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Adaptive Auxiliary Loop for Output-Based Compensation of Perturbations in Linear Systems

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Abstract—The problem of output-feedback compensation of bounded additive perturbations affecting a minimum-phase linear system with unknown parameters is considered. An adaptive auxiliary loop is developed, which does not require to know the perturbation model and allows one to: *a*) separate the processes of estimation of parametric and additive perturbations, *b*) estimate and compensate for the additive perturbation with any given accuracy if the conditions of the parametric identifiability are met. The above-mentioned separated estimation of two disturbances of different nature is achieved by augmentation of the A.M. Tsykunov auxiliary loop method with the law to identify the unknown parameters, which is based on the instrumental variables approach and the procedure of dynamic regressor extension and mixing (DREM). The obtained system of the additive perturbations compensation has a certain potential to be used together with the conventional industrial PI-, PID-controllers. The theoretical results of this study are validated via mathematical modelling.

Keywords: disturbance, estimation, auxiliary loop, identification, instrumental variables, convergence, overparameterization, dynamic extension and mixing

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1. INTRODUCTION

The problem of external perturbation compensation has been attracting considerable attention of specialists in radio engineering, electrical engineering, control theory, etc. for many years. To date, two basic principles of such compensation have been proposed—indirect and direct ones.

Considering the indirect compensation, a characteristic polynomial of a closed-loop system is chosen so that the component of the system forced motion caused by the perturbation is reduced as much as possible in comparison with the one associated with the reference signal. However, it is impossible to ensure the same quality of compensation for perturbations with significantly different spectra by certain choice of the characteristic polynomial of a closed-loop system, and the problem of how to choose it on the basis of *a priori* data on the perturbation spectrum or in an optimal way (*e.g.*, in terms of $\mathcal{H}_2$-, $\mathcal{H}_\infty$-norms, invariant ellipsoid metrics, etc.) is faced. In this sense, it is necessary to admit the limitations of classical feedback.

A natural response to this challenge was the development of the direct compensation principle, according to which the control signal is decomposed into two components. The first one corrects the characteristic polynomial of the closed-loop system, while the second is to be equal to the perturbation with the opposite sign. If the disturbance is matched with the control signal and measurable, this approach achieves its full compensation. At first glance, the matching condition seems to be restrictive, because in practice the disturbance and the control signal are often unmatched. However, in fact, by following the equivalent-input-disturbance approach [1] and adopting some rather weak differentiability conditions for the original perturbation, the matching condition can always be satisfied. For example, consider the following system:

$$\dot{y} = a_1 y + b_1 x + b_1 f,$$
$$\dot{x} = a_2 x + b_2 u,$$

then, if the perturbation f is differentiable, the application of the equivalent-input-disturbance approach allows one to obtain a system with matched perturbation:

$$\begin{aligned}\dot{y} &= a_1 y + b_1 \zeta,\\ \dot{\zeta} &= a_2 x + b_2 u + \dot{f} \pm a_2 f = a_2 \zeta + b_2 \left[u + \underbrace{b_2^{-1}\left(\dot{f} - a_2 f\right)}_{f_{eq}} \right],\end{aligned}$$

where $a_1,\ a_2,\ b_1,\ b_2$ are some scalars, y stands for a measurable output signal, x denotes unmeasurable state vector, $\zeta = x + f$ is a virtual state, u denotes a control signal, f stands for the original disturbance, f_{eq} is an equivalent input disturbance.

Thus, we will hereafter refer only to matched perturbations, assuming that the principle of equivalent-input-disturbance has already been implemented.

Much more restrictive requirement is that the perturbation is measurable. To relax it, various disturbance observers have been proposed in the literature that allow one to reconstruct the value of the disturbance with some (usually arbitrary) accuracy from measurements of the control and system output signals. Without pretending to provide an exhaustive review, some of the existing perturbation observers are considered further. We recommend an interested reader to study the reviews [2–5] to become aware of the full variety of methods.

Existing perturbation observers can be classified as follows:

1) methods that require to know both the system parameters and the perturbation model and parameters (extended Luenberger observer) [6],
2) methods that require knowledge of the system parameters and the disturbance model with parametric uncertainty [7–11],
3) methods that require the disturbance model to be known, while the system and disturbance parameters can be unknown [12–16],
4) methods requiring only knowledge of the system parameters [1, 17–20],
5*) methods that require neither knowledge of the disturbance model nor the system parameters [21–26].

The algorithms that belong to group 5* are the subject of interest of this study since, compared to other solutions, they require minimum amount of *a priori* information about the system and the perturbation. A detailed analysis of such algorithms shows that, in fact, there are no observers in the literature that allow one to estimate the additive perturbation if the system parameters are unknown. This is due to the fact that existing algorithms are not able to separate parametric and additive disturbances. Instead, the algorithms from the group 5* estimate an augmented perturbation consisting of their sum. Let us illustrate this statement with an example.

A first-order system is considered:

$$\dot{x} = u + \theta^\top \varphi(x) + f,$$

where $\theta^\top \varphi(x)$ is a parametric disturbance, f denotes an additive perturbation.

If the parameters θ are known and $\dot{f}$ is bounded, then a trivial observer ($s\colon = \frac{d}{dt}$):

$$\hat{f} = \frac{s}{ls+1}\left[x\right] - \theta^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right] - \frac{1}{ls+1}\left[u\right],$$

according to [26, p. 196; 27], allows one to estimate and compensate for the additive disturbance f with an arbitrary accuracy.

If the parameters θ are unknown, then only augmented disturbance can be estimated:

$$\frac{s}{ls+1}\left[x\right] - \frac{1}{ls+1}\left[u\right] = \frac{1}{ls+1}\left[f\right] + \theta^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right].$$

In order to single out an additive disturbance from the augmented one, the estimate of the parameters θ is to be obtained and used:

$$\hat{f} = \frac{s}{ls+1}\left[x\right] - \frac{1}{ls+1}\left[u\right] - \hat{\theta}^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right] = \frac{1}{ls+1}\left[f\right] - \tilde{\theta}^{\top}\frac{1}{ls+1}\left[\varphi\left(x\right)\right],$$

from which it follows that the separation of two types of perturbations is possible if and only if the parametric error $\tilde{\theta}$ converges to zero asymptotically. However, in case of perturbations, the existing identification laws provide only the parametric error boundedness [28, p. 556], which does not allow one to achieve complete separation even by augmenting the disturbance observer with known parameter estimation algorithms. Incomplete attempts to estimate additive and parametric uncertainties apart from each other can be found in [24, 29, 30].

The considered separation problem is of a fundamental nature and does not depend on the specific type of applied perturbation observer. Therefore, the existing observers estimate and compensate for an augmented perturbation represented as a sum of parametric and additive disturbances. Such an approach certainly deserves the right to exist and proved itself in comparison with the conventional PI- and PID-controllers [22] a long time ago. However, firstly, in such control systems the synergetic principle of least action [31] is violated, since the whole dynamics of the system is compensated without any reflection on its "usefulness" or "armfulness" to achieve the control objective; secondly, considering some practical scenarios, the baseline stabilising component of the control law (*e.g.*, PI- or PID-controller) has already been chosen by robust methods taking into account the parametric uncertainty ($\theta^{\top}\varphi\left(x\right)$ in the above-given example), and it is necessary to estimate and compensate only for the additive perturbation (f in the mentioned example). Therefore, the problem of design of the additive perturbation observer in the presence of parametric disturbance is actual.

In this study we consider the problem of estimation and output-based compensation of bounded additive perturbations affecting a minimum-phase linear system with unknown parameters. The solution of this problem is proposed to be obtained on the basis of the indirect adaptive control framework, according to which the control design procedure is decomposed into two stages. At the first one, a control law is introduced that ensures the achievement of the control objective assuming that the system parameters are known (in this study, we use the method of A.M. Tsykunov auxiliary loop [26, p. 196] to design such a law). At the second stage, the identification law is developed, and all unknown parameters of the control law defined at the first stage are substituted with their dynamic estimates. The key structural element of such an adaptive control system is the above-mentioned law, which is required to ensure asymptotic convergence of the unknown parameter estimates to their true values in a closed loop affected by an additive disturbance under the weakest possible regressor excitation requirements. In this paper, a recently proposed algorithm [32] based on the instrumental variables method [33] and the dynamic regressor extension and mixing (DREM) procedure [34] is proposed to solve the online identification problem. The convergence conditions of the parameter identification process using such a law are given below:

—a controller that stabilises the system affected by the additive and parametric perturbations is known,
—a control signal to compensate for the disturbance is bounded (for instance, via sat $\{.\}$ function),
—a reference signal includes not less than n different frequencies (where n is the system order),
—the reference and disturbance signals spectra have no common frequencies.

It is shown that when these conditions are met, the parametric error convergences to zero despite the presence of the additive disturbance, and the proposed system of adaptive compensation of such perturbation ensures its asymptotic estimation and compensation. The potential of application of the proposed system together with the conventional industrial PI-, PID-controllers is shown via numerical experiments, the fact that the conditions of parametric convergence are met when typical reference signals are used is also illustrated.

2. PROBLEM STATEMENT

The following perturbed linear dynamic system is considered:

$$\begin{aligned} y(t) &= \frac{Z(\theta,\ s)}{R(\theta,\ s)}\left[u(t)+f(t)\right], \\ Z(\theta,\ s) &= b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0, \\ R(\theta,\ s) &= s^n + a_{n-1}s^{n-1} + \ldots + a_0, \end{aligned} \tag{2.1}$$

where $y(t)$ is a measurable output signal, $u(t)$ stands for a control signal to be designed, $f(t)$ denotes an unknown bounded disturbance, $\theta \in D_\theta \subset \mathbb{R}^{n+m+1}$ is a vector of unknown parameters of an open-loop system, $R(\theta,\ s)$, $Z(\theta,\ s)$ are polynomials of order n and $m \leqslant n-1$, respectively, $s[.] := \frac{d}{dt}[.]$ denotes a differential operator.

Further, the control signal $u(t)$ is assumed to be chosen as:

$$\begin{aligned} u(t) &= u_b(t) + u_c(t), \\ u_b(t) &= \frac{P_y(\kappa,s)}{Q_y(\kappa,\ s)}[y(t)] + \frac{P_r(\kappa,s)}{Q_r(\kappa,\ s)}[r(t)] = \frac{P_y(\kappa,s)}{Q_y(\kappa,\ s)}[y(t)] + r_f(t), \end{aligned} \tag{2.2}$$

where $u_b(t)$ is a baseline component to stabilize the system, $u_c(t)$ stands for a summand to compensate for a disturbance $f(t)$, $\kappa \in D_\kappa \subset \mathbb{R}^{n_\kappa}$ denotes known time-invariant parameters of the control law, $r(t)$ is a reference signal, $m_y \leqslant n_y$ and $m_r \leqslant n_r$ are orders of pairs of polynomials $P_y(\kappa,\ s)$, $Q_y(\kappa,\ s)$ and $P_r(\kappa,\ s)$, $Q_r(\kappa,\ s)$, respectively.

For a rigorous formal problem statement, together with the system (2.1), its parametrization in the form of a linear regression equation is also considered:

$$z(t) = \varphi^\top(t)\,\theta + w(t), \tag{2.3}$$

where

$$\begin{gathered} z(t) = \frac{s^n}{\Lambda(s)}y(t),\ \varphi(t) = \begin{bmatrix} -\frac{\alpha_{n-1}^\top(s)}{\Lambda(s)}[y(t)] & \frac{\alpha_m^\top(s)}{\Lambda(s)}[u(t)] \end{bmatrix}^\top, \\ w(t) = \begin{bmatrix} b_m & b_{m-1} & \ldots & b_0 \end{bmatrix} \frac{\alpha_m(s)}{\Lambda(s)}[f(t)], \\ \alpha_{n-1}^\top(s) = \begin{bmatrix} s^{n-1} & \cdots & s & 1 \end{bmatrix},\ \alpha_m^\top(s) = \begin{bmatrix} s^m & \cdots & s & 1 \end{bmatrix}, \end{gathered}$$

$\varphi(t) \in \mathbb{R}^{m+n+1}$ is a measurable regressor, $z(t) \in \mathbb{R}$ stands for a measurable regressand, $\Lambda(s)$ denotes a stable polynomial of order n.

We adopt the following assumptions with respect to the stabilizing component of the control law and the disturbance.

Assumption 1. The transfer function of the closed-loop system (θ_{cl}: $\mathbb{R}^{n+m+1} \times \mathbb{R}^{n_\kappa} \mapsto \mathbb{R}^{n_{cl}+m_{cl}+1}$ are unknown parameters of the closed-loop system):

$$W_{cl}(\theta_{cl},\ s)[.] = \frac{Z(\theta,\ s)\, Q_y(\kappa,\ s)}{[Q_y(\kappa,\ s)\, R(\theta,\ s) - Z(\theta,\ s)\, P_y(\kappa,\ s)]}[.] = \frac{Z_{cl}(\theta_{cl},\ s)}{R_{cl}(\theta_{cl},\ s)}[.] \tag{2.4}$$

has Hurwitz polynomials $Z_{cl}(\theta_{cl},\ s)$ and $R_{cl}(\theta_{cl},\ s)$ for certain time-invariant κ from D_κ and any θ from D_θ.

Assumption 2. There exists a known function μ: $[t_0,\ \infty) \mapsto \mathbb{R}_>$ such that for

$$\lambda(t) = \frac{-1}{m_0 \mu^{n+1}(t)} \left(\frac{d^{n+1}}{dt^{n+1}}[f(t)] + \sum_{i=1}^{n} m_i \mu^{n-(i-1)}(t) \frac{d^i}{dt^i}[f(t)] \right)$$

for any $m_i,\ i = 0, \ldots,\ n$ it holds that $\mu\lambda \in L_\infty$ and $\lambda \in L_p \cap L_\infty$ for $p \in [1,\ \infty)$.

Assumption 3. An instrumental variable $\zeta(t) \in \mathbb{R}^{n+m+1}$, which is defined as follows (where $\theta_{iv} \in D_\theta \subset \mathbb{R}^{n+m+1}$ are known parameters of the instrumental variable):

$$\begin{aligned}
\zeta(t) &= \left[-\tfrac{\alpha_{n-1}^\top(s)}{\Lambda(s)}[y_{iv}(t)] \quad \tfrac{\alpha_m^\top(s)}{\Lambda(s)}[u_{iv}(t)] \right]^\top, \\
y_{iv}(t) &= \frac{Z(\theta_{iv}, s)}{R(\theta_{iv},\ s)}[u_{iv}(t)], \\
u_{iv}(t) &= \frac{P_y(\kappa, s)}{Q_y(\kappa,\ s)}[y_{iv}(t)] + \frac{P_r(\kappa,\ s)}{Q_r(\kappa,\ s)}[r(t)],
\end{aligned} \tag{2.5}$$

and a filtered disturbance $w(t)$ are independent, *i.e.*:

$$\forall t \geqslant t_0 \left| \int_{t_0}^{t} \zeta_i(s)\, w(s)\, ds \right| \leqslant c < \infty\ \forall i = 1, \ldots,\ n + m + 1. \tag{2.6}$$

Based on closed-loop equation (2.4), the required behavior of the system is defined as:

$$y_{ref}(t) = W_{cl}(\theta_{cl},\ s)[r_f(t)]. \tag{2.7}$$

The aim is to design the compensation control signal $u_c(t)$, which does not include the output signal $y(t)$ derivatives, in such a way that the following equalities hold:

$$\overline{\lim_{t\to\infty}} |y(t) - y_{ref}(t)| = \overline{\lim_{t\to\infty}} |\tilde{y}(t)| = 0. \tag{2.8}$$

Therefore, the problem of estimation and compensation of an unknown perturbation that affects a minimum-phase system with unknown parameters is stated in this study.

Remark 1. Considering practical scenarios, we almost always know the stabilizing controller $u_b(t)$, which ensures the that the characteristic polynomial of the closed-loop system $R_{cl}(\theta_{cl},\ s)$ is Hurwitz one (*e.g.*, in some situations it can be chosen via trial and error). The polynomial $Z_{cl}(\theta_{cl}, s)$ to be Hurwitz requires the plant (2.1) to be minimum phase.

Remark 2. It should be noted that the output of the reference model (2.4) is not measurable, and the reference model itself defines different control quality for each specific parameter vector of the system θ from D_θ. These facts essentially distinguish the problem solved in this study from the classical problem of model reference adaptive control, in which the reference model is the same for all θ.

Remark 3. Assumption 2 restricts a class of admissible external disturbances. As for practical scenarios, there almost always exists $p \in [1, \infty)$ such that $\frac{d^i}{dt^i} f(t) \in L_p$ for all $i = 1, \ldots, n+1$, and, to meet such assumption, it is sufficient to choose $\mu(t) = \text{const} = \mu > 0$. If $\frac{d^i}{dt^i} f(t) \notin L_p$ for all $p \in [1, \infty)$, but $\sup\limits_t \left| \frac{d^i}{dt^i} f(t) \right| < \infty$, then it is sufficient to choose $\mu(t) = \mu_0 t + \mu_1$ with arbitrary $\mu_0 > 0$, $\mu_1 > 0$. Unfortunately, if the disturbance $f(t)$ has unbounded time derivatives, some additional *a priori* information is required for reasonable choice of $\mu(t)$.

