering solutions. This field of research includes works by V.A. Granovskiy [3, 4], who proposed to use test signals for the correction of noise in dynamic measurements. The work by S. Engelberg [5], which implies the introduction of additional filters to reduce the negative effects from noise, as well as the work [6], describe a new approach to data collection. A separate field of research on the processing of noise-contaminated dynamic signals is related to the use of the theory and methods of inverse problems solving. In this field, works were written by G.N. Solopchenko [7–9] (who proposed to use the A.N. Tikhonov regularization methods for dynamic measurements), the work by A.F. Verlan [10] (where the problem of processing of noise-contaminated information is reduced to solving Fredholm equations of the first kind), as well as the work by A. Forbes [11] (where the problem of processing of dynamic measurements is presented as an inverse problem being solved by using Gaussian process). In this paper, we propose a method of processing noise-contaminated dynamic measurements, which is based on the effect of self-regularization and does not require significant re-adjustment of the measuring system parameters.

2. CLOSED-LOOP MEASURING SYSTEM MODEL

At their input measuring systems have a primary measuring transducer (sensor), that is why the input signal $U(t)$ cannot be measured directly. Due to the time lag in the measuring system and in order to learn the actual system status, we need to restore the input signal $U(t)$ using the sensor's output signal. One of the methods of the input signal restoration was proposed in [12] and includes a closed-loop measuring system model. The structural model of such system is presented in Fig. 1.

In this model several groups of coefficients may be distinguished. Group one includes coefficients a_i, related to the output signal, and group two—coefficients b_j, related to the input signal. The next group includes coefficients d_j, which act as filter characteristics. The user can adjust these characteristics depending on the nature of the input signal noise. The final group includes k_i, which are feedback coefficients and are used for the correction of the dynamic error values.

The significant difficulty in using of the closed-loop model is that when the noise level of the output signal changes, arises a need to correct all the dynamic characteristics of the measuring system, and new values must be selected for coefficients k_i and d_j. Meanwhile, the changes in characteristics of one group do not automatically lead to changes in characteristics of other groups, and each characteristics within one group is corrected independently for each parameter. Thus, the process of re-adjustment of dynamic characteristics results in a more complicated algorithm of