Remark 4. In Assumption 3, the necessary conditions of asymptotic convergence of the identification law [32] are assumed to be met. In a stationary case, the inequality (2.6) is satisfied if the spectra of $r(t)$ and $f(t)$ do not have common frequencies.

3. MAIN RESULT

The description of the proposed method to solve the problem (2.8) is decomposed into three parts. In the first one, the filtered equivalent of the perturbation is parameterized as a function of the control and output signals and some unknown parameters calculated via θ. In the second part, an identification law for the unknown parameters is designed on the basis of the perturbation parametrization. In the third part, based on the obtained parametrization of the filtered perturbation and identification law, an adaptive signal for disturbance compensation is introduced and the achievement of the objective (2.8) is proved.

3.1. Disturbance Parametrization

Temporarily, the parameters θ are assumed to be known and, following [26], an auxiliary model is introduced:

$$\hat{y}^*(t) = \frac{Z(\theta, s)}{R(\Theta, s)} [u(t)], \tag{3.1}$$

where $\Theta \in \mathbb{R}^n$ are known parameters of the auxiliary model such that the system (3.1) is stable.

Owing to equations (3.1) and (2.1), the error $\varepsilon^*(t) = y(t) - \hat{y}^*(t)$ is written as follows:

$$\varepsilon^*(t) = \frac{Z(\theta, s)}{R(\Theta, s)} [f(t)] + \frac{R(\Theta, s) - R(\theta, s)}{R(\Theta, s)} [y(t)]. \tag{3.2}$$

Then, if $n - m$ derivatives of the signals $\varepsilon^*(t)$ and $y(t)$ are measurable, and the polynomial $Z(\theta, s)$ is Hurwitz one, then the following signal

$$u_c(t) = -\frac{R(\Theta, s)}{Z(\theta, s)} [\varepsilon^*(t)] + \frac{R(\Theta, s) - R(\theta, s)}{Z(\theta, s)} [y(t)] = -f(t) \tag{3.3}$$

ensures full compensation of the disturbance:

$$y(t) = \varepsilon^*(t) + \hat{y}^*(t) = \frac{Z(\theta, s)}{R(\theta, s)} [u_b(t)] = W_{cl}(\theta_{cl}, s) [r_f(t)] = y_{ref}(t). \tag{3.4}$$

The perturbation and its compensator (3.3) can be represented in the following equivalent form:

$$f(t) = -u_c(t) = \psi_a^\top(\Theta)\,\alpha(s)\,[\varepsilon_f(t)] + \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)\alpha(s)\,[y_f(t)], \tag{3.5a}$$

$$\begin{array}{ll} \dot{\xi}_\varepsilon(t) = A_b\xi_\varepsilon(t) + \rho e_m\varepsilon^*(t), & \dot{\xi}_y(t) = A_b\xi_y(t) + \rho e_m y(t), \\ \begin{cases} \varepsilon_f(t) = e_1^\top\xi_\varepsilon(t), \text{ if } m \geqslant 1 \\ \varepsilon_f(t) = \rho\varepsilon^*(t), \text{ if } m = 0, \end{cases} & \begin{cases} y_f(t) = e_1^\top\xi_y(t), \text{ if } m \geqslant 1 \\ y_f(t) = \rho y(t), \text{ if } m = 0, \end{cases} \end{array} \tag{3.5b}$$

$$\begin{cases} \dot{\hat{x}}^*(t) = A_0\hat{x}^*(t) - \Theta\hat{y}^*(t) + \theta_b u(t) \\ \hat{y}^*(t) = e_1^\top\hat{x}^*(t), \end{cases} \tag{3.5c}$$

where

$$A_b = A_0 - \psi_b(\theta)\,e_1^\top,\ \rho = \frac{1}{b_m} = \frac{1}{e_1^\top\theta_b},$$

$$\psi_b(\theta) = \begin{bmatrix}\frac{b_{m-1}}{b_m} & \frac{b_{m-2}}{b_m} & \cdots & \frac{b_0}{b_m}\end{bmatrix}^\top = \rho\mathcal{L}_\psi\theta_b,$$

$$\psi_a(\theta_a) = \begin{bmatrix}\mathcal{I}_n\theta_a & 1\end{bmatrix}^\top,$$

$$\theta_a = \begin{bmatrix}a_{n-1} & a_{n-2} & \cdots & a_0\end{bmatrix}^\top = \mathcal{L}_a\theta,$$

$$\theta_b = \begin{bmatrix}b_m & b_{m-1} & \cdots & b_0\end{bmatrix}^\top = \mathcal{L}_b\theta,$$

$$\mathcal{L}_a = \begin{bmatrix}I_{n\times n} & 0_{n\times(m+1)}\end{bmatrix},\ \mathcal{L}_b = \begin{bmatrix}0_{(m+1)\times n} & I_{(m+1)\times(m+1)}\end{bmatrix},$$

$$\mathcal{L}_\psi = \begin{bmatrix}0_{m\times 1} & I_{m\times m}\end{bmatrix},\ \alpha(s) = \begin{bmatrix}1 & \cdots & s^{n-1} & s^n\end{bmatrix},$$

and $\mathcal{I}_n$ is a matrix, which secondary diagonal contains ones, while all other elements are zeros, e_i denotes a vector, which ith element is one, while all other ones are zeros, A_0 stands for an upper-shift matrix of respective dimension, $\theta_a,\ \theta_b$ are components of the vector $\theta = \begin{bmatrix}\theta_a^\top & \theta_b^\top\end{bmatrix}^\top$ and simultaneously parameters of the polynomials $R(\theta,\ s)$ and $Z(\theta,\ s)$.

According to the problem statement, the parameters θ are unknown and the derivatives of the signals $\varepsilon^*(t)$ and $y(t)$ are not directly measurable, which means that the compensator (3.3), (3.5a) can not be implemented. The requirement to know the derivatives of the mentioned signals can be relaxed by design of filtered derivative observers.

Statement 1. *Define:* 1) *the observers of the ith filtered derivative of the signals $y_f(t)$ and $\varepsilon_f(t)$*

$$\left.\begin{array}{l} \begin{cases} \dot{H}_i^\varepsilon(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)H_i^\varepsilon(t) + e_{n+1}\left(K_0(t)\,\varepsilon_f(t) - v_\varepsilon(t)\right) \\ h_i^\varepsilon(t) = e_{i+1}^\top H_i^\varepsilon(t),\ v_\varepsilon(t) = \sum\limits_{j=0}^{i} v_j^\varepsilon(t,\ M_0,\ K_0), \end{cases} \\ \begin{cases} \dot{H}_i^y(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)H_i^y(t) + e_{n+1}\left(K_0(t)\,y_f(t) - v_y(t)\right) \\ h_i^y(t) = e_{i+1}^\top H_i^y(t),\ v_y(t) = \sum\limits_{j=0}^{i} v_j^y(t,\ M_0,\ K_0), \end{cases} \end{array}\right. \quad \forall i = 0, \ldots, n,$$

where

$$M_0^\top(t) = \begin{bmatrix}-m_0\mu^{n+1}(t) & \ldots & -m_{n-1}\mu^2(t) & -m_n\mu(t)\end{bmatrix},$$

$$G_0 = \begin{bmatrix}0_{n+1} & \begin{matrix}I_{n\times n} \\ 0_{1\times n}\end{matrix}\end{bmatrix},\ K_0(t) = m_0\mu^{n+1}(t),$$

and

$$\begin{cases} v_0^{\varepsilon}(t,\ M_0,\ K_0) = 0 \\ v_1^{\varepsilon}(t,\ M_0,\ K_0) = s^{-1}\left[s\left[M_0^{\top}(t)\right]H_i^{\varepsilon}(t)\right] + s^{-1}\left[s\left[K_0(t)\right]\varepsilon_f(t)\right] \\ \vdots \\ v_j^{\varepsilon}(t,\ M_0,\ K_0) = v_{j-1}^{\varepsilon}(t,\ M_0,\ K_0) - s^{-1}\left[v_{j-1}^{\varepsilon}\left(t,\ \dot{M}_0,\ \dot{K}_0\right)\right], \quad j = 2, \ldots i, \end{cases}$$

$$\begin{cases} v_0^{y}(t,\ M_0,\ K_0) = 0 \\ v_1^{y}(t,\ M_0,\ K_0) = s^{-1}\left[s\left[M_0^{\top}(t)\right]H_i^{y}(t)\right] + s^{-1}\left[s\left[K_0(t)\right]y_f(t)\right] \\ \vdots \\ v_j^{y}(t,\ M_0,\ K_0) = v_{j-1}^{y}(t,\ M_0,\ K_0) - s^{-1}\left[v_{j-1}^{y}\left(t,\ \dot{M}_0,\ \dot{K}_0\right)\right], \quad j = 2, \ldots i, \end{cases}$$

and scalars $m_0,\ m_1,\ \ldots, m_n$ *are coefficients of a stable polynomial;*

2) *the filtered disturbance:*

$$\begin{gathered} f_f(t) := \psi_a^{\top}(\Theta)\, h_{\varepsilon}(t) + \left(\psi_a^{\top}(\theta_a) - \psi_a^{\top}(\Theta)\right) h_y(t), \\ h_y(t) = \begin{bmatrix} h_0^y(t) & \cdots & h_i^y(t) & \cdots & h_n^y(t) \end{bmatrix}^{\top}, \\ h_{\varepsilon}(t) = \begin{bmatrix} h_0^{\varepsilon}(t) & \cdots & h_i^{\varepsilon}(t) & \cdots & h_n^{\varepsilon}(t) \end{bmatrix}^{\top}. \end{gathered} \tag{3.6}$$

Then, if Assumptions 1 and 2 are met, then:

1) *the error* $\tilde{f}(t) = f_f(t) - f(t)$ *converges asymptotically to zero* $\lim\limits_{t\to\infty} \tilde{f}(t) = 0$,

2) $\tilde{f} \in L_p \cap L_\infty$ *for* $p \in [1,\ \infty)$.

Proof of Proposition 1 is postponed to Appendix.

Note that in the stationary case $\mu(t) = \text{const}$, the filtered derivative observers proposed in Proposition 1 are reduced to proper differentiators. According to Proposition 1, if the parameters θ are known, then instead of the true perturbation $f(t)$, using measurable signals only, it is possible to calculate some filtered perturbation $f_f(t)$, which asymptotically converges to the true perturbation if Assumptions 1 and 2 are met. In this case, the signal

$$u_c(t) = -f_f(t) \tag{3.7}$$

allows one to obtain the following result.

Theorem 1. *If the parameters* θ *are known, and Assumptions 1 and 2 are met, then the control law* (2.2) + (3.7) *ensures* $\tilde{y} \in L_\infty$ *and* $\lim\limits_{t\to\infty} |\tilde{y}(t)| = 0$.

Proof of Theorem 1 is given in Appendix.

The requirement to know the parameters θ is relaxed by application of the identification law proposed in [32] and design of an adaptive auxiliary model (3.1) and an adaptive version of the compensation signal (3.7).

3.2. Unknown Parameters Identification

First of all, in order to implement the adaptive compensation signal, the estimates $\hat{\theta}(t)$, $\hat{\rho}(t)$, $\hat{\psi}_b(t)$ of the system unknown parameters and the disturbance parametrization (3.6) are required to be obtained. It should be noted that we do not need to identify the parameters $\psi_a(\theta_a)$, as the function $\psi_a\colon \mathbb{R}^{n+m+1} \mapsto \mathbb{R}^{n+m+2}$ obviously satisfies the Lipschitz condition, and, consequently, the estimate $\hat{\psi}_a(t)$ can be obtained via direct substitution $\hat{\psi}_a(t) := \psi_a\left(\hat{\theta}_a\right)$, where $\hat{\theta}_a(t) = \mathcal{L}_a\hat{\theta}(t)$. To make the effect of the adaptive law of perturbation compensation, which is based on dynamic estimates, asymptotically equivalent to the effect of the ideal compensation signal (3.1), (3.7), at least asymptotic convergence of parametric errors to zero is necessary to be ensured. For this purpose, the identification law developed in [32] will be used.

Applying the instrumental variable (2.5), the regression equation (2.3) is extended by means of averaging and sliding window filters:

$$\begin{gathered}\dot{\vartheta}(t)=\zeta_{iv}(t)\,z(t)-\zeta_{iv}(t-T)\,z(t-T)\,,\ \vartheta(t_0)=0_{n+m+1},\\ \dot{\psi}(t)=\zeta_{iv}(t)\,\varphi^{\top}(t)-\zeta_{iv}(t-T)\,\varphi^{\top}(t-T)\,,\ \psi(t_0)=0_{(n+m+1)\times(n+m+1)},\end{gathered}\tag{3.8}$$

$$\begin{gathered}\dot{Y}(t)=-\frac{1}{F(t)}\dot{F}(t)\left(Y(t)-\vartheta(t)\right),\ Y(t_0)=0_{n+m+1},\\ \dot{\Phi}(t)=-\frac{1}{F(t)}\dot{F}(t)\left(\Phi(t)-\psi(t)\right),\ \Phi(t_0)=0_{(n+m+1)\times(n+m+1)},\\ \dot{F}(t)=pt^{p-1},\ F(t_0)=F_0>0,\end{gathered}\tag{3.9}$$

where $T>0$ denotes a sliding window width, $p\geqslant 1,\ F_0\geqslant t_0^p$ stand for the filter parameters.

Application of filtration (3.8) and (3.9) in case $\theta=\mathrm{const}$ allows one to obtain the following regression equation [32]:

$$Y(t)=\Phi(t)\,\theta+W(t)\,,\tag{3.10}$$

where the disturbance $W(t)$ satisfies the following equations:

$$\begin{gathered}\dot{W}(t)=-\frac{1}{F(t)}\dot{F}(t)\left(W(t)-\varepsilon(t)\right),\ W(t_0)=0_{n+m+1},\\ \dot{\varepsilon}(t)=\zeta_{iv}(t)\,w(t)-\zeta_{iv}(t-T)\,w(t-T)\,,\ \varepsilon(t_0)=0_{n+m+1}.\end{gathered}\tag{3.11}$$

Having multiplied (3.10) by $\operatorname{adj}\{\Phi(t)\}$, the set of scalar regression equations is obtained:

$$\begin{gathered}\mathcal{Y}(t)=\Delta(t)\,\theta+\mathcal{W}(t)\,,\\ \mathcal{Y}(t):=\operatorname{adj}\{\Phi(t)\}\,Y(t)\,,\quad \Delta(t):=\det\{\Phi(t)\}\,,\quad \mathcal{W}(t):=\operatorname{adj}\{\Phi(t)\}\,W(t)\,.\end{gathered}\tag{3.12}$$

Based on the regression equation (3.10), the estimate of the parameters θ can be obtained via application of the gradient descent method. However, in order to implement the compensation component (3.1), (3.7), the estimates of the parameters $\psi_b(\theta)$ and ρ are required. It should be noted that, owing to the equality $\psi_b(\theta)=\rho\mathcal{L}_\psi\mathcal{L}_b\theta$, it is sufficient to have estimates of the parameters ρ and θ to obtain $\hat{\psi}_b(t)$. The value of $\hat{\theta}(t)$ can be calculated via (3.12), and therefore, now we need to derive a regression equation with respect to ρ.

Having multiplied (3.12) by $e_1^{\top}\mathcal{L}_b$, we have

$$\begin{gathered}e_1^{\top}\mathcal{L}_b\mathcal{Y}(t)=\Delta(t)\,e_1^{\top}\mathcal{L}_b\theta+e_1^{\top}\mathcal{L}_b\mathcal{W}(t)\\ \Rightarrow e_1^{\top}\mathcal{L}_b\mathcal{Y}(t)=\Delta(t)\,b_m+e_1^{\top}\mathcal{L}_b\mathcal{W}(t)\,,\end{gathered}$$

from which the required regression equation is obtained via multiplication by ρ

$$\begin{gathered}\mathcal{Y}_\rho(t)=\mathcal{M}_\rho(t)\,\rho+\mathcal{W}_\rho(t)\,,\\ \mathcal{Y}_\rho(t):=\Delta(t)\,,\quad \mathcal{M}_\rho(t):=e_1^{\top}\mathcal{L}_b\mathcal{Y}(t)\,,\quad \mathcal{W}_\rho(t)=-\rho e_1^{\top}\mathcal{L}_b\mathcal{W}(t)\,.\end{gathered}\tag{3.13}$$

Based on the regression equations (3.12) and (3.13), the laws are designed to estimate all parameters that are necessary for the implementation of (3.1), (3.7):

$$\begin{gathered}\dot{\hat{\theta}}(t)=-\gamma\Delta(t)\left(\Delta(t)\,\hat{\theta}(t)-\mathcal{Y}(t)\right),\ \hat{\theta}(t_0)=\hat{\theta}_0,\\ \dot{\hat{\rho}}(t)=-\gamma_\rho\mathcal{M}_\rho(t)\left(\mathcal{M}_\rho(t)\,\hat{\rho}(t)-\mathcal{Y}_\rho(t)\right),\ \hat{\rho}(t_0)=\hat{\rho}_0,\\ \hat{\psi}_b(t)=\hat{\rho}(t)\,\mathcal{L}_\psi\mathcal{L}_b\hat{\theta}(t)\,.\end{gathered}\tag{3.14}$$

Theorem 2. *If Assumption 3 is met and additionally*

C1) $y(t)$ *and* $u_c(t)$ *are bounded,*

C2) $\Delta \notin L_2$,

then the identification laws (3.14) *ensure that:*

1) *the parametric errors* $\tilde{\theta}(t) = \hat{\theta}(t) - \theta$, $\tilde{\rho}(t) = \hat{\rho}(t) - \rho$, $\tilde{\psi}_b(t) = \hat{\psi}_b(t) - \psi_b(\theta)$ *converge asymptotically to zero:*

$$\lim_{t\to\infty} \tilde{\theta}(t) = 0,\ \ \lim_{t\to\infty} \tilde{\rho}(t) = 0,\ \ \lim_{t\to\infty} \tilde{\psi}_b(t) = 0,$$

2) $\tilde{\theta} \in L_2 \cap L_\infty$, $\tilde{\rho} \in L_2 \cap L_\infty$, $\tilde{\psi}_b \in L_2 \cap L_\infty$.

Proof of Theorem 2 is presented in Appendix.

As, according to the problem statement, the component $u_b(t)$ of the control law is stabilising and the polynomial $R_{cl}(\theta_{cl},\ s)$ is Hurwitz one, then C1 requires to form a bounded compensation signal $u_c(t)$, for example, using a standard saturation function (see Section 3.3). Following proof from [32], a sufficient condition to meet C2 is that the reference signal $r(t)$ is sufficiently rich, *e.g.*, when $m = n - 1$, it has to include at least n different frequencies. Assumption 3 (inequality (2.6)) is met, for example, if the spectra of the reference $r(t)$ and the perturbation $f(t)$ signals do not have common frequencies. The convergence conditions for identification laws of type (3.14) are given in more detail in [32].

One of the freedom degrees of the used parameterization (2.3) and the whole proposed identification scheme is the choice of the type of regressor $\varphi(t)$. If the initial values of the unknown parameters estimates are such that $|u_c(t) + f(t)| > |f(t)|$ (in the sense of mean integral value or variance), then use of (2.3) results in a parameterization with a smaller perturbation value $w(t)$ in a similar sense. In contrast, if the initial values of the unknown parameters estimates are such that $|u_c(t) + f(t)| < |f(t)|$ is achieved even without parametric adaptation (in the sense of mean integral value or variance), then the choice of the following regressor

$$\varphi(t) = \left[-\frac{\alpha_{n-1}^\top(s)}{\Lambda(s)} z(t) \ \ \frac{\alpha_m^\top(s)}{\Lambda(s)} u_b(t) \right]^\top$$

allows one to obtain a parametrization with smaller value of the disturbance $w(t)$ in a similar sense.

The further parameterization as well as the premises and results of Theorem 2 do not depend on whether the regressor is calculated using $u_b(t)$ or $u(t)$.

The states of filters (3.9) lose their awareness to new values of $\vartheta(t)$ and $\psi(t)$ signals, and thus, the parameters θ too, at the rate of $\frac{\dot{F}(t)}{F(t)}$. It is currently not possible to completely prevent such loss of awareness, since the coefficient $\frac{\dot{F}(t)}{F(t)}$ ensures that the parametric error converges to zero in the presence of an external perturbation.

However, by redefining:

$$\begin{aligned} Y(t) &:= \frac{1}{T}\vartheta(t), \\ \Phi(t) &:= \frac{1}{T}\psi(t) \end{aligned} \tag{3.15}$$

an identification law can be derived, which properties are more acceptable for practical scenarios.

Theorem 3. *If Assumption 3 is met and additionally*

C1) $y(t)$ *and* $u_c(t)$ *are bounded,*

C2) *there exist scalars* $\Delta_{\mathrm{UB}} \geqslant \Delta_{\mathrm{LB}} > 0$ *such that* $\Delta_{\mathrm{LB}} \leqslant |\Delta(t)| \leqslant \Delta_{\mathrm{UB}}$, $\forall t \geqslant t_e$, *then there exist a scalar* $\delta_0 > 0$ *and a signal* $\delta_1 \in L_1$, $\lim\limits_{t\to\infty} \delta_1(t) = 0$ *such that*

$$\left\| \begin{bmatrix} \tilde{\theta}(t) \\ \tilde{\rho}(t) \\ \tilde{\psi}_b(t) \end{bmatrix} \right\| \leqslant \delta_1(t) + \delta_0 T^{-1}, \ \forall t \geqslant t_0. \tag{3.16}$$

Proof of Theorem 3 can be found in Appendix.

Thus, the identification law (3.14) based on the signals (3.8) + (3.15) provides awareness to new values of the parameters θ, but at the same time parametric errors converge not to zero, but to its neighbourhood, which is proportional to the parameter $T > 0$, and under a more strict condition in comparison with $\Delta \notin L_2$.

3.3. Adaptive Auxiliary Loop

Being motivated by (3.1), (3.5a)–(3.5c), (3.6) and requirement C1 and using the estimates (3.14), the compensation component of the control law is formed as:

$$\begin{aligned} &u_c(t) = \mathrm{sat}_{f_{\max}}\left\{-\hat{f}(t)\right\}, \\ &\hat{f}(t) = \psi_a^\top(\Theta)\,\hat{h}_\varepsilon(t) + \left(\hat{\psi}_a^\top(t) - \psi_a^\top(\Theta)\right)\hat{h}_y(t), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} &\begin{cases} \dot{\hat{H}}_i^\varepsilon(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)\hat{H}_i^\varepsilon(t) + e_{n+1}\left(K_0(t)\hat{\varepsilon}_f(t) - \hat{v}_\varepsilon(t)\right) \\ \hat{h}_i^\varepsilon(t) = e_{i+1}^\top \hat{H}_i^\varepsilon(t), \ \hat{v}_\varepsilon(t) = \sum\limits_{j=0}^{i} \hat{v}_j^\varepsilon(t,\ M_0,\ K_0), \end{cases} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall i = 0,\ldots,n, \\ &\begin{cases} \dot{\hat{H}}_i^y(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)\hat{H}_i^y(t) + e_{n+1}\left(K_0(t)\hat{y}_f(t) - \hat{v}_y(t)\right) \\ \hat{h}_i^y(t) = e_{i+1}^\top \hat{H}_i^y(t), \ \hat{v}_y(t) = \sum\limits_{j=0}^{i} \hat{v}_j^y(t,\ M_0,\ K_0), \end{cases} \end{aligned} \tag{3.18a}$$

$$\begin{aligned} &\dot{\hat{\xi}}_\varepsilon(t) = \left(A_0 - \hat{\psi}_b(t)\,e_1^\top\right)\hat{\xi}_\varepsilon(t) + \hat{\rho}(t)\,e_m\varepsilon(t), \\ &\qquad \begin{cases} \hat{\varepsilon}_f(t) = e_1^\top\hat{\xi}_\varepsilon(t), \ \text{if } m \geqslant 1 \\ \hat{\varepsilon}_f(t) = \hat{\rho}(t)\,\varepsilon(t), \ \text{if } m = 0, \end{cases} \\ &\dot{\hat{\xi}}_y(t) = \left(A_0 - \hat{\psi}_b(t)\,e_1^\top\right)\hat{\xi}_y(t) + \hat{\rho}(t)\,e_m y(t), \\ &\qquad \begin{cases} \hat{y}_f(t) = e_1^\top\hat{\xi}_y(t), \ \text{if } m \geqslant 1 \\ \hat{y}_f(t) = \hat{\rho}(t)\,y(t), \ \text{if } m = 0, \end{cases} \end{aligned} \tag{3.18b}$$

$$\begin{cases} \dot{\hat{x}}(t) = A_0\hat{x}(t) - \Theta\hat{y}(t) + \hat{\theta}_b(t)\,u(t) \\ \hat{y}(t) = e_1^\top\hat{x}(t), \end{cases} \tag{3.18c}$$

and $\varepsilon(t) = y(t) - \hat{y}(t)$, $\hat{\psi}_a(t) := \psi_a\left(\hat{\theta}_a\right)$, $\hat{\theta}_a(t) = \mathcal{L}_a\hat{\theta}(t)$, $\mathrm{sat}_{f_{\max}}\{.\}$ is a conventional saturation function to bound the absolute value of the signal $u_c(t)$ by $f_{\max}$. The signals $\hat{v}_\varepsilon(t)$ and $\hat{v}_y(t)$ are calculated via equations given in Proposition 1.

Theorem 4. *If*

1) *Assumptions 1–3 are met and the premise* C2 *from Theorem 2 is satisfied,*
2) $|f(t)| < f_{\max}$ *for all* $t \geqslant t_0$ *and a sufficiently large scalar* $f_{\max} > 0$,

then the control signal (2.2) + (3.17) + (3.14) *ensures* $\lim\limits_{t\to\infty} |\tilde{y}(t)| = 0$.

Proof of Theorem 4 is given in Appendix.

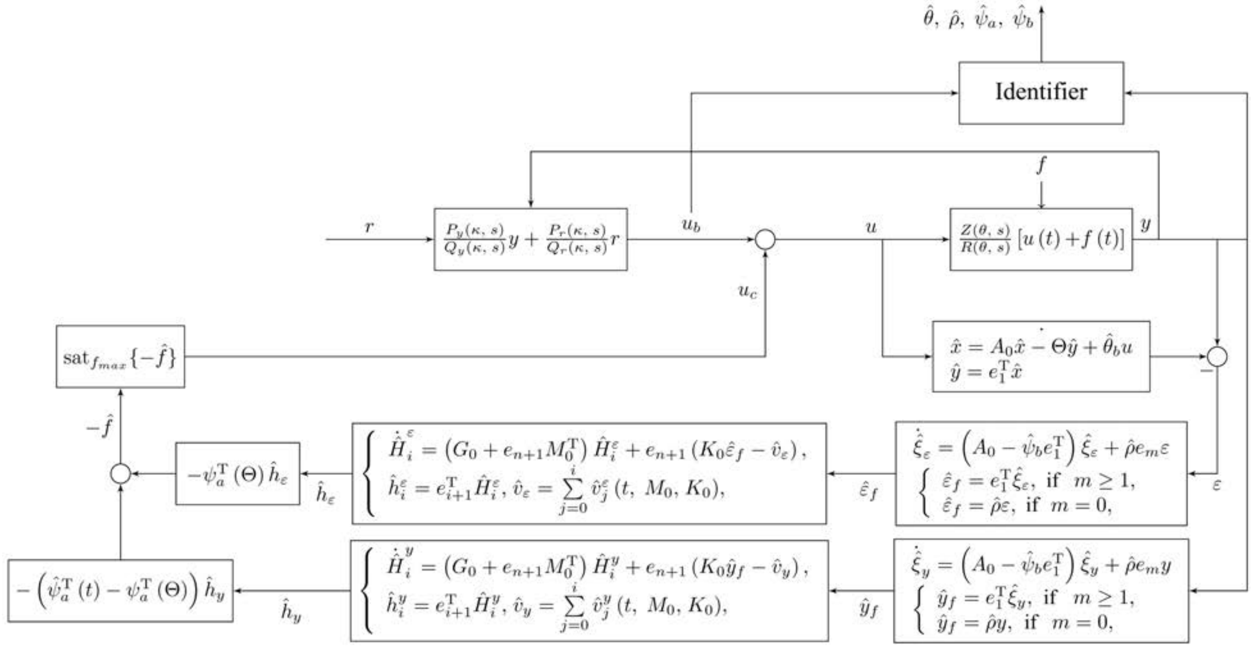

Fig. 1. Structural scheme of adaptive auxiliary loop.

The application of time-varying filters (3.18a) may be inappropriate for practical scenarios, for example, due to the requirement of the control signal noise resistance or inaccuracies related to the discretization/numerical solution of time-varying differential equations (3.18a). Therefore, in Theorem 5 we investigate the stability of the closed-loop control system for the case when $\mu\left(t\right) = \text{const}$ and Assumption 2 is not satisfied (in the sense that $\lambda \notin L_p$). Note that the situation when $\mu\left(t\right) = const$ but Assumption 2 is met ($i.e.$, $\lambda \notin L_p$) has already been considered in Theorem 4.

Theorem 5. *If*

1) *Assumptions 1, 3 are met and the premise* C2 *from Theorem 2 is satisfied,*
2) $\mu\left(t\right) = \mu > 0,\ \lambda \in L_\infty,$
3) $\left|f\left(t\right)\right| < f_{\max}$ *for all* $t \geqslant t_0$ *and sufficiently large scalar* $f_{\max} > 0,$

then the control signal (2.2) + (3.17) + (3.14) *ensures* $\lim\limits_{t\to\infty}\left|\tilde{y}\left(t\right)\right| \leqslant \epsilon$ *for arbitrarily small value of* $\epsilon > 0.$

Proof of Theorem is postponed to Appendix.

Therefore, if $(n+1)$ derivatives of the perturbation are bounded, then the proposed adaptive compensation system (3.17), (3.18a)–(3.18c), (3.9)+(3.14) of the external disturbance with time-invariant observers of the derivatives (3.18a) ensures convergence of the tracking error to an arbitrarily small neighbourhood of zero.

Now we use the identifier (3.14) based on (3.15) instead of (3.9) to adjust the parameters of the compensator (3.17), (3.18a)–(3.18c).

Theorem 6. *If*

1) *Assumptions 1, 3 are met and the premise* C2 *from Theorem 3 is satisfied,*
2) $\mu\left(t\right) = \mu > 0,\ \lambda \in L_\infty,$
3) $\left|f\left(t\right)\right| < f_{\max}$ *for all* $t \geqslant t_0$ *and sufficiently large scalar* $f_{\max} > 0,$

then the control signal (2.2) + (3.17) *with the identification laws* (3.14) + (3.15) *ensures* $\lim\limits_{t\to\infty}|\tilde{y}(t)| \leqslant \epsilon$ *for an arbitrarily small scalar* $\epsilon > 0.$

Proof of Theorem 6 can be found in Appendix.

The proposed adaptive system to compensate for an external bounded perturbation consists of an adaptive auxiliary model (3.18c), adaptive filtering algorithms (3.18b), filtered derivative observers with high-gain feedback (3.18a), compensation law (3.17) and identification algorithms (3.14) based on the signals calculated with the help of the instrumental model (2.5) and filters (2.3), (3.8), (3.9) or (2.3), (3.8), (3.15). The structural diagram of such a control system is shown in Fig. 1.

Unlike the existing solutions [2–26], if the conditions of parametric convergence are met, the proposed adaptive compensator separates estimation of the additive and parametric perturbations and asymptotically compensates for the perturbation $f(t)$, which affects the system with unknown parameters. At the same time, the stabilising component of the controller is designed independently of the compensation one, which allows one to obtain a control system with two degrees of freedom. In fact, three different schemes of external perturbation compensation are proposed in this study, which can be classified as follows:

—time-varying observers of filtered derivatives (3.18a) + the identifier (3.14) based on the signals obtained with the help of the filtering with averaging (3.9),

—time-invariant observers of the filtered derivatives (3.18a) + the identifier (3.14) based on the signals obtained with the help of the filtering with averaging (3.9),

—time-invariant observers of the filtered derivatives (3.18a) + the identifier (3.14) based on the signals obtained with the help of the sliding window filtering (3.8) + (3.15).

The first two schemes are of theoretical significance, but due to the loss of awareness of the (3.9) filters to the unknown parameters changes and the potentially infinitely large gain of the filtered derivatives observers (3.18a), they are of little practical value. The third scheme is free from the disadvantages of the first two schemes, but ensures convergence of the tracking error only to a bounded neighbourhood of the equilibrium point and under a stricter condition (C2 from Theorem 3 instead of C2 from Theorem 2).

Remark 5. The use of time-varying derivative observers (3.18a) together with an identifier based on signals obtained by sliding window filtering (3.15) is not reasonable, as, due to the properties of the identifier, the convergence of the tracking error will be ensured only to a bounded set.