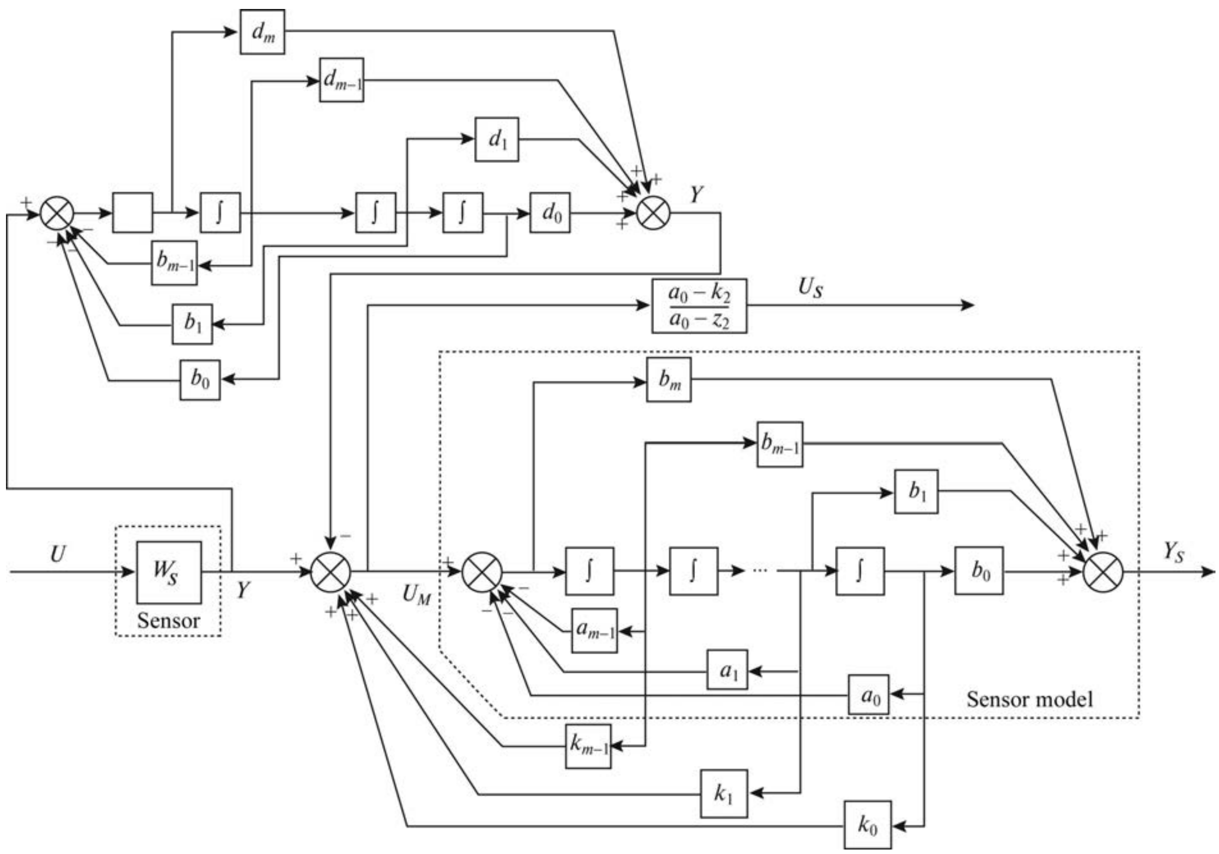

Fig. 1. Structural model of the measuring system.

signal processing. To solve the problem of making the algorithms of output signal processing less complicated, we need to develop a measuring system model, in which the number of groups of the parameters set by the user is reduced to a minimum.

3. OPEN-LOOP MEASURING SYSTEM MODEL

In this paper, we propose a discrete model of an open-loop measuring system, which allows to restore signals by using the noise-contaminated input data and avoid the use of feedback and additional filters. In other words, the model structure excludes the parameters of feedback k_i and the parameters of filters d_j, what allows to reduce the number of the parameters being set. In the proposed model, the correction block and additional filters are replaced with a block of the input signal restoration. The block model is given in Fig. 2, where $U(t)$ is the input signal, $U_\delta(t)$ is the restored signal, $Y_M(t)$ is the model's output signal, c_l, g_j are the coefficients of the measuring system model, $l = \overline{0,n}$, $j = \overline{0,m}$.

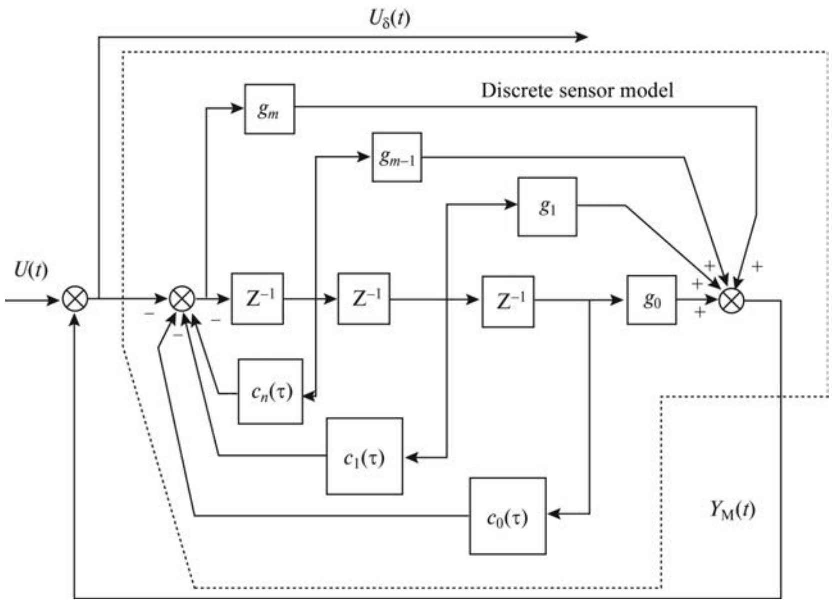

Fig. 2. Block of the input signal restoration in the open-loop discrete model. Nomenclature: $U(t)$ input signal, $U_\delta(t)$ restored signal, $Y_M(t)$ model's output signal, c_l, g_j coefficients of the measuring system model, $l = \overline{0,n}, j = \overline{0,m}$.