4. NUMERICAL EXPERIMENTS

A differential equation has been considered:

$$m\frac{d^2}{dt^2}y(t) = \mathcal{F}(y,\ u,\ t).$$

We assumed that $\frac{1}{m}\mathcal{F}(y,\ u,\ t)$ could be approximated as:

$$\frac{1}{m}\mathcal{F}(y,\ u,\ t) := \theta_2\frac{d}{dt}y(t) + \theta_1 y(t) + \theta_3(u + f(t)),\ \theta_3 := \frac{1}{m},$$

from which, owing to $s[.] := \frac{d}{dt}[.]$, the second-order dynamic system was obtained:

$$y(t) = \frac{\theta_3}{s^2 + \theta_2 s + \theta_1}[u(t) + f(t)]. \tag{4.1}$$

The stabilizing component of the control law was defined as a PID-controller with proper differential summand:

$$u_b(t) = K_P(r(t) - y(t)) + \frac{K_I}{s}[r(t) - y(t)] + \frac{K_D s}{K_F \mathrm{s} + 1}[r(t) - y(t)]. \tag{4.2}$$

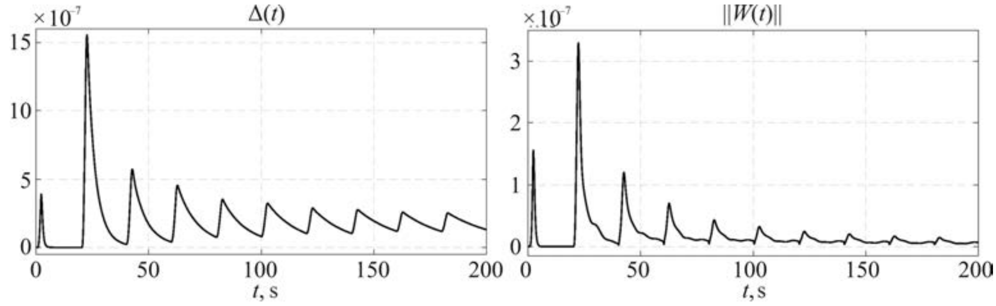

Fig. 2. Behavior of $\Delta(t)$ and $||\mathcal{W}(t)||$.

The parameters of the system (4.1) and the control law (4.2) were chosen as:

$$\begin{gathered}\theta_1 = 2, \quad \theta_2 = 5, \quad \theta_3 = 1, \\ K_P = 6, \quad K_I = 2.5, \quad K_D = 1.5, \quad K_F = 0.01, \\ f(t) = 2.5 + 1.25\cos(\pi t) + 2.5\sin(0.3\pi t), \\ r(t) = 10\mathrm{sgn}(\sin(0.05\pi t)).\end{gathered} \tag{4.3}$$

The parameters of the adaptive auxiliary loop (3.18a)–(3.18c), the parametrization (2.3), the instrumental model (2.5), the filters (3.8), (3.9) and the identification laws (3.14) were set as follows:

$$\begin{gathered}\Theta = \begin{bmatrix}20 & 100\end{bmatrix}^\top, \; m_0 = 1, \; m_1 = 3, \; m_3 = 3, \; \mu = 10^4, \\ \Lambda(s) = s^2 + 20s + 100, \; Z(\theta_{iv}, \; s) = 4, \\ R(\theta_{iv}, \; s) = s^2 + 2s + 4, \; p = 2, \; T = 4, \\ \hat{\theta}_0 = \begin{bmatrix}2 & 4 & 2\end{bmatrix}^\top, \; \hat{\rho}_0 = 0.5, \; \gamma = \gamma_\rho = 10^{13}, \; \overline{f}_{\max} = 10.\end{gathered} \tag{4.4}$$

The parameters of the adaptive auxiliary loop (3.18a)–(3.18c), the parameterization (2.3) and the instrumental model (2.5) were chosen to guarantee the stability of the corresponding differential equations. The values of gains γ, γ_ρ were picked by trial and error so as to ensure approximately the following proportionality:

$$\gamma \sim \frac{1}{\Delta^2(t)}, \; \gamma_\rho \sim \frac{1}{\mathcal{M}_\rho^2(t)}. \tag{4.5}$$

Considering practical scenarios, we need *a priori* data on the amplitude values of the regressor $\Delta(t)$ calculated for typical system trajectories at fixed p and T to choose γ, γ_ρ, γ_ρ, γ_{ψ_b}. In case such information is not available, the parameters (4.5) have to be picked online by trial and error.

Figure 2 presents the behavior of the regressor $\Delta(t)$ and the norm of the disturbance $\mathcal{W}(t)$ from the regression equation (3.12).

The transients presented in Fig. 2 demonstrate that throughout the simulation, starting from $t = T = 4$, the regressor $\Delta(t)$ was bounded away from zero and the perturbation $\mathcal{W}(t)$ was asymptotically decreasing. According to the analysis from [32], these two observations verify that Assumption 3 and the premise C2 from Theorem 2 were met.

Figure 3a depicts behavior of the tracking error $|\tilde{y}(t)|$. In Fig. 3b transients of the compensation error $|u_c(t) + f(t)|$ are given. Figures 3c and 3d are to show behavior of estimates $\hat{\theta}(t)$ and $\hat{\rho}(t)$, respectively.

The presented transients illustrate the result proved in Theorem 4. If the parametric convergence conditions C2 from Theorem 2 and Assumption 3 are met (see comments to Fig. 2), then the asymptotic convergence of the estimates $\hat{\theta}(t)$ and $\hat{\rho}(t)$ to the unknown parameters θ and ρ is

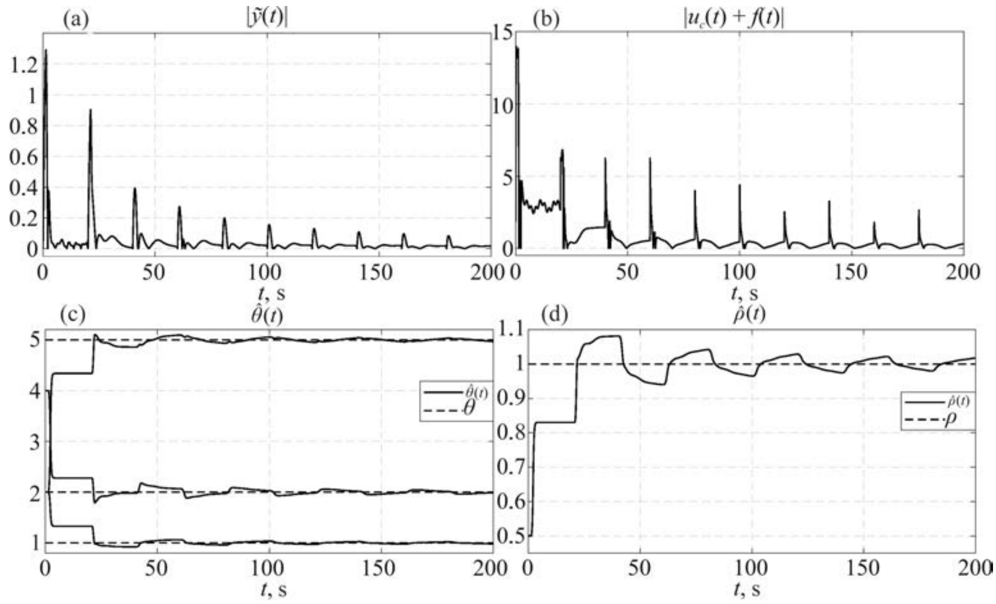

Fig. 3. Behavior of $|\tilde{y}(t)|$, $|u_c(t) + f(t)|$ and $\hat{\theta}(t)$, $\hat{\rho}(t)$.

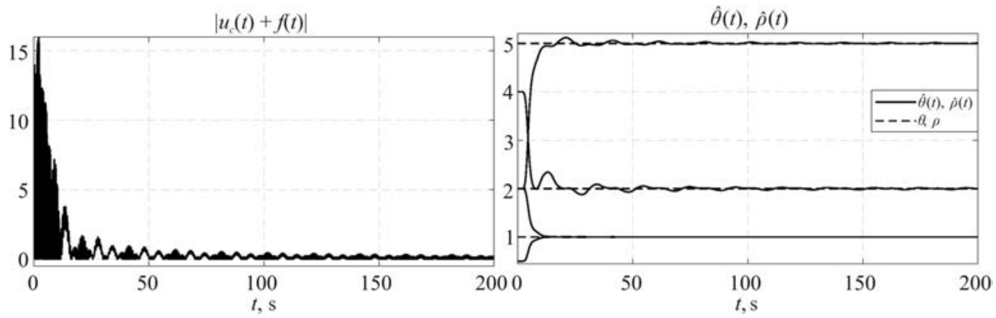

Fig. 4. Behavior of $|u_c(t) + f(t)|$ and $\hat{\theta}(t)$, $\hat{\rho}(t)$.

ensured, which leads to the asymptotic convergence to zero of the compensation error $u_c(t) + f(t)$, which, in its turn, results in asymptotic compensation of the effect of the perturbation $f(t)$ on the transient quality. Note that from the practical point of view, due to the use of the integral summand in the control signal (4.2), it is sufficient to guarantee $u_c(t) + f(t) \approx \text{const}$ to compensate for the perturbation $f(t)$. Considering the conducted experiment, the parametric error convergence was achieved when we used a typical for practical scenarios reference signal in the form of a rectangular periodic signal. This result allows one to conclude that the conditions of parametric error convergence are not restrictive and met for typical practical cases.

The next aim was to demonstrate that, if the reference signal had a special form, then the transient quality for $\hat{\theta}(t)$, $\hat{\rho}(t)$ and $u_c(t) + f(t)$ could be improved. So $r(t)$ and gains γ, γ_ρ were chosen as:

$$r(t) = 10\left[\sin(0.2\pi t) + \sin(3\pi t)\right], \ \gamma = \gamma_\rho = 10^{10}, \tag{4.6}$$

whereas all other parameters values were set in accordance with (4.3) and (4.4).

The reference signal (4.6) spectrum has no common frequencies with the one of the disturbance, and $r(t)$ includes $n = 2$ frequencies. According to [32], together with the boundedness of $y(t)$, $u_c(t)$, it immediately results in satisfaction of the premise C2 from Theorem 2 and Assumption 3.

Figure 4 depicts behavior of $u_c(t) + f(t)$ and $\hat{\theta}(t)$, $\hat{\rho}(t)$ for the experiment with the reference signal (4.6).

Such choice of the reference signal improved the transient quality for $\hat{\theta}(t)$, $\hat{\rho}(t)$ and $u_c(t)+f(t)$. Therefore, the conducted numerical experiments validated the theoretical conclusions made in this study and illustrated the properties of the proposed system for different reference signals.

Remark 6. Results of numerical experiments, which were not included in the paper, showed that for any choice of initial conditions $\hat{\theta}_0,\ \hat{\rho}_0$ the proposed solution provided asymptotic perturbation compensation. However, the transients quality could be arbitrarily poor in this case. Therefore, for practical scenarios, it is recommended to make the initial conditions $\hat{\theta}_0,\ \hat{\rho}_0$ equal to the 'nominal' parameters of the system, for example, the ones for which the stabilising component of the control law was designed. In this case, it is possible not only to ensure that the goal of (2.8) is achieved, but also to obtain an admissible transient quality.

5. CONCLUSION

A method of output-based compensation of a bounded additive perturbation affecting a linear minimum-phase system with unknown parameters is developed. The basis of the proposed solution is the perturbation compensation algorithm grounded on A.M. Tsykunov's auxiliary loop method [26, p. 196], which requires knowledge of the system parameters for asymptotic estimation and compensation of the additive perturbation. To relax this requirement, the identification law based on the method of instrumental variables and the DREM procedure is used to provide exact online asymptotic identification of the system unknown parameters in a closed loop. The convergence conditions of the parameter identification process are:

C1) boundedness of the compensation signal (for example, via $\text{sat}\{.\}$ function),
C2) sufficiently rich reference signal,
C3) no common frequencies in spectra of the reference and disturbance signals.

The obtained estimates are used instead of the unknown parameters in the perturbation compensation algorithm based on the auxiliary loop method [26]. It is shown that the substitution of unknown parameters with their estimates is feasible in the sense that under the conditions C1)–C3) the proposed system of perturbation adaptive compensation, as well as its non-adaptive equivalent, provides asymptotic estimation and compensation of the disturbance.

The properties of the proposed solution are demonstrated via numerical simulation. It is shown that the conditions C1)–C3) are not restrictive and met for typical control systems with PI-, PID-controllers and conventional reference signals. The authors believe that the proposed approach has a potential for practical application, but note that the method has a drawback related to the need for trial and error selection of some parameters of the algorithm for identification and estimation of unmeasured derivatives.

The scope of further research is to improve the quality of transients for perturbation estimation and increase the convergence rate of the unknown parameters estimates to their true values.

APPENDIX A

This appendix contains auxiliary lemmas, which are axiomatically used to prove main result of this study.

Lemma A1. *For any stable and proper operator* $\mathcal{H}(t,\ s)[.]$ *and corresponding signal* $y(t) = \mathcal{H}(t,\ s)[u(t)]$ *the following holds:* $u \in L_p \Rightarrow y \in L_p,\ p \in [1,\ \infty]$.

Proof is presented in [35, p. 75].

Lemma A2. *If* $\dot{f} \in L_\infty$ *and* $f \in L_p,\ p \in (0,\ \infty)$, *then* $\lim\limits_{t\to\infty} f(t) = 0$.

Proof is given in [35, p. 80].

Lemma A3. *Consider the scalar system defined by*

$$\dot{x}(t) = -a^2(t)\,x(t) + b(t),\ x(t_0) = x_0,$$

where $x(t) \in \mathbb{R}$ *and* $a,\ b\colon \mathbb{R}_+ \mapsto \mathbb{R}$ *are piecewise continuous bounded functions. If* $a \notin L_2$ *and* $b \in L_1$*, then* $\lim\limits_{t\to\infty} x(t) = 0$.

Proof can be found in [36, Section 3.A.1].

Lemma A4. *Consider the nonnegative scalar functions* $f\colon [t_0,\ \infty) \mapsto \mathbb{R}$, $g\colon [t_0,\ \infty) \mapsto \mathbb{R}$*. If* $f(t) \leqslant g(t)$ *for all* $t \geqslant t_0$ *and* $g \in L_p$, $p \in (0,\ \infty)$*, then* $f \in L_p$.

Proof is presented in [28, p. 74].

Lemma A5. *If* $f \in L_p$, $1 \leqslant p < \infty$*, then* $g(t) = H(s)[f(t)] \in L_\infty$ *and* $\lim\limits_{t\to\infty} g(t) = 0$ *for any stable and strictly proper transfer function* $H(s)[.]$.

Proof is given in [35, p. 83].

APPENDIX B

Proof of Statement 1. A linear time-varying operator is defined as:

$$\mathcal{H}(t,s)[.] = e_1^\top \left(sI_{n+1} - G_0 - e_{n+1}M_0^\top(t)\right)^{-1} e_{n+1}K_0(t)[.].$$

In order to prove the proposition under consideration, firstly, an auxiliary lemma is proved for the above-given time-varying operator.

Lemma B1. *Define signals* $\left(s[.] := \frac{d}{dt}[.]\right)$:

$$Y^*(t) = \mathcal{H}(t,s)\left[s^i[U(t)]\right], \tag{B.1a}$$

$$Y_i(t) = e_{i+1}^\top\left(sI_{n+1} - G_0 - e_{n+1}M_0^\top(t)\right)^{-1} e_{n+1}K_0(t)[U(t) - v(t)], \tag{B.1b}$$

where

$$\begin{aligned}
&v(t) = \sum_{j=0}^{i} v_j(t,\ M_0,\ K_0),\\
&v_0(t,\ M_0,\ K_0) = 0,\\
&v_1(t,\ M_0,\ K_0) = s^{-1}\left[s\left[M_0^\top(t)\right]X(t)\right] + s^{-1}\left[s\left[K_0(t)\right]U(t)\right],\\
&\vdots\\
&v_j(t,\ M_0,\ K_0) = v_{j-1}(t,\ M_0,\ K_0) - s^{-1}\left[v_{j-1}\left(t,\ \dot{M}_0,\ \dot{K}_0\right)\right],\ j = 2, \ldots i,
\end{aligned} \tag{B.2}$$

and $X(t) = \left(sI_{n+1} - G_0 - e_{n+1}M_0^\top(t)\right)^{-1} e_{n+1}K_0(t)[U(t) - v(t)]$.

Then for all $i = 0, \ldots, n$ *and* $t \geqslant t_0$ *it holds that* $Y^*(t) = Y_i(t)$.