Let us build the mathematical model of the open-loop measuring system as follows. Based on the concept proposed in the theory of dynamic measurements [2], transfer function $W(p)$ of the open-loop measuring system is:

$$W(p) = \frac{b_m p^m + b_{m-1} p^{m-1} + \ldots + b_1 p + b_0}{a_n p^n + a_{n-1} p^{n-1} + \ldots + a_1 p + a_0} = \frac{y(p)}{u(p)}. \tag{1}$$

At stage one of building the mathematical model of the open-loop system, let us represent transfer function (1) by differential equation:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \ldots + b_1 u' + b_0 u. \tag{2}$$

Due to the output signal distortion, the following conditions correspond to the system status at the initial moment of time $t = 0$:

$$y(0) = q_0, \quad y'(0) = q_1, \ \ldots, \ y^{(n-2)}(0) = q_{n-2}.$$

Let us denote the following:

$$U = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \ldots + b_1 u' + b_0 u.$$

Thus, we find that the open-loop mathematical model (2) will look as follows:

$$a_n y^{(n)} + a_{(n-1)} y^{(n-1)} + \ldots + a_1 y' + a_0 y = U, \tag{3}$$

$$y(0) = q_0,\ y'(0) = q_1,\ \ldots,\ y^{(n-2)}(0) = q_{n-2}. \tag{4}$$

The proposed mathematical model (3), (4) is multifunctional. On one hand, it serves as grounds for validation of the discrete open-loop model, when based on the known input signal $U(t)$, we need to find function $Y_M(t)$, which corresponds to the output signal of the open-loop measuring system, and compare the obtained results to the output signal $Y_S(t)$, generated by the closed-loop system. On the other hand, equation (3) is used to develop the computational scheme for restoration of the input signal $U(t)$ by the known output signal $Y(t)$. It is worth noting that when restoring the input signal, it is necessary to take into account the fact that noise will be inevitably present in the measured output signal, so the information sufficient for the restoration of the input signal by the proposed method implies that its overall level does not exceed a certain level δ. We can present this situation mathematically as follows. In the problem on restoring the input signal we need to use the noise-contaminated output signal $Y(t)$ to find the input signal $U_\delta(t)$, provided that deviation $Y(t)$ from the accurate values of the output signal does not exceed the value δ.

4. OPEN-LOOP MODEL VALIDATION

At stage one of the validation, the known input signal $U(t)$ is fed to the closed-loop system and to the open-loop system. In the closed-loop system, output signal $Y_S(t)$ is formed, and in the open-loop system—output signal $Y_M(t)$. At the next stage of the validation, deviation of $Y_M(t)$ from $Y_S(t)$ is evaluated. If the deviation value does not exceed level δ, the validation is considered to be successful.

To construct a method of forming of output signal $Y_M(t)$, the following approach is proposed. Based on the idea presented in [13], let us define functions z_k as follows:

$$\begin{gathered} y = z_1, \\ y' = z_1' = z_2, \\ y'' = z_2' = z_3, \\ \ldots \\ y^{(n-2)} = z_{n-3}' = z_{n-2}. \end{gathered}$$

Then, equation (3) can be transformed into a system of differential equations:

$$\begin{cases} y = z_1, \\ y' = z_1' = z_2, \\ y'' = z_2' = z_3, \\ \ldots \\ a_n z_{n-2}'' + a_{n-1} z_{n-2}' + a_{n-2} z_{n-2} + \ldots + a_2 z'' + a_1 z' + a_0 z = U, \end{cases}$$

and the conditions of (4) will look as follows:

$$z_1(0) = q_0,\ z_2(0) = q_1,\ \ldots,\ z_{n-2}'(0) = q_{n-2}.$$

After transforming the last equation in the system, we finally find that the following system can serve as the basis for the validation of the open-loop model:

$$\begin{cases} y = z_1, \\ y' = z_1' = z_2, \\ y'' = z_2' = z_3, \\ \ldots \\ a_n z''_{n-2} + a_{n-1} z'_{n-2} + a_{n-2} z_{n-2} = U - a_{n-3} z_{n-3} - \ldots - a_2 z'' - a_1 z' - a_0 z \end{cases} \tag{5}$$

with the following initial conditions

$$z_1(0) = q_0,\ z_2(0) = q_1,\ \ldots,\ z'_{n-2}(0) = q_{n-2}. \tag{6}$$

From (5), (6) we need to find functions $z_k(t)$, $k = \overline{1, n-2}$. It is worth noting that problem (5), (6) pertains to the class of inverse problems, the specific feature of which is that the presence of noise or slight deviations in the input data can lead to significant misrepresentation of the final result, so regularization must be used to ensure the method stability in terms of the noise. According to the theory of inverse problems, regularization is fulfilled either by introducing additional stabilizing functional with a certain regularization parameter to the equation [14], or 'when solving some types of problems, it is possible to use only discretization, omitting the regularization step; this is called the problem self-regularization by discretization' [15]. In this paper, we propose a method, in which discretization interval acts as the regularization parameter.

The main stages of forming of $Y_M(t)$ include the following. First, we need to select some value of discretization interval τ, by dividing segment $[0; T]$ into R parts, $R = (T - 0)/\tau$. Then $t_i = (i-1)\tau$, and the value of functions $z_k(t)$ in the moment of time t_i corresponds to $z_k(t_i) = z_k^i$, $k = \overline{1, n-2}$, $i = \overline{1, R+1}$. The initial conditions will look like $z_k^1 = q_k$.

Next, based on the finite-difference representations of derivatives

$$z'_{n-2}(t_i) = \frac{z_k^i - z_k^{i-1}}{\tau},\quad z''_{n-2}(t_i) = \frac{z_{n-2}^i - 2z_{n-2}^{i-1} + z_{n-2}^{i-2}}{\tau^2},\quad i = \overline{1, K},\quad k = \overline{1, n-2},$$

we transform the last equation in system (5). We obtain:

$$z_{n-2}^i = \left(U_{i-2} - c_{n-2} z_{n-2}^{i-2} - c_{n-1} \left(\frac{z_{n-2}^{i-1}}{\tau} \right) \right) c_n, \tag{7}$$

where coefficients c_{n-2}, c_{n-1}, c_n are the coefficients of the open-loop model and are defined by the following formulas:

$$c_{n-2} = \frac{a_n}{\tau^2} - \frac{a_{n-1}}{\tau} + a_{n-2},\quad c_{n-1} = \frac{-2a_n}{\tau} + a_{n-1},\quad c_n = \frac{\tau^2}{a_n},$$

values U_i are formed from input signal $U = b_m u^{(m)} + b_{m-1} u^{(m-1)} + \ldots + b_1 u' + b_0 u$. Next, from the remaining equations of system (5) with the use of (7), we find values z_k^i, $k = \overline{n-3, 1}$ all the way to values z_1^i, which correspond to values $Y_M(t_i)$.

The validation process is considered to be successful if we meet the condition $|Y_M(t_i) - Y_S(t_i)| \leqslant \delta$ in each moment of time t_i; if otherwise, we must return to the initial stage of the computational scheme with a new parameter τ.

5. METHOD OF THE INPUT SIGNAL RESTORATION

The method of restoration of input signal $U_\delta(t)$ by using the known noise-contaminated output signal $Y(t)$ is based on the finite-difference analogue of equation (2) with fixed τ, obtained at the stage of validation of the open-loop model, and the following initial conditions:

$$u(0) = r_0, \quad u'(0) = r_1, \ \ldots, \ u^{(m-2)}(0) = r_{m-2}. \tag{8}$$

To build the computational scheme for restoration of input signal $U_\delta(t)$, let us introduce the following $Y = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0$. By applying Y to equation (2), we obtain the following:

$$b_m u^{(m)} + b_{m-1} u^{(m-1)} + \ldots + b_1 u' + b_0 u = Y.$$

Next, like at the validation stage, we use the method of reduction of order and substitute the variables as follows:

$$\begin{aligned} u &= v_1, \\ u' &= v_1' = v_2, \\ u'' &= v_2' = v_3, \\ &\ldots \\ u^{(n-2)} &= v_{n-3}' = v_{n-2}. \end{aligned}$$

As a result, we obtain the following system:

$$\begin{cases} u = v_1, \\ u' = v_1' = v_2, \\ u'' = v_2' = v_3, \\ \ldots \\ b_m v_{m-2}'' + b_{m-1} v_{m-2}' + b_{m-2} v_{m-2} = Y - b_{m-3} v_{m-3} - \ldots - b_2 v'' - b_1 v' - b_0 v \end{cases} \tag{9}$$

with the following initial conditions: $v(0) = r_0, v'(0) = r_1, \ldots, v_{m-2}'(0) = r_{m-2}$.