Proof. Equations (B.1a) and (B.1b) are represented in the state-space form:

$$\begin{cases} \dot{X}^*(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)X^*(t) + e_{n+1}K_0(t)\,s^i[U(t)] \\ Y^*(t) = e_1^\top X^*(t), \end{cases}$$

$$\begin{cases} \dot{X}(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)X(t) + e_{n+1}\left(K_0(t)U(t) - v(t)\right) \\ Y_i(t) = e_{i+1}^\top X(t). \end{cases}$$

Owing to the structure of the matrix $G_0 + e_{n+1}M_0^{\top}(t)$, it is easy to check that the following equations hold:

$$\begin{aligned}
&e_1^{\top} X(t) = Y_0(t),\\
&e_1^{\top} s[X(t)] = e_1^{\top}\left(G_0 + e_{n+1}M_0^{\top}(t)\right)X(t) = e_2^{\top}X(t) = Y_1(t),\\
&e_1^{\top} s^2[X(t)] = e_1^{\top} s\left[\left(G_0 + e_{n+1}M_0^{\top}(t)\right)X(t)\right] = e_1^{\top}\left(G_0 + e_{n+1}M_0^{\top}(t)\right)\dot{X}(t)\\
&\qquad\qquad = e_1^{\top}\left(G_0 + e_{n+1}M_0^{\top}(t)\right)^2 X(t) = e_2^{\top} s[X(t)] = e_3^{\top}X(t) = Y_2(t),\\
&\vdots\\
&e_1^{\top} s^i[X(t)] = e_i^{\top} s[X(t)] = e_{i+1}^{\top}X(t) = Y_i(t), \quad i = 0, \ldots, n,
\end{aligned}$$

which allows one to define the error vector as:

$$\begin{aligned}
E(t) &= s^i[X(t)] - X^*(t),\\
e_1^{\top}E(t) &= Y_i(t) - Y^*(t).
\end{aligned}$$

Equations for $s^{i+1}[X(t)]$ for all $i = 0, \ldots, n$ are written as:

$$\begin{aligned}
&s[X(t)] = \left(G_0 + e_{n+1}M_0^{\top}(t)\right)X(t) + e_{n+1}\left(\tilde{v}_0(t) + K_0(t)U(t) - \sum_{j=1}^{i} v_j(t)\right),\\
&s^2[X(t)] = \left(G_0 + e_{n+1}M_0^{\top}(t)\right)s[X(t)] + e_{n+1}\left(\tilde{v}_1(t) + K_0(t)s[U(t)] - s\left[\sum_{j=2}^{i} v_j(t)\right]\right),\\
&s^3[X(t)] = \left(G_0 + e_{n+1}M_0^{\top}(t)\right)s^2[X(t)] + e_{n+1}\left(\tilde{v}_2(t) + K_0(t)s^2[U(t)] - s^2\left[\sum_{j=3}^{i} v_j(t)\right]\right),\\
&s^4[X(t)] = \left(G_0 + e_{n+1}M_0^{\top}(t)\right)s^3[X(t)] + e_{n+1}\left(\tilde{v}_3(t) + K_0(t)s^3[U(t)] - s^3\left[\sum_{j=4}^{i} v_j(t)\right]\right),\\
&\vdots\\
&s^{i+1}[X(t)] = \left(G_0 + e_{n+1}M_0^{\top}(t)\right)s^i[X(t)] + e_{n+1}\left(\tilde{v}_i(t) + K_0(t)s^i[U(t)]\right),
\end{aligned}$$

where

$$\begin{aligned}
&\tilde{v}_0(t) = v_0(t),\\
&\tilde{v}_1(t) = \dot{\tilde{v}}_0(t) + s\left[M_0^{\top}(t)\right]X(t) + s[K_0(t)]U(t) - s[v_1(t)],\\
&\tilde{v}_2(t) = \dot{\tilde{v}}_1(t) + s\left[M_0^{\top}(t)\right]s[X(t)] + s[K_0(t)]s[U(t)] - s^2[v_2(t)],\\
&\tilde{v}_3(t) = \dot{\tilde{v}}_2(t) + s\left[M_0^{\top}(t)\right]s^2[X(t)] + s[K_0(t)]s^2[U(t)] - s^3[v_3(t)],\\
&\vdots\\
&\tilde{v}_i(t) = \dot{\tilde{v}}_{i-1}(t) + s\left[M_0^{\top}(t)\right]s^{i-1}[X(t)] + s[K_0(t)]s^{i-1}[U(t)] - s^i[v_i(t)],\ i = 1, \ldots,\ n.
\end{aligned}$$

Owing to the recurrent sequence (B.2), it is obtained:

$$\begin{aligned}
&\tilde{v}_0(t) = v_0(t),\\
&s[v_1(t)] = s\left[M_0^\top(t)\right]X(t) + s[K_0(t)]U(t),\\
&s^2[v_2(t)] = s^2\left[M_0^\top(t)\right]X(t) + s^2[K_0(t)]U(t) + s\left[M_0^\top(t)\right]s[X(t)]\\
&\qquad + s[K_0(t)]s[U(t)] - s^2\left[M_0^\top(t)\right]X(t) - s^2[K_0(t)]U(t)\\
&\qquad = s\left[M_0^\top(t)\right]s[X(t)] + s[K_0(t)]s[U(t)],\\
&\vdots\\
&s^j[v_j(t)] = s\left[M_0^\top(t)\right]s^{j-1}[X(t)] + s[K_0(t)]s^{j-1}[U(t)], \quad j = 3, \dots i,
\end{aligned}$$

from which $\tilde{v}_i(t) = 0$ for all $i = 0, \dots, n$.

Then the derivative of $E(t)$ is written as:

$$\begin{aligned}
\dot{E}(t) &= s^{i+1}[X(t)] - \dot{X}^*(t)\\
&= \left(G_0 + e_{n+1}M_0^\top(t)\right)s^i[X(t)] + e_{n+1}\left(\tilde{v}_i(t) + K_0(t)s^i[U(t)]\right)\\
&\quad - \left(G_0 + e_{n+1}M_0^\top(t)\right)X^*(t) - e_{n+1}K_0(t)s^i[U(t)]\\
&= \left(G_0 + e_{n+1}M_0^\top(t)\right)E(t),\ E(t_0) = 0_{n+1},
\end{aligned}$$

and therefore, owing to the facts that $G_0 + e_{n+1}M_0^\top(t)$ is a Hurwitz matrix and $E(t_0) = 0_{n+1}$, it is obtained that $Y^*(t) = Y_i(t)$ for all $i = 0, \dots, n$ and $t \geqslant t_0$.

Proof of Lemma B1 is completed.

Now we are in position to continue proof of Proposition 1. Having applied the operator $\mathcal{H}(t,s)[.]$ to the left- and right-hand sides of (3.5a), it is obtained:

$$f_f(t) := \mathcal{H}(t,s)[f(t)] = \psi_a^\top(\Theta)h_\varepsilon(t) + \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)h_y(t),$$

where

$$h_y(t) = \begin{bmatrix} \mathcal{H}(t,s)[y_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[y_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[y_f(t)]] \end{bmatrix},\ h_\varepsilon(t) = \begin{bmatrix} \mathcal{H}(t,s)[\varepsilon_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[\varepsilon_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[\varepsilon_f(t)]] \end{bmatrix}.$$

The application of Lemma B1 yields:

$$h_y(t) = \begin{bmatrix} \mathcal{H}(t,s)[y_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[y_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[y_f(t)]] \end{bmatrix} = \begin{bmatrix} h_0^y(t)\\ \vdots\\ h_i^y(t)\\ \vdots\\ h_n^y(t) \end{bmatrix},\ h_\varepsilon(t) = \begin{bmatrix} \mathcal{H}(t,s)[\varepsilon_f(t)]\\ \vdots\\ \mathcal{H}(t,s)\left[s^{n-1}[\varepsilon_f(t)]\right]\\ \mathcal{H}(t,s)[s^n[\varepsilon_f(t)]] \end{bmatrix} = \begin{bmatrix} h_0^\varepsilon(t)\\ \vdots\\ h_i^\varepsilon(t)\\ \vdots\\ h_n^\varepsilon(t) \end{bmatrix},$$

so, consequently, the definition (3.6) is obtained.

The original $f(t)$ and filtered $f_f(t)$ disturbances are represented in the state-space form:

$$\begin{cases} \dot{\mathcal{F}}(t) = G_0\mathcal{F}(t) + e_{n+1}s^{n+1}[f(t)] \\ f(t) = e_1^\top \mathcal{F}(t), \end{cases}$$

$$\begin{cases} \dot{\mathcal{F}}_f(t) = \left(G_0 + e_{n+1}M_0^\top(t)\right)\mathcal{F}_f(t) + e_{n+1}K_0(t)[f(t)] \\ f_f(t) = e_1^\top \mathcal{F}_f(t). \end{cases}$$

Then the error $\tilde{f}(t)$ satisfies the below-given equation:

$$\begin{gathered} \dot{\tilde{\mathcal{F}}} = \left(G_0 + e_{n+1}M_0^\top(t)\right)\mathcal{F}_f(t) + e_{n+1}K_0(t)[f(t)] \\ -\left(G_0 \pm e_{n+1}M_0^\top(t)\right)\mathcal{F}(t) - e_{n+1}s^{n+1}[f(t)] = \left(G_0 + e_{n+1}M_0^\top(t)\right)\tilde{\mathcal{F}}(t) \\ + e_{n+1}K_0(t)\left(f(t) - \frac{1}{m_0\mu^{n+1}(t)}\left(s^{n+1}[f] + \sum_{i=0}^{n} m_i\mu^{n-(i-1)}(t)\,s^i[f(t)]\right)\right) \\ = \left(G_0 + e_{n+1}M_0^\top(t)\right)\tilde{\mathcal{F}}(t) + e_{n+1}K_0(t)\lambda(t), \\ \tilde{f}(t) = e_1^\top\tilde{\mathcal{F}}(t), \end{gathered}$$

or, representing it in the operator form, we have:

$$\tilde{f}(t) = \mathcal{H}(t,s)[\lambda(t)]. \tag{B.3}$$

In accordance with Assumption 2, it holds that $\lambda \in L_p \cap L_\infty$ for some $p \in [1,\ \infty)$, and then application of Lemma A1 allows one to obtain $\tilde{f} \in L_p \cap L_\infty$.

Equation (B.3) is rewritten in the observer canonical form:

$$\begin{cases} \dot{E}(t) = \left(G_0 + \mathcal{I}_{n+1}M_0(t)\,e_1^\top\right)E(t) + e_{n+1}K_0(t)\lambda(t) \\ \tilde{f}(t) = e_1^\top E(t). \end{cases} \tag{B.4}$$

The normalized error is defined as

$$\eta(t) = \Gamma^{-1}(t)\,E(t),$$

where $\Gamma(t) = \text{diag}\{1, \mu(t), \ldots, \mu^n(t)\}$, and therefore, $\tilde{f}(t) = e_1^\top\eta(t) = e_1^\top E(t)$.

Having differentiated $\eta(t)$ and used (B.4), it is obtained:

$$\begin{aligned} \dot{\eta}(t) &= \frac{d\Gamma^{-1}(t)}{dt}E(t) + \Gamma^{-1}(t)\left(G_0 + \mathcal{I}_{n+1}M_0(t)\,e_1^\top\right)E(t) + \Gamma^{-1}(t)\,e_{n+1}K_0(t)\lambda(t) \\ &= \left(\frac{d\Gamma^{-1}(t)}{dt}\Gamma(t) + \mu(t)\,G\right)\eta(t) + e_{n+1}m_0\mu(t)\lambda(t),\ G = G_0 - \mathcal{I}_{n+1}Me_1^\top, \end{aligned}$$

where the following equalities are used (they can be easily checked via substitution):

$$\Gamma^{-1}(t)\,G_0\Gamma(t) = \mu(t)\,G_0,\quad \Gamma^{-1}(t)\,\mathcal{I}_{n+1}M_0(t) = -\mu(t)\,\mathcal{I}_{n+1}M,\quad e_1^\top\Gamma(t) = e_1^\top.$$

According to Assumption 2, it holds that $\mu\lambda \in L_\infty$, and, as the autonomous differential equation

$$\dot{x}(t) = \left(\frac{d\Gamma^{-1}(t)}{dt}\Gamma(t) + \mu(t)\,G\right)x(t),\ x(t_0) = x_0$$

is asymptotically stable, then $\mu\lambda \in L_\infty \Rightarrow \dot{\eta} \in L_\infty$, therefore, owing to $\dot{\tilde{f}}(t) = \dot{\eta}_1(t)$ it holds that $\dot{\tilde{f}} \in L_\infty$, and, following Lemma A2, we have $\lim\limits_{t\to\infty} \tilde{f}(t) = 0$.

Proof of Theorem 1. Having substituted (2.2) + (3.7) into (2.1) and subtracted (2.7) from the obtained result, it is written:

$$\tilde{y}(t) = W_{cl}(\theta_{cl},\ s)\left[\tilde{f}(t)\right].$$

According to Proposition 1, we have $\tilde{f} \in L_p \cap L_\infty$ for $p \in [1,\ \infty)$, then $\tilde{y} \in L_\infty$ and, following Lemma A5, it is obtained that $\lim\limits_{t\to\infty} \tilde{y}(t) = 0$.

Proof of Theorem 2. The identification laws (3.14) are rewritten in general terms:

$$\dot{\hat{\kappa}}(t) = -\gamma\mathcal{M}_\kappa(t)\left(\mathcal{M}_\kappa(t)\hat{\kappa}(t) - \mathcal{Y}_\kappa(t)\right),\ \hat{\kappa}(t_0) = \hat{\kappa}_0, \tag{B.5}$$

where $\kappa \in \{\theta,\ \rho\}$, $\hat{\kappa} \in \left\{\hat{\theta},\ \hat{\rho}\right\}$ and $\mathcal{Y}_\kappa(t) = \mathcal{M}_\kappa(t)\kappa + \mathcal{W}_\kappa(t)$.

Taking into consideration Proposition 5 from [32], if C1 and Assumption 3 are met, then $\mathcal{W} \in L_2$. Thus, as $\mathcal{M}_\rho \in L_\infty$, $\mathcal{Y}_\rho \in L_\infty$ (owing to C1), then on the basis of (3.12) and (3.13) we have $\mathcal{W}_\rho \in L_2$. As, owing to (3.13), it holds that

$$\Delta \notin L_2 \Rightarrow \mathcal{M}_\rho \notin L_2,$$

then Theorem 2 can be proved via analysis of properties of the general law (B.5) in case the conditions $\mathcal{W}_\kappa \in L_2$ and $\mathcal{M}_\kappa \notin L_2$ are satisfied.

Considering Theorem 2 from [32] and Proposition 1 from [37], if $\mathcal{W}_\kappa \in L_2$ and $\mathcal{M}_\kappa \notin L_2$, then $\lim\limits_{t\to\infty} \tilde{\kappa}(t) = 0$, $\tilde{\kappa} \in L_\infty$. Therefore, to complete proof, we need to show $\tilde{\kappa} \in L_2$.

To that end, the following upper bound of the derivative of the function $V = \frac{1}{2}\tilde{\kappa}^\top\tilde{\kappa}$ is obtained:

$$\begin{aligned}\dot{V}(t) &\leqslant -\gamma\tilde{\kappa}^\top\mathcal{M}_\kappa^2(t)\tilde{\kappa} + \gamma\tilde{\kappa}^\top\mathcal{M}_\kappa(t)\mathcal{W}_\kappa(t)\\ &\leqslant -\gamma(1-\chi)\tilde{\kappa}^\top\mathcal{M}_\kappa^2(t)\tilde{\kappa} + \gamma\chi^{-1}\|\mathcal{W}_\kappa(t)\|^2\\ &\leqslant -2\gamma\mathcal{M}_\kappa^2(t)(1-\chi)V(t) + \gamma\chi^{-1}\|\mathcal{W}_\kappa(t)\|^2,\end{aligned} \tag{B.6}$$

where $\chi \in (0, 1)$.

Having integrated (B.6), it is written:

$$V(t) \leqslant V(t_0) - \gamma(1-\chi)\int\limits_{t_0}^{t}\mathcal{M}_\kappa^2(s)\|\tilde{\kappa}(s)\|^2 ds + \gamma\chi^{-1}\int\limits_{t_0}^{t}\|\mathcal{W}_\kappa(s)\|^2 ds.$$

As $V \in L_\infty$ (it was shown above) and $\mathcal{W}_\kappa \in L_2$, then the following integral is bounded:

$$\int\limits_{t_0}^{t}\mathcal{M}_\kappa^2(s)\|\tilde{\kappa}(s)\|^2 ds \leqslant \frac{-V(t) + V(t_0) + \gamma\chi^{-1}\int_{t_0}^{t}\|\mathcal{W}_\kappa(s)\|^2 ds}{\gamma(1-\chi)} < \infty.$$

There exists the following lower bound for the integrand (here $\mathcal{M}_\kappa(t) > 0$ is assumed to hold almost everywhere and, consequently, $\operatorname{ess}\inf\limits_t \mathcal{M}_\kappa^2(t) \neq 0$):

$$\operatorname{ess}\inf_t \mathcal{M}_\kappa^2(t)\|\tilde{\kappa}(t)\|^2 \leqslant \mathcal{M}_\kappa^2(t)\|\tilde{\kappa}(t)\|^2,$$

which, following Lemma A4, means that $\tilde{\kappa} \in L_2$.