The main idea of the proposed method implies that at each step of the iteration process, first, by using the finite-difference analogue of the last equation of system (9), we find value $v_{m-2}'(t_i)$ in the current moment of time t_i according to formula

$$v_{m-2}^i = \left(Y_{i-2} - g_{m-2} v_{m-2}^{i-2} - g_{m-1} \left(\frac{v_{m-2}^{i-1}}{\tau} \right) \right) g_m, \tag{10}$$

where coefficients g_{m-2}, g_{m-1}, g_m are the coefficients of the open-loop model and are defined by the following formulas:

$$g_{m-2} = \frac{b_m}{\tau^2} - \frac{b_{m-1}}{\tau} + b_{m-2}, \quad g_{m-1} = \frac{-2b_m}{\tau} + b_{m-1}, \quad g_m = \frac{\tau^2}{b_m}.$$

Next, using the finite-difference analogues of derivatives, we obtain the solution for system (9), and find value $v_{m-2}'(t_i)$ in the current moment of time t_i. Upon completion of the iteration process, we obtain all values v_1^i, $\overline{1, K}$. These values correspond to the restored signal $U_\delta(t_i)$.

The main advantage of the proposed method of the input signal restoration is that the dynamic error is corrected thanks to the effect of self-regularization, and the noise level in the restored input signal $U_\delta(t)$ does not exceed the noise level in the output signal.

In [16, 17], the theoretical evaluation of the error of the input signal restoration method was performed. As a result, the criterion was formulated for discretization interval τ:

$$\tau\delta^2 \leqslant \frac{\frac{c_1}{c_2}+\frac{c_0}{c_2}}{\frac{g_1}{g_2}+\frac{g_0}{g_2}}. \tag{11}$$

Condition (11) proves the dependence of the method stability on τ. According to the concept presented in [15], discretization interval τ is a regularization parameter, and the proposed method has the effect of self-regularization.

The advantage of the proposed open-loop model and the input signal restoration method is that these do not require the re-adjustment of the coefficients of dynamic characteristics k_i and d_j, and this allows to make the computational scheme significantly more simple.

6. VERIFICATION OF THE MEASURING SYSTEM MODEL AND THE INPUT SIGNAL RESTORATION METHOD

The verification of the measuring system model and the input signal restoration method was performed through experimental research, including computational experiments. In the course of the computational experiments, first, we used simulation modelling to form the test values of input signal $U(t)$. We used the test values for comparative analysis with the calculated values $U_\delta(t)$. In the course of further experimenting, we restored input signal $U_\delta(t)$ by the experimental values of output signal Y, using the discretization interval obtained at the previous stages of the research. Next, we compared the restored signal to input signal $U(t)$.

6.1. Computational Experiment Technique

The computational experiment included two stages. The goal of stage one was the numerical validation of the open-loop model. At this stage, discretization interval τ was selected. The goal of stage two was the numerical verification of the input signal restoration method. The experiment scheme is given in Fig. 3.

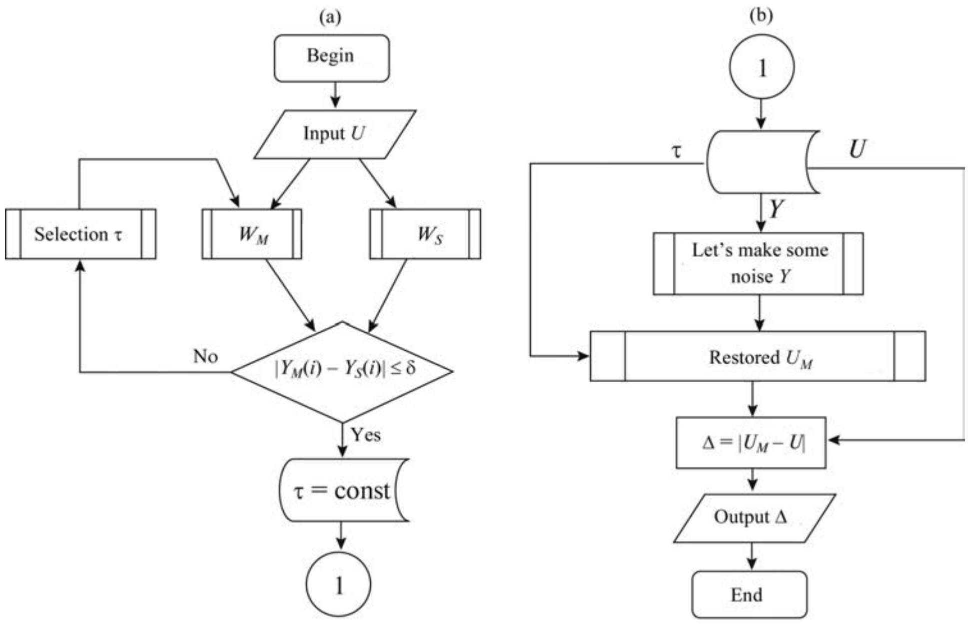

Fig. 3. Structural scheme of the computational experiment: *a* model adjustment stage; *b* signal restoration stage.

Stage I. Model validation. First, signal $U(t)$ was fed to the model with correcting feedback (W_S) and to the model without feedback (W_M) independently of each other. Next, we formed output signals Y_M and Y_S. Then, we did the calculation:

$$\Delta_Y = \max_{t\in[0,T]} |Y_M(t) - Y_S(t)|.$$

If $\Delta_Y > \delta$, then we selected a new value τ according to conditions of (11), a new output signal Y_M was formed, and we recalculated Δ. Upon the attainment of the condition $\Delta_Y \leqslant \delta$, value τ was fixed, and we moved to stage two of the experiment.

Stage II. Signal restoration. At this stage, noise-contaminated signal Y was fed to open-loop model W_M. Next, using parameter τ obtained at stage one, we found values U_δ, which corresponded to the restored input signal. Finally, we evaluated deviation of U_δ from U with the help of value

$$\Delta_U = \max_{t\in[0,T]} |U_\delta(t) - U(t)|.$$

Value Δ_U is the estimation of the accuracy of the input signal restoration method.

6.2. Computational Experiment Results

In this paper, we have presented the results of the experimental research for various orders of measuring systems regarding the restoration of the input signal by using the noise-contaminated data.

In the computational experiment, as the input signal, we took function $U(t) = 1$, and noise level $\delta = 5\%$. The experiment was conducted for measuring systems of the following orders, as given in the Table:

Computational experiment parameters

Measuring system order	Transfer function
II+I	$\dfrac{b_1p + b_0}{a_2p^2 + a_1p + a_0}$
IV+0	$\dfrac{b_0}{(a_2p^2 + a_1p + a_0)(a_5p^2 + a_4p + a_3)}$
V+III	$\dfrac{(b_1p + b_0)(b_4p^2 + b_3p + b_2)}{(a_2p^2 + a_1p + a_0)(a_5p^2 + a_4p + a_3)(a_7p + a_6)}$

Experiment results for the measuring system with the second order for the output signal and with the first order for the input signal (II+I) are presented in Fig. 4.

Deviation of the restored signal from the input one in the experiment equaled no more than 5% at $\tau = 1.2 \times 10^{-3}$.

Experiment results for the measuring system with the fourth order for the output signal and with the zeroth order for the input signal (IV+0) are presented in Fig. 5.

Deviation of the restored signal from the input one in this experiment equaled no more than 5% at $\tau = 1.75 \times 10^{-3}$.

Experiment results for the measuring system with the fifth order for the output signal and with the third order for the input signal (V+III) are presented in Fig. 6.

Deviation of the restored signal from the input one in the experiment equaled no more than 5% at $\tau = 1.75 \times 10^{-3}$.

The experiment results prove the stability of the method of the input signal restoration by using the open-loop model for systems of various orders, i.e., the noise level in the restored signal remains within the controlled limits.