Considering the error $\tilde{\psi}_b(t)$, the following is obtained:

$$\begin{aligned}\tilde{\psi}_b(t) &= \hat{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\hat{\theta}(t) - \rho\mathcal{L}_\psi\mathcal{L}_b\theta \pm \hat{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta \\ &= \hat{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) + \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta \pm \rho\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) \\ &= \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) + \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta + \rho\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t),\end{aligned}$$

from which it is written that

$$\left.\begin{array}{l}\tilde{\theta} \in L_2 \cap L_\infty \\ \tilde{\rho} \in L_2 \cap L_\infty\end{array}\right\} \Rightarrow \tilde{\psi}_b \in L_2 \cap L_\infty,\ \left.\begin{array}{l}\lim\limits_{t\to\infty}\tilde{\theta}(t) = 0 \\ \lim\limits_{t\to\infty}\tilde{\rho}(t) = 0\end{array}\right\} \Rightarrow \lim_{t\to\infty}\tilde{\psi}_b(t) = 0.$$

Proof of Theorem 3. Owing to (3.11) and using (3.15), it is written for $W(t)$:

$$W(t) = \frac{1}{T}\varepsilon(t),$$

$$\varepsilon(t) = \int\limits_{\max(t_0, t-T)}^{t} \zeta_{iv}(s)\, w(s)\, ds,$$

from which, if Assumption 3 is met, it is obtained that $|W_i(t)| \leqslant \frac{1}{T}c$.

If C1 is satisfied, the following holds for the regressor $\Phi(t)$:

$$\|\Phi(t)\| = \frac{1}{T}\left\|\int\limits_{\max(t_0, t-T)}^{t} \zeta_{iv}(s)\,\varphi^\top(s)\, ds\right\| \leqslant \frac{1}{T}T\sup_t\left\|\zeta_{iv}(t)\,\varphi^\top(t)\right\| = \sup_t\left\|\zeta_{iv}(t)\,\varphi^\top(t)\right\|,$$

therefore, there exists a scalar $c_{\mathcal{W}} > 0$ such that

$$\|\mathcal{W}(t)\| \leqslant \frac{1}{T}c_{\mathcal{W}},\ \|\mathcal{W}_\rho(t)\| \leqslant \frac{1}{T}c_{\mathcal{W}}. \tag{B.7}$$

Based on (B.7), the errors $\tilde{\theta}(t),\ \tilde{\rho}(t),\ \tilde{\psi}_b(t)$ are defined as:

$$\begin{aligned}\tilde{\theta}(t) &= \phi_{\tilde{\theta}}(t, t_0)\,\tilde{\theta}(t_0) + \int\limits_{t_0}^{t}\phi_{\tilde{\theta}}(t,\tau)\,\Delta(\tau)\,\mathcal{W}(\tau)\, d\tau, \\ \tilde{\rho}(t) &= \phi_{\tilde{\rho}}(t,\ t_0)\,\tilde{\theta}(t_0) + \int\limits_{t_0}^{t}\phi_{\tilde{\rho}}(t,\tau)\,\mathcal{M}_\rho(\tau)\,\mathcal{W}_\rho(\tau)\, d\tau, \\ \tilde{\psi}_b(t) &= \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t) + \tilde{\rho}(t)\mathcal{L}_\psi\mathcal{L}_b\theta + \rho\mathcal{L}_\psi\mathcal{L}_b\tilde{\theta}(t),\end{aligned} \tag{B.8}$$

where $\phi_{\tilde{\theta}}(t,\tau) = e^{-\gamma\int_\tau^t \Delta^2(s)ds}$, $\phi_{\tilde{\rho}}(t,\tau) = e^{-\gamma\int_\tau^t \Delta^2(s)ds}$.

Owing to (B.7), (3.13), if C1 is met, then it holds that $\Delta\mathcal{W},\ \mathcal{M}_\rho\mathcal{W}_\rho \in L_\infty$, and if C2 is satisfied, then $\phi_{\tilde{\theta}},\ \phi_{\tilde{\rho}} \in L_1$, from which, using (B.8) and owing to (B.7), it immediately follows that there exist $\delta_0 > 0$ and $\delta_1 \in L_1,\ \lim\limits_{t\to\infty}\delta_1(t) = 0$ such that the inequality (3.16) is satisfied.

Proof of Theorem 4. Having substituted (2.2) + (3.17) into (2.1) and subtracted (2.7) from the obtained result, it is written:

$$\begin{aligned}\tilde{y}(t) &= W_{cl}(\theta_{cl},\ s)\left[\mathrm{sat}_{f_{\max}}\left\{-\hat{f}(t) \pm f_f(t) \pm f(t)\right\} + f(t)\right] \\ &= W_{cl}(\theta_{cl},\ s)\left[\mathrm{sat}_{f_{\max}}\left\{\tilde{f}_f(t) - \tilde{f}(t) - f(t)\right\} + f(t)\right],\end{aligned} \tag{B.9}$$

where

$$\begin{aligned}
\tilde{f}_f(t) &= -\hat{f}(t) + f_f(t) \\
&= -\psi_a^\top(\Theta)\hat{h}_\varepsilon(t) - \left(\hat{\psi}_a^\top(t) - \psi_a^\top(\Theta)\right)\hat{h}_y(t) \pm \hat{\psi}_a^\top(t) h_y(t) \\
&\quad + \psi_a^\top(\Theta) h_\varepsilon(t) + \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right) h_y(t) \\
&= -\psi_a^\top(\Theta)\tilde{h}_\varepsilon(t) - \tilde{\psi}_a^\top(t) h_y(t) - \left(\hat{\psi}_a^\top(t) - \psi_a^\top(\Theta)\right)\tilde{h}_y(t) \\
&= -\psi_a^\top(\Theta)\tilde{h}_\varepsilon(t) - \tilde{\psi}_a^\top(t) h_y(t) - \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)\tilde{h}_y(t) - \tilde{\psi}_a^\top(t)\tilde{h}_y(t),
\end{aligned} \tag{B.10}$$

and (owing to Lemma B1)

$$\tilde{h}_y(t) = \begin{bmatrix} \mathcal{H}(t,s)[\tilde{y}_f(t)] \\ \vdots \\ \mathcal{H}(t,s)\left[s^{n-1}[\tilde{y}_f(t)]\right] \\ \mathcal{H}(t,s)\left[s^n[\tilde{y}_f(t)]\right] \end{bmatrix}, \quad \tilde{h}_\varepsilon(t) = \begin{bmatrix} \mathcal{H}(t,s)[\tilde{\varepsilon}_f(t)] \\ \vdots \\ \mathcal{H}(t,s)\left[s^{n-1}[\tilde{\varepsilon}_f(t)]\right] \\ \mathcal{H}(t,s)\left[s^n[\tilde{\varepsilon}_f(t)]\right] \end{bmatrix}. \tag{B.11}$$

As (B.9)–(B.11) depend from the errors $\tilde{\varepsilon}_f(t) = \hat{\varepsilon}_f(t) - \varepsilon_f(t)$ and $\tilde{y}_f(t) = \hat{y}_f(t) - y_f(t)$, then, using (3.5a)–(3.5c) and (3.18b), the differential equations for such errors are obtained:

$$\begin{aligned}
&\Sigma_1\colon \begin{cases} \dot{\tilde{\xi}}_y = \left(A_b - \tilde{\psi}_b e_1^\top\right)\tilde{\xi}_y - \tilde{\psi}_b y_f + \tilde{\rho} e_m y \\ \tilde{y}_f = \begin{cases} e_1^\top \tilde{\xi}_y, \text{ if } m \geqslant 1 \\ \tilde{\rho} y, \text{ if } m = 0, \end{cases} \end{cases} \\
&\Sigma_2\colon \begin{cases} \dot{\tilde{\xi}}_\varepsilon = \left(A_b - \tilde{\psi}_b e_1^\top\right)\tilde{\xi}_\varepsilon - \tilde{\psi}_b \varepsilon_f + (\rho + \tilde{\rho}) e_m \tilde{\varepsilon} + \tilde{\rho} e_m \varepsilon^* \\ \tilde{\varepsilon}_f = \begin{cases} e_1^\top \tilde{\xi}_\varepsilon, \text{ if } m \geqslant 1 \\ \tilde{\rho}\tilde{\varepsilon} + \rho\tilde{\varepsilon} + \tilde{\rho}\varepsilon^*, \text{ if } m = 0. \end{cases} \end{cases}
\end{aligned} \tag{B.12}$$

The system Σ_3 depends form the error $\tilde{\varepsilon}(t) = \varepsilon(t) - \varepsilon^*(t) = y(t) - \hat{y}(t) - y(t) + \hat{y}^*(t)$, thus the differential equation for it is obtained as:

$$\Sigma_3\colon \begin{cases} \dot{\tilde{x}}(t) = \left(A_0 - \Theta e_1^\top\right)\tilde{x}(t) + \tilde{\theta}_b(t) u(t) \\ \tilde{\varepsilon}(t) = e_1^\top \tilde{x}(t). \end{cases} \tag{B.13}$$

Further proof of this theorem is based on Lemma A5. In order to apply it, first of all, we need to show that $\tilde{f}_f \in L_2 \cap L_\infty$ and $\lim\limits_{t\to\infty} \tilde{f}_f(t) = 0$. To that end, equation (B.13) is rewritten in the operator form as

$$\tilde{\varepsilon}(t) = e_1^\top \left(sI - A_0 + \Theta e_1^\top\right)\left[\tilde{\theta}_b(t) u(t)\right],$$

and, as $\tilde{\theta}_b \in L_2 \cap L_\infty$ (see Theorem 2) and $u \in L_\infty$ (owing to Assumption 1 and equation (3.17)), then, using Lemma A1, it holds that $\tilde{\varepsilon} \in L_2 \cap L_\infty$.

Further proof differs for cases $m = 0$ and $1 \leqslant m \leqslant n-1$.

A) If $m = 0$, then we have

$$\left\{ \begin{array}{c} \tilde{\rho},\ \tilde{\varepsilon} \in L_2 \cap L_\infty \\ y,\ \varepsilon^* \in L_\infty \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \tilde{\varepsilon}_f \in L_2 \cap L_\infty \\ \tilde{y}_f \in L_2 \cap L_\infty \end{array} \right\}.$$

B) As for $1 \leqslant m \leqslant n-1$, the following quadratic form is introduced:

$$V = \tilde{\xi}_\varepsilon^\top P \tilde{\xi}_\varepsilon, \tag{B.14}$$

where $A_b^\top P + P A_b = -Q,\ Q = Q^\top > 0$.

Considering the system Σ_2, the derivative of (B.14) is written as:

$$\begin{aligned}\dot{V} &= \tilde{\xi}_\varepsilon^\top \left(P\left[A_b - \tilde{\psi}_b e_1^\top\right] + \left[A_b - \tilde{\psi}_b e_1^\top\right]^\top P\right)\tilde{\xi}_\varepsilon + 2\tilde{\xi}_\varepsilon^\top P\left(-\tilde{\psi}_b \varepsilon_f + (\rho+\tilde{\rho})\, e_m \tilde{\varepsilon} + \tilde{\rho} e_m \varepsilon^*\right) \\ &= -\tilde{\xi}_\varepsilon^\top \left(Q + P\tilde{\psi}_b e_1^\top + e_1 \tilde{\psi}_b^\top P\right)\tilde{\xi}_\varepsilon + 2\tilde{\xi}_\varepsilon^\top P\left(-\tilde{\psi}_b \varepsilon_f + (\rho+\tilde{\rho})\, e_n \tilde{\varepsilon} + \tilde{\rho} e_n \varepsilon^*\right) \\ &\leqslant -\left(\lambda_{\min}(Q) - 2\lambda_{\max}(P)\left\|\tilde{\psi}_b\right\| - \chi_1^{-1}\lambda_{\max}^2(P)\right)\left\|\tilde{\xi}_\varepsilon\right\|^2 + \epsilon,\end{aligned} \tag{B.15}$$

where $\chi_1 > 0$ and $\epsilon = \chi_1\left(\left\|\tilde{\psi}_b \varepsilon_f\right\| + \left\|(\rho+\tilde{\rho})\, e_n \tilde{\varepsilon}\right\| + \left\|\tilde{\rho} e_n \varepsilon^*\right\|\right)^2$.

For all Q and P there exist scalars $\chi_1 > 0$ and $\chi_2 > 0$ such that

$$\lambda_{\min}(Q) - \chi_1^{-1}\lambda_{\max}^2(P) \geqslant \chi_2 > 0.$$

This fact allows one to rewrite (B.15) as:

$$\dot{V} \leqslant -\left(\chi_2 - 2\lambda_{\max}(P)\left\|\tilde{\psi}_b\right\|\right)\left\|\tilde{\xi}_\varepsilon\right\|^2 + \epsilon. \tag{B.16}$$

As $\tilde{\psi}_b \in L_\infty$ and $\tilde{\psi}_b(t) \to 0$ when $t \to \infty$ (see Theorem 2), then there always exist scalars $\sigma_1 > 0$ and $\sigma_2 > 0$ and time instant $\infty > t_V \geqslant t_0$ such that

$$0 < \frac{-\left(\chi_2 - 2\lambda_{\max}(P)\left\|\tilde{\psi}_b\right\|\right)}{\lambda_{\max}(P)} \leqslant \sigma_1, \text{ for all } t \leqslant t_V,$$
$$\frac{\chi_2 - 2\lambda_{\max}(P)\left\|\tilde{\psi}_b\right\|}{\lambda_{\max}(P)} \geqslant \sigma_2 > 0, \text{ for all } t \geqslant t_V.$$

Using it, equation (B.16) is rewritten as follows:

$$\dot{V}(t) \leqslant \begin{cases} \sigma_1 V(t) + \epsilon(t), \text{ for all } t \leqslant t_V \\ -\sigma_2 V(t) + \epsilon(t), \text{ for all } t > t_V. \end{cases} \tag{B.17}$$

As $\tilde{\psi}_b,\ \tilde{\rho},\ \tilde{\varepsilon} \in L_2 \cap L_\infty$ (see Theorem 2 and above-given proof) and $\varepsilon_f \in L_\infty$, $\varepsilon^* \in L_\infty$ (owing to Assumption 1 and equation (3.17)), then $\epsilon \in L_1 \cap L_\infty$. As $\infty > t_V \geqslant t_0$ and $\epsilon \in L_1 \cap L_\infty$, then $V(t)$ is bounded for all $t \in [t_0,\ t_V]$. Given $\epsilon \in L_1 \cap L_\infty$, then, using Lemma A1, the solution of (B.17) is also bounded for all $t \geqslant t_V$ and, consequently, $V \in L_\infty$.

Having integrated (B.17), it is obtained:

$$V(t) \leqslant V(t_V) - \sigma_2 \int\limits_{t_V}^{t} V(s)\, ds + \int\limits_{t_V}^{t} \epsilon(s)\, ds. \tag{B.18}$$

As $V \in L_\infty$ and $\epsilon \in L_1$, then the following integral is bounded:

$$\sigma_2 \int\limits_{t_V}^{t} V(s)\, ds \leqslant -V(t) + V(t_V) + \int\limits_{t_V}^{t} \epsilon(s)\, ds < \infty,$$

from which it holds that $\tilde{\varepsilon}_f \in L_2 \cap L_\infty$.

Having repeated the above reasoning (B.14)–(B.18), it is obtained that $\tilde{y}_f \in L_2 \cap L_\infty$. Therefore, using Lemma A1, for all $0 \leqslant m \leqslant n-1$ in the sense of the following implication

$$\left.\begin{array}{l}\tilde{\varepsilon}_f \in L_2 \cap L_\infty\\ \tilde{y}_f \in L_2 \cap L_\infty\end{array}\right\} \Rightarrow \left.\begin{array}{l}s^i\left[\tilde{\varepsilon}_f(t)\right] \in L_2 \cap L_\infty\\ s^i\left[\tilde{y}_f(t)\right] \in L_2 \cap L_\infty\end{array}\right\} \forall i = 0, \ldots, n \tag{B.19}$$

we almost always have $\tilde{h}_\varepsilon \in L_2 \cap L_\infty$, $\tilde{h}_y \in L_2 \cap L_\infty$ (for example, if $\frac{d}{dt}\left[\mu(t)\right] = 0$, then $\mathcal{H}(t,s)\left[s^i[.]\right] = \mathcal{H}(t,s)\, s^i[.]$ and $\tilde{h}_\varepsilon \in L_2 \cap L_\infty$, $\tilde{h}_y \in L_2 \cap L_\infty$ follow directly from Lemma A1).