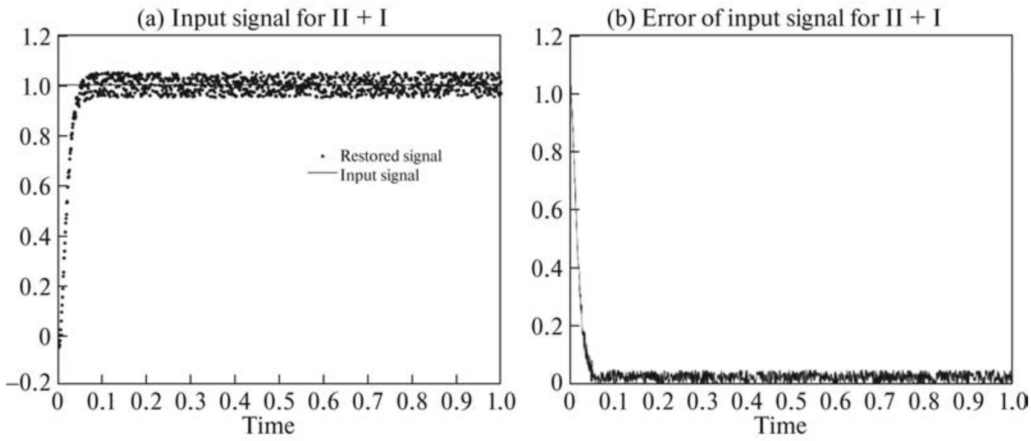

Fig. 4. (a) Charts of the source input signal $U(t)$ and the restored signal $U_\delta(t)$. (b) Deviation of the restored signal from the input one for the system with the second order for the output signal and with the first order for the input signal Δ_U.

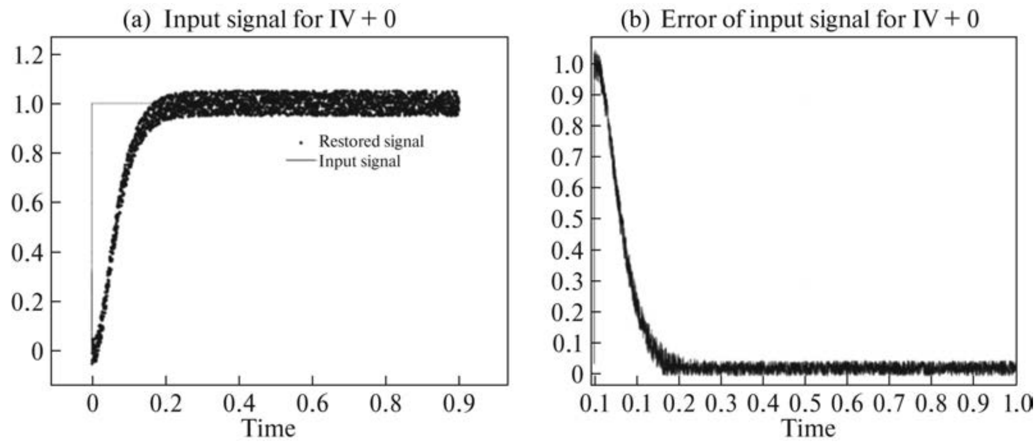

Fig. 5. (a) Charts of the source input signal $U(t)$ and the restored signal $U_\delta(t)$. (b) Deviation of the restored signal from the input one for the system with the fourth order for the output signal and with the zeroth order for the input signal Δ_U.

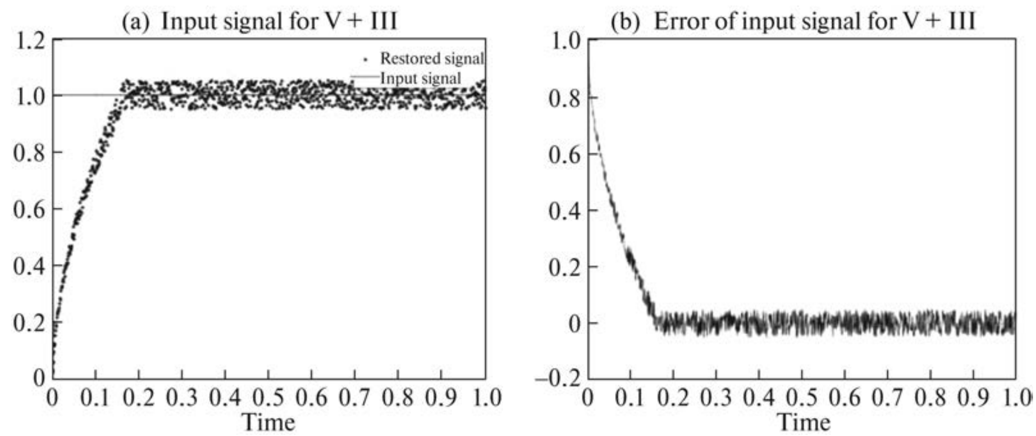

Fig. 6. (a) Charts of the source input signal $U(t)$ and the restored signal $U_\delta(t)$. (b) Deviation of the restored signal from the input one for the system with the fifth order for the output signal and with the third order for the input signal Δ_U.