Taking into consideration $\tilde{\psi}_a \in L_2 \cap L_\infty$ (see Theorem 2), on the basis of equation (B.10) it is finally obtained that $\tilde{f}_f \in L_2 \cap L_\infty$. As $\tilde{f}_f \in L_2 \cap L_\infty$, $\tilde{f} \in L_p \cap L_\infty$ for $p \in [1,\ \infty)$ (see Proposition 1), then there exists a sufficiently large scalar $f_{\max}$ such that the following holds

$$\mathrm{sat}_{f_{\max}}\left\{\tilde{f}_f(t) - \tilde{f}(t) - f(t)\right\} = \tilde{f}_f(t) - \tilde{f}(t) - f(t),$$

thus, equation (B.9) is rewritten as:

$$\tilde{y}(t) = W_{cl}\left(\theta_{cl},\ s\right)\left[\tilde{f}_f(t) - \tilde{f}(t)\right],$$

from which, owing to $\tilde{f}_f \in L_2 \cap L_\infty$, $\tilde{f} \in L_p \cap L_\infty$ and using Lemma A5, it is obtained that $\lim\limits_{t\to\infty} \tilde{y}(t) = 0$.

Proof of Theorem 5. In accordance with proof of Proposition 1 and Theorem 4, when $\mu(t) = \mu > 0$, the closed-loop system is described by the following equations:

$$\begin{cases}\dot{\eta}(t) = \mu G\eta(t) + e_{n+1}m_0\mu\lambda(t)\\ \tilde{f}(t) = e_1^\top\eta(t),\end{cases} \tag{B.20a}$$

$$\tilde{y}(t) = W_{cl}\left(\theta_{cl},\ s\right)\left[\mathrm{sat}_{f_{\max}}\left\{\tilde{f}_f(t) - \tilde{f}(t) - f(t)\right\} + f(t)\right], \tag{B.20b}$$

$$\tilde{f}_f(t) = -\psi_a^\top(\Theta)\,\tilde{h}_\varepsilon(t) - \tilde{\psi}_a^\top(t)\,h_y(t) - \left(\psi_a^\top(\theta_a) - \psi_a^\top(\Theta)\right)\tilde{h}_y(t) - \tilde{\psi}_a^\top(t)\,\tilde{h}_y(t), \tag{B.20c}$$

where $\tilde{f} \in L_\infty$ as $\mu\lambda \in L_\infty$ and G is a Hurwitz matrix, and, if Assumption 1 is met, repeating proof (B.12)–(B.19) from Theorem 4, we have for $\tilde{f}_f(t)$ that

$$\tilde{\rho},\ \tilde{\theta},\ \tilde{\psi}_b \in L_2 \cap L_\infty \Rightarrow \tilde{f}_f \in L_2 \cap L_\infty.$$

Then there exists a sufficiently large scalar $f_{\max}$ such that the following holds:

$$\tilde{y}(t) = \underbrace{W_{cl}\left(\theta_{cl},\ s\right)\left[\tilde{f}_f(t)\right]}_{\tilde{y}_1(t)} - \underbrace{W_{cl}\left(\theta_{cl},\ s\right)\left[\tilde{f}(t)\right]}_{\tilde{y}_2(t)},$$

and, owing to $\tilde{f}_f \in L_2 \cap L_\infty$ and using Lemma A5, it is obtained that $\lim\limits_{t\to\infty} \tilde{y}_1(t) = 0$.

In order to complete the proof, equation (B.20a) is considered together with the system

$$\begin{array}{l}\dot{z}(t) = A_{cl}z(t) + e_{n_{cl}}\tilde{f}(t),\\ \tilde{y}_2(t) = \left[0_{1\times(n_{cl}-(m_{cl}+1))}\ \ \theta_{b.cl}^\top\right]z(t),\end{array} \tag{B.21}$$

where

$$A_{cl} = A_0 + e_{n_{cl}}\theta_{a.cl}^\top,\ A_0 = \begin{bmatrix}0_{n_{cl}} & \begin{matrix}I_{n_{cl}-1}\\ 0_{1\times(n_{cl}-1)}\end{matrix}\end{bmatrix},\ e_{n_{cl}} = \begin{bmatrix}0_{n_{cl}-1}\\ 1\end{bmatrix},$$

and the parameters $\theta_{a.cl},\ \theta_{b.cl}$ are equivalents of $\theta_a,\ \theta_b$ and match the parameters of the numerator and denominator of the transfer function of the closed-loop system (2.4).

The following quadratic form is introduced to analyze stability of (B.20a) + (B.21):

$$V = z^\top P_z z + \frac{1}{\mu}\eta^\top P_\eta \eta, \tag{B.22}$$

where $P_z A_{cl} + A_{cl}^\top P_z = -Q_z,\ P_\eta G + G^\top P_\eta = -Q_\eta$ and matrices $Q_z = Q_z^\top > 0,\ Q_\eta = Q_\eta^\top > 0$ are chosen such that for some scalars $\chi_z \in (0,1)\,,\ \chi_\eta \in (0,1)$ it holds that:

$$\begin{gathered} 0 < c_z \leqslant \lambda_{\min}\left(Q_z - \chi_z^{-1} P_z e_{n_{cl}} e_{n_{cl}}^\top P_z\right), \\ 0 < c_\eta \leqslant \lambda_{\min}\left(Q_\eta - m_0 \chi_\eta^{-1} P_\eta e_{n+1} e_{n+1}^\top P_\eta - \chi_z e_1 e_1^\top\right). \end{gathered}$$

Owing to equations (B.20a) and (B.21), the derivative of (B.22) is written as:

$$\begin{aligned} \dot V &= z^\top \left[P_z A_{cl} + A_{cl}^\top P_z\right] z + 2z^\top P_z e_{n_{cl}} e_1^\top \eta + \eta^\top \left[P_\eta G + G^\top P_\eta\right]\eta + 2\eta^\top P_\eta e_{n+1} m_0 \lambda \\ &= -z^\top Q_z z - \eta^\top Q_\eta \eta + 2z^\top P_z e_{n_{cl}} e_1^\top \eta + 2\eta^\top P_\eta e_{n+1} m_0 \lambda. \end{aligned} \tag{B.23}$$

Having applied the inequalities:

$$\begin{gathered} 2z^\top P_z e_{n_{cl}} e_1^\top \eta \leqslant \chi_z^{-1} z^\top P_z e_{n_{cl}} e_{n_{cl}}^\top P_z z + \chi_z \eta^\top e_1 e_1^\top \eta, \\ 2\eta^\top P_\eta e_{n+1} \lambda \leqslant \chi_\eta^{-1} \eta^\top P_\eta e_{n+1} e_{n+1}^\top P_\eta \eta + \chi_\eta \lambda^2, \end{gathered}$$

it is obtained that:

$$\begin{aligned} \dot V =& - z^\top \left[Q_z - \chi_z^{-1} P_z e_{n_{cl}} e_{n_{cl}}^\top P_z\right] z \\ & - \eta^\top \left[Q_\eta - m_0 \chi_\eta^{-1} P_\eta e_{n+1} e_{n+1}^\top P_\eta - \chi_z e_1 e_1^\top\right]\eta + \chi_\eta m_0 \lambda^2 \\ \leqslant& - \min\left\{\frac{c_z}{\lambda_{\max}(P_z)}, \frac{\mu c_\eta}{\lambda_{\max}(P_\eta)}\right\} V + \chi_\eta m_0 \lambda^2. \end{aligned} \tag{B.24}$$

As, owing to Assumption 2, it holds that $\lim\limits_{\mu\to\infty} \lambda^2(\mu) = 0$, then we have from (B.24) that $\lim\limits_{t\to\infty} \tilde y_1(t) = 0$ and $\lim\limits_{t\to\infty} |\tilde y(t)| \leqslant \epsilon$ for arbitrarily small scalar $\epsilon > 0$.

Proof of Theorem 6. In case $\mu(t) = \mu > 0$, owing to Theorem 5, the closed-loop control system is described by equations (B.20a)–(B.20c) and (B.21). Moreover, considering forced motion component $\tilde y_2(t)$ and repeating analysis (B.21)–(B.24), we have $\lim\limits_{t\to\infty} |\tilde y_2(t)| \leqslant \epsilon_2$ for arbitrarily small scalar $\epsilon_2 > 0$. Using the upper bound (3.16) and equations (B.10)–(B.13), the following upper bound for $\tilde f_f(t)$ is obtained via simple but tedious reasoning:

$$\left|\tilde f_f(t)\right| \leqslant \tilde f_{1f}(t) + T^{-1} \tilde f_{0f}, \tag{B.25}$$

where $\tilde f_{1f} \in L_2$ and $0 < \tilde f_{0f} < \infty$.

Then, in order to complete proof, the following system remains to be considered

$$\begin{aligned} \dot z(t) &= A_{cl} z(t) + e_{n_{cl}} \tilde f_f(t), \\ \tilde y_1(t) &= \begin{bmatrix} 0_{1\times(n_{cl}-(m_{cl}+1))} & \theta_{b.cl}^\top \end{bmatrix} z(t). \end{aligned} \tag{B.26}$$

The below-given quadratic form is introduced to analyze stability of (B.26):

$$V = z^{\top} P_z z, \tag{B.27}$$

where $P_z A_{cl} + A_{cl}^{\top} P_z = -Q_z$ and the matrix $Q_z = Q_z^{\top} > 0$ is chosen such that for some scalar $\chi_z \in (0,\ 1)$ it holds that $0 < c_z \leqslant \lambda_{\min}(Q_z) - \chi_z^{-1} \|P_z e_{n_{cl}}\|^2$.

Owing to (B.26), the derivative of (B.27) is written as:

$$\dot{V} = z^{\top} \left[P_z A_{cl} + A_{cl}^{\top} P_z \right] z + 2 z^{\top} P_z e_{n_{cl}} \tilde{f}_f. \tag{B.28}$$

Using (B.25), the following is obtained:

$$\begin{aligned}
&2 \left\| z^{\top} P_z e_{n_{cl}} \tilde{f}_f \right\| \leqslant 2 \|z\| \|P_z e_{n_{cl}}\| \left| \tilde{f}_f \right|, \\
&2 \|z\| \|P_z e_{n_{cl}}\| \left| \tilde{f}_f \right| \leqslant \chi_z^{-1} \|P_z e_{n_{cl}}\|^2 \|z\|^2 + \chi_z \tilde{f}_f^2, \\
&\tilde{f}_f^2 \leqslant \tilde{f}_{1f}^2(t) + 2T^{-1} \tilde{f}_{1f} \tilde{f}_{0f} + T^{-2} \tilde{f}_{0f}^2, \\
&2T^{-1} \tilde{f}_{1f} \tilde{f}_{0f} \leqslant \tilde{f}_{1f}^2(t) + T^{-2} \tilde{f}_{0f}^2,
\end{aligned}$$

and therefore, we have:

$$\dot{V} \leqslant -c_z \|z\|^2 + 2\chi_z \tilde{f}_{1f}^2(t) + 2T^{-2} \chi_z \tilde{f}_{0f}^2. \tag{B.29}$$

As $\tilde{f}_{1f} \in L_2$ and $\tilde{f}_{0f} < \infty$, then from (B.29) it holds that $\lim\limits_{t\to\infty} |\tilde{y}_1(t)| \leqslant \epsilon_1$ for arbitrarily small scalar $\epsilon_1 > 0$, which allows one to conclude that $\lim\limits_{t\to\infty} |\tilde{y}(t)| \leqslant \epsilon$ for arbitrarily small scalar $\epsilon > 0$.

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Stabilization of Nonlinear Continuous–Discrete Dynamic Systems with a Constant Sampling Step

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Abstract—This paper considers nonlinear continuous–discrete (hybrid) systems containing two subsystems of differential and difference equations, respectively, and one-dimensional (scalar) or multidimensional (vector) control. The transition from a nonlinear hybrid system with a constant sampling step $h > 0$ to an equivalent, in a natural sense, nonlinear discrete dynamic system is presented. Sufficient conditions are established, first, for reducing the first approximation systems of nonlinear discrete systems to the Brunovský canonical form and, second, for stabilizing such systems and nonlinear hybrid systems with control of different dimensions. Algorithms for constructing stabilizing control laws for nonlinear hybrid systems are developed. Numerical examples are provided to illustrate the effectiveness of this approach to stabilizing nonlinear hybrid dynamic systems.

Keywords: continuous–discrete system, discrete system, hybrid system, controlled system, equilibrium, stabilizable system, Brunovský canonical form

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1. INTRODUCTION

Stabilization of dynamic systems is one of the most important problems in control theory, which is due to the demands of control practice and open (unsolved) scientific issues in this area [1–4]. The solution of this problem will ensure stable operating modes of dynamic systems and contribute to solution of control problems for these systems.

Hybrid dynamic systems serve as mathematical models of real mechanical, technical, technological, and other processes of a heterogeneous nature that cannot be described only by differential equations. For example, control of aircraft and electric trains can be modeled only using a discrete thrust regulator [5, 6]. Therefore, the main feature of such hybrid (continuous–discrete) dynamic control systems is their adequacy to modeled objects [7], which, as a rule, have nonlinear operation processes. Hence, the corresponding hybrid dynamic systems are nonlinear continuous–discrete systems.

Among hybrid systems, there is a large class of systems stabilized by switching at appropriate time instants [8]. However, the issues of stabilizing real dynamic control systems are inseparably connected with their controllability issues.

Methods for stabilizing controlled dynamic systems have been created and developed for over 150 years [4]. Much experience has been gained in stabilization of continuous systems and discrete systems; for example, see [9–16]. There are R&D results in the field of stabilizing linear hybrid [3, 7, 17] and nonlinear hybrid systems described by differential equations with different nonlinearities and a discrete state- or output-feedback controller [18–22]. However, the issues of stabilizing nonlinear hybrid systems with operation processes described by differential and difference equations and states containing both continuous and discrete components have not been sufficiently studied so far.

This paper presents a general approach to stabilizing such nonlinear continuous–discrete (hybrid) systems with one-dimensional (scalar) or multidimensional (vector) control and a constant sampling step. The approach is based on the transition from a given nonlinear hybrid system to an equivalent, in the natural sense, nonlinear discrete dynamic system.

The novelty of this study is as follows:

1) The concept of the Brunovský canonical form is introduced for the first approximation system of a nonlinear discrete dynamic system with scalar control.

2) Sufficient conditions are established for reducing the first approximation system of an equivalent nonlinear discrete system with scalar (vector) control to the Brunovský canonical form (i.e., to a set of independent subsystems each having the Brunovský canonical form).

3) Algorithms for reducing the first approximation systems of equivalent nonlinear discrete systems (with both scalar and vector control) to the Brunovski canonical form are developed and demonstrated by examples.

4) Sufficient conditions for stabilizing the first approximation systems of equivalent nonlinear discrete dynamic control systems (both with scalar and vector control) are established.

5) Sufficient conditions for stabilizing nonlinear hybrid dynamic control systems with control of different dimensions are established.

6) Algorithms for constructing stabilizing control laws for nonlinear hybrid systems with control of different dimensions are developed and demonstrated by examples.

2. PROBLEM STATEMENT

Consider a nonlinear continuous–discrete control system of the form

$$\begin{cases} x'(t) = f(x(t), y(t_k)), \quad t_k \leqslant t < t_{k+1} \\ y(t_{k+1}) = g(x(t_{k+1}), y(t_k), u(t_k)), \quad k = 0, 1, 2, \ldots \end{cases} \tag{1}$$

with initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{2}$$

where $x \in R^n$ and $y \in R^m$ are the state vectors of system (1) characterizing the behavior of its continuous and discrete parts, respectively; $u \in R^q$ is the control vector (input) of system (1); the time instants t_k define a uniform grid on R with a constant step $h > 0$, i.e., $t_{k+1} - t_k = h > 0$, $k = 0, 1, 2, \ldots$ and $t_k = kh$; the functions $f(x, y)$ and $g(x, y, u)$ are continuously differentiable in the aggregate of variables.

Assume that without control ($u = 0$),

$$f(0, 0) = 0, \quad g(0, 0, 0) = 0. \tag{3}$$

In other words, system (1) with $u = 0$ has the trivial equilibrium $x = 0$, $y = 0$.

System (1), (2) with a chosen control law $u = u(t_k)$, $k = 0, 1, 2, \ldots$, operates in accordance with the following standard scheme:

1) Initial conditions $z_0 = (x_0, y_0)$ are specified.
2) The solution $x = \varphi_0(t)$ of the Cauchy problem $x' = f(x, y_0)$, $x(t_0) = x_0$, is found, and the vectors $x_1 = \varphi_0(t_1)$ and $y_1 = g(x_1, y_0, u_0)$, where $u_0 = u(t_0)$, are constructed.
3) The solution $x = \varphi_1(t)$ of the Cauchy problem $x' = f(x, y_1)$, $x(t_1) = x_1$, is found, and the vectors $x_2 = \varphi_1(t_2)$ and $y_2 = g(x_2, y_1, u_1)$, where $u_1 = u(t_1)$, are constructed. And so on.