7. CONCLUSION

In this paper, we have proposed the open-loop measuring system model and the method of the input signal restoration by the noise-contaminated output signal for a random order dynamic system. The input signal restoration method is based on using of regularization approaches. It has been demonstrated that this method has the effect of self-regularization. On the grounds of the built computational schemes, the computational experiment has been conducted and the comparative analysis has been performed regarding the results of restoration of the input signal with test functions. The experiment results prove that the proposed method maintains the level of error in the restored input signal at the level of error of the input data for various orders of measuring system.

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OBITUARY

Alexander M. Krasnosel'skii
(April 30, 1955–March 29, 2024)

Chief Researcher of M.S. Pinsker Laboratory No. 1 (the Institute for Information Transmission Problems, the Russian Academy of Sciences) and our dear friend, colleague, coauthor, and interlocutor, Alexander Krasnosel'skii passed away on March 29, 2024, at the age of 68 after a severe operation.

Alexander Krasnosel'skii, in full Alexander Markovich Krasnosel'skii, was born on April 30, 1955, in the family of Mark Alexandrovich Krasnosel'skii, one of the founders of nonlinear functional analysis in the USSR. Krasnosel'skii's house was always full of mathematicians of various ages (Mark's students and colleagues), and the friendly atmosphere in the house was largely the merit of Alexander's mother, Sarra Izrailevna. Undoubtedly, that atmosphere influenced Alexander's lifestyle: in any company, he became one of the centers of communication.

Having graduated from Moscow State University, Krasnosel'skii Jr. (further referred to as Krasnosel'skii) worked for almost twelve years in the Laboratory of Mathematical Methods at the All-Union Cardiology Research Center, the USSR Academy of Medical Sciences. He was a member of the group to design an automatic diagnosis system for heart rhythm disorders. During that time, Krasnosel'skii defended his candidate's and doctoral dissertations and, in fact, pioneered the harmonic balance method in nonlinear analysis. The results of his research were published in a monograph, which was later translated and published in English.

Upon moving to the Institute for Information Transmission Problems (IITP), the USSR Academy of Sciences, Alexander's scientific interests were closely connected with the development of topological methods of nonlinear analysis for studying automatic regulation and control systems as well as systems arising in information transmission networks. Also, he was actively involved in elaborating the mathematical theory of hysteresis and the stability theory of asynchronous control systems and related numerical methods.

Since 1991 and until the last time, Krasnosel'skii worked in M.S. Pinsker Laboratory No. 1 (Theory of Information Transmission and Control) of IITP as a leading executor on all research topics of the Laboratory. In recent years, Alexander was an invariable member of a dissertation council and the Academic Council of IITP. He was a member of the editorial board of *Automation and Remote Control*.

Krasnosel'skii loved life very much and was very fond of mathematics, including elementary mathematics. He loved to discuss it with friends and especially with the young. In the last decades, one of Alexander's main hobbies was active teaching, first at the Jewish State Academy and then at the Higher School of Economics, where he headed IITP's basic department and was a member of the Academic Council of the Faculty of Mathematics. Clearly, he always lacked teaching when working in the academic institution. Krasnosel'skii always sincerely admired the abilities of young mathematicians born at the Faculty of Mathematics with his participation.

Alexander was an outstanding and versatile personality: he was passionate about books, loved traveling, and was seriously engaged in bicycle touring. Avoiding conflicts, he was at the same time firm in matters of principle. Alexander's self-comprehension as a member of a research team was his peculiarity: in this respect, he could always be relied upon.

Alexander was a deeply religious man. He tenderly loved his four children and seven grandchildren. He loved to make friends and knew how to do it.

May Alexander's memory live on forever!

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