Hence, the solution of system (1) with the initial point $z_0 = (x_0, y_0)$ and a chosen control law $u = u(t_k)$, $k = 0, 1, 2, \ldots$, is the function

$$z(t) = (x(t), y(t)) = \begin{cases} (\varphi_0(t), y_0), & t_0 \leqslant t < t_1 \\ (\varphi_1(t), y_1), & t_1 \leqslant t < t_2 \\ (\varphi_2(t), y_2), & t_2 \leqslant t < t_3 \\ \qquad\vdots \end{cases} \tag{4}$$

The components of the solution (4) have the following features: the function $x(t)$ is continuous for all $t \geqslant 0$, continuously differentiable on any interval (t_k, t_{k+1}), but not necessarily differentiable at time instants $t = t_k$, where $k = 0, 1, 2, \ldots$; the function $y(t)$ is piecewise constant and changes its values at time instants $t = t_k$, $k = 0, 1, 2, \ldots$.

The main problem of this paper is to establish sufficient conditions for stabilizing systems (1) and to design stabilizing control laws for such systems. The solution of this problem is based on the transition from the nonlinear hybrid system (1) to an equivalent nonlinear discrete dynamic system.

3. TRANSITION TO THE DISCRETE SYSTEM. CONTROLLABILITY AND STABILIZABILITY OF SYSTEMS

Due to the relations (3), the functions $f(x, y)$ and $g(x, y, u)$ can be represented as

$$f(x, y) = A_1 x + B_1 y + a(x, y), \quad g(x, y, u) = A_2 x + B_2 y + Cu + b(x, y, u),$$

where $A_1 = f'_x(0,\, 0)$, $B_1 = f'_y(0,\, 0)$, $A_2 = g'_x(0,\, 0,\, 0)$, $B_2 = g'_y(0,\, 0,\, 0)$, and $C = g'_u(0,\, 0,\, 0)$ are matrices of compatible dimensions and the smooth nonlinearities $a(x, y)$ and $b(x, y, u)$ satisfy the relations

$$a(x, y) = o(\|x\| + \|y\|) \quad \text{as} \quad \|x\| + \|y\| \to 0,$$
$$b(x, y, u) = o(\|x\| + \|y\| + \|u\|) \quad \text{as} \quad \|x\| + \|y\| + \|u\| \to 0.$$

By denoting $x(t_k) = x_k$, $y(t_k) = y_k$, and $u(t_k) = u_k$, we write system (1) as the equivalent hybrid system

$$\begin{cases} x'(t) = A_1 x(t) + B_1 y_k + a(x(t), y_k)\,, & t_k \leqslant t < t_{k+1} \\ y_{k+1} = A_2 x_{k+1} + B_2 y_k + Cu_k + b(x_{k+1}, y_k, u_k), & k = 0, 1, 2, \ldots. \end{cases} \tag{5}$$

Let $\det A_1 \neq 0$; by applying the shift operator along the trajectories of system $x' = f(x, y)$ over the time from $t = 0$ to $t = h > 0$ (see [23]), we perform the transition from system (5) to a discrete system of the form

$$\begin{cases} x_{k+1} = e^{A_1 h} x_k + A_1^{-1}(e^{A_1 h} - I) B_1 y_k + \varepsilon(x_k, y_k; h) \\ y_{k+1} = A_2 e^{A_1 h} x_k + (A_2 A_1^{-1}(e^{A_1 h} - I) B_1 + B_2) y_k + Cu_k + \delta(x_k, y_k, u_k; h), \end{cases} \tag{6}$$

where

$$\varepsilon(x_k, y_k; h) = e^{(t_k + h) A_1} \int\limits_{t_k}^{t_k + h} e^{-sA_1} a(p(s, x_k, y_k), y_k) ds,$$

$x = p(t, x_k, y_k)$ is the solution of the Cauchy problem

$$x' = A_1 x + B_1 y_k + a(x, y_k), \quad x(t_k) = x_k\,;$$
$$\delta(x_k, y_k, u_k; h) = A_2 \varepsilon(x_k, y_k; h) + b(x_{k+1}, y_k, u_k). \tag{7}$$

Note that the case of a singular matrix A_1 can also be considered, but it will lead to substantially more complicated formulas.

The discrete system (6) can be written in the compact form

$$z_{k+1} = A(h)z_k + Bu_k + \xi(z_k, u_k, h), \quad k = 0, 1, 2, \dots, \tag{8}$$

where

$$z_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad B = \begin{bmatrix} O \\ C \end{bmatrix}, \quad \xi(z_k, u_k, h) = \begin{bmatrix} \varepsilon(x_k, y_k; h) \\ \delta(x_k, y_k, u_k; h) \end{bmatrix},$$

$A(h)$ is a block matrix of order $(n+m)$:

$$A(h) = \begin{bmatrix} e^{A_1 h} & A_1^{-1}(e^{A_1 h} - I)B_1 \\ A_2 e^{A_1 h} & A_2 A_1^{-1}(e^{A_1 h} - I)B_1 + B_2 \end{bmatrix}. \tag{9}$$

System (6) with $u = 0$ (equivalently, system (8)) has the trivial equilibrium.

Based on [23] and condition (7), it can be established that

$$\varepsilon(x, y; h) = o(\|x\| + \|y\|) \quad \text{as} \quad \|x\| + \|y\| \to 0,$$
$$\delta(x, y, u; h) = o(\|x\| + \|y\| + \|u\|) \quad \text{as} \quad \|x\| + \|y\| + \|u\| \to 0\,.$$

Then the function $\xi(z, u, h)$ in system (8) satisfies the condition

$$\xi(z, u, h) = o(\|z\| + \|u\|) \quad \text{as} \quad \|z\| + \|u\| \to 0\,. \tag{10}$$

For a chosen control law $u = u(t_k)$, $k = 0, 1, 2, \dots$, the hybrid system (1) and the discrete system (6) are equivalent in the following sense [8]:

• If $(x(t), y_k)$ is the solution of system (1), (x_k, y_k) will be the solution of system (6), where $x_k = x(t_k)$.

• If (x_k, y_k) is the solution of system (6), $(x(t), y_k)$ will be the solution of system (1), where $x(t)$ is the solution of the Cauchy problem $x' = f(x, y_k), x(t_k) = x_k$.

Therefore, systems (1) and (8) are equivalent as well.

Following [24–26], we introduce several notions.

Definition 1. The hybrid system (1) is said to be controllable for $h = h_0 > 0$ if for any vectors $z^{(0)}$, $z^{(1)} \in R^{n+m}$, there exists a control law $u = u(t_k)$, $k = 0, 1, \dots, l-1$ $(l \in N)$, such that $z(t_l) = z^{(1)}$, $t_l = t_0 + lh_0$, for the solution $z = z(t)$ of system (1) with the initial condition $z(t_0) = z^{(0)}$.

Definition 2. The discrete system (8) is said to be controllable for $h = h_0 > 0$ if for any states $z^{(0)}$, $z^{(1)} \in R^{n+m}$, there exists a control law u_k, $k = 0, 1, \dots, l-1$ $(l \in N)$, such that $z_l = z^{(1)}$, $z_l = z(t_l) = z(t_0 + lh_0)$, for the solution z_k, $k = 0, 1, \dots, l$, of system (8) with the initial condition $z_0 = z^{(0)}$.

For some $h > 0$, the equilibrium of systems (1) and (8) may be unstable.

Definition 3. The hybrid system (1) is said to be stabilizable for $h = h_0 > 0$ if there exists a piecewise constant function $u = \varphi(t) \in R^q$, $t_k \leqslant t < t_{k+1}$, $k = 0, 1, 2, \dots$, such that system (1) with the control law $u = \varphi(t)$ has the asymptotically stable solution $x = 0$, $y = 0$ for $h = h_0 > 0$.

Definition 4. The discrete system (8) is said to be stabilizable for $h = h_0 > 0$ if there exists a control law $u_k = \Phi(z_k)$, $k = 0, 1, 2, \dots$, such that system (8) has the asymptotically stable solution $z = 0$ for $h = h_0 > 0$.

Direct comparison of Definitions 1 and 2 yields the following result.

Theorem 1. *The hybrid system* (1) *is controllable for* $h = h_0 > 0$ *if and only if the discrete system* (8) *is controllable for* $h = h_0 > 0$.

The proof of this theorem, as well as those of other important results of the paper, is postponed to the Appendix.

The linear discrete system

$$z_{k+1} = A(h)z_k + Bu_k, \quad k = 0, 1, 2, \ldots, \tag{11}$$

is the first approximation system of the nonlinear discrete system (8).

Now we proceed to the issues of stabilizing system (1) considering its control peculiarities.

4. STABILIZATION OF THE HYBRID SYSTEM (1) WITH SCALAR CONTROL

Let $u \in R^1$ in system (1); then the first approximation system of the corresponding discrete system (8) takes the form

$$z_{k+1} = A(h)z_k + bu_k, \quad k = 0, 1, 2, \ldots, \tag{12}$$

where $z \in R^{n+m}$, $A(h)$ is the matrix (9), b is a matrix of dimensions $(n + m) \times 1$, and $u \in R^1$.

Assume that for $h = h_0 > 0$, system (12) satisfies the complete reachability condition, i.e.,

$$\operatorname{rank}\,[b, A(h_0)b, A^2(h_0)b, \ldots, A^{n+m-1}(h_0)b] = n + m. \tag{13}$$

which implies the controllability of system (12) for $h = h_0 > 0$.

Note that condition (13) can be relaxed, e.g., by imposing the complete controllability condition on system (12) for $h = h_0 > 0$; for details, see [11, pp. 268–269].

We have the following result.

Theorem 2. *Assume that the linear discrete system* (12) *satisfies condition* (13). *Then there exists a transformation*

$$z_k^* = Sz_k, \quad u_k^* = \alpha z_k + u_k, \tag{14}$$

where $\det S \neq 0$ *and* α *is a matrix of dimensions* $1 \times (n + m)$, *reducing this system to*

$$\begin{cases} z^*_{k+1,1} = z^*_{k,2} \\ z^*_{k+1,2} = z^*_{k,3} \\ \vdots \\ z^*_{k+1,n+m-1} = z^*_{k,n+m} \\ z^*_{k+1,n+m} = u^*_k, \end{cases} \tag{15}$$

where $z^*_{k,i}$ *denotes the* i*th component of the vector* z^*_k.

System (15) will be called the *Brunovský canonical form of system* (12) *for* $h = h_0 > 0$.

Theorem 3. *Assume that for* $h = h_0 > 0$, *system* (12) *satisfies condition* (13). *Then it is stabilizable for* $h = h_0 > 0$.

From the proof of Theorem 3 (see the Appendix) and equalities (14) we arrive at the following fact.

Corollary 1. *A control law* $u_k = S^* z_k$, $k = 0, 1, 2, \ldots,$ *where* $S^* = pS - \alpha$, $u^*_k = pz^*_k$, *stabilizes the linear discrete system* (12), *and all eigenvalues of the matrix* $(A(h_0) + bS^*)$ *are smaller than 1 by their magnitude.*

Theorem 4. *If the first approximation system* (12) *of the corresponding nonlinear discrete system* (8) *satisfies condition* (13), *the corresponding hybrid system* (1) *will be stabilizable for* $h = h_0 > 0$.

Example 1. It is required to construct a stabilizing control law for a hybrid system of the form

$$\begin{cases} x'(t) = -x(t) + y_{k,1} + a(x(t), y_k), \quad t_k \leqslant t < t_{k+1} \\ y_{k+1,1} = x(t_{k+1}) + y_{k,2} - u_k + b_1(x(t_{k+1}), y_k, u_k) \\ y_{k+1,2} = -x(t_{k+1}) + y_{k,1} + 0.5u_k + b_2(x(t_{k+1}), y_k, u_k), \quad k = 0, 1, 2, \ldots, \end{cases} \tag{16}$$

where $x \in R^1$, $y \in R^2$, $u \in R^1$, $a(x, y)$, $b_1(x, y, u)$, and $b_2(x, y, u)$ are smooth nonlinearities in accordance with system (5), by reducing the corresponding discrete first approximation system to the Brunovský canonical form.

For a nonlinear discrete system equivalent to (16), the first approximation system is a discrete system of the form (12) with the matrices

$$A(h) = \begin{bmatrix} e^{-h} & 1 - e^{-h} & 0 \\ e^{-h} & 1 - e^{-h} & 1 \\ -e^{-h} & e^{-h} & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \\ 0.5 \end{bmatrix}.$$

It can be proved that this discrete system with all $h:\ 0 < h \neq ln\,4$ is controllable.

Letting $h_0 = ln2$ yields $A(h_0) = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 1 \\ -0.5 & 0.5 & 0 \end{bmatrix}$; note that the solution $x = 0,\ y = 0$ of system (16) with $u = 0$ is stable (but not asymptotically stable). We compile the matrix $F(h_0) = [b, A(h_0)b, A^2(h_0)b]$:

$$F(h_0) = \begin{bmatrix} 0 & -0.5 & -0.25 \\ -1 & 0 & -0.75 \\ 0.5 & -0.5 & 0.25 \end{bmatrix}; \quad \text{then } F^{-1}(h_0) = \begin{bmatrix} 6 & -4 & -6 \\ 2 & -2 & -4 \\ -8 & 4 & 8 \end{bmatrix},$$

$$f(h_0) = [-8 \ \ 4 \ \ 8], \quad S = \begin{bmatrix} f(h_0) \\ f(h_0)A(h_0) \\ f(h_0)A^2(h_0) \end{bmatrix} = \begin{bmatrix} -8 & 4 & 8 \\ -6 & 2 & 4 \\ -4 & 0 & 2 \end{bmatrix}.$$

Then $z_k^* = Sz_k$, which corresponds to the system

$$\begin{cases} z_{k,1}^* = -8x_k + 4y_{k,1} + 8y_{k,2} \\ z_{k,2}^* = -6x_k + 2y_{k,1} + 4y_{k,2} \\ z_{k,3}^* = -4x_k + 2y_{k,2}. \end{cases}$$

Consequently,

$$\begin{cases} z_{k+1,1}^* = -8x_{k+1} + 4y_{k+1,1} + 8y_{k+1,2} = -6x_k + 2y_{k,1} + 4y_{k,2} = z_{k,2}^* \\ z_{k+1,2}^* = -6x_{k+1} + 2y_{k+1,1} + 4y_{k+1,2} = -4x_k + 2y_{k,2} = z_{k,3}^* \\ z_{k+1,3}^* = -4x_{k+1} + 2y_{k+1,2} = -3x_k - y_{k,1} + u_k = u_k^*. \end{cases}$$

Thus, for $h_0 = ln2$, the Brunovský canonical form of the discrete first approximation system corresponding to system (16) is given by

$$\begin{cases} z_{k+1,1}^* = z_{k,2}^* \\ z_{k+1,2}^* = z_{k,3}^* \\ z_{k+1,3}^* = u_k^*, \end{cases}$$

where $u_k^* = -3x_k - y_{k,1} + u_k$. As is easily verified, $u_k^* = \alpha z_k + u_k$, where $\alpha = f(h_0)A^3(h_0)$.

Let us find a stabilizing control law for system (16).

We have $u_k = S^* z_k$, $S^* = pS - \alpha$, $u_k^* = p z_k^*$, where $p = [p_1 \ \ p_2 \ \ p_3]$, p_1, p_2, and p_3 are the coefficients of the characteristic equation of the last system of equations:

$$\lambda^3 - p_3\lambda^2 - p_2\lambda - p_1 = 0.$$

Since the choice of these coefficients is arbitrary, we assign them so that the roots λ_1, λ_2, and λ_3 of the characteristic equation lie inside the unit circle.

Taking $\lambda_{1,2,3} = \frac{1}{2}$ yields $\lambda^3 - \frac{3}{2}\lambda^2 + \frac{3}{4}\lambda - \frac{1}{8} = 0$, and $p = [\frac{1}{8} \ \ -\frac{3}{4} \ \ \frac{3}{2}]$, $\alpha = [-3 \ \ -1 \ \ 0]$; hence, $S^* = [\frac{1}{2} \ \ 0 \ \ 1]$, and $u_k = \frac{1}{2}x_k + y_{k,2}$ is the desired control law.

Indeed, for $h_0 = ln2$, substituting this control law into the discrete system (12) corresponding to system (16) gives the matrix $(A(h_0) + bS^*)$, for which all eigenvalues are equal to $\frac{1}{2}$ (smaller than 1 by their magnitude). Based on the proof of Theorem 4 (see the Appendix), we conclude that system (16) is stabilizable for $h_0 = ln2$ by the control law $u_k = \frac{1}{2}x_k + y_{k,2}$.

Consider an important special case of system (1) in which $n = m = q = 1$. In this case, system (1) contains scalar equations and is equivalent to the hybrid system

$$\begin{cases} x'(t) = a_1 x(t) + b_1 y_k + a(x(t), y_k), & t_k \leqslant t < t_{k+1} \\ y_{k+1} = a_2 x_{k+1} + b_2 y_k + c u_k + b(x_{k+1}, y_k, u_k), & k = 0, 1, 2, \ldots, \end{cases} \tag{17}$$

where $a_1 = f'_x(0,0)$, $b_1 = f'_y(0,0)$, $a_2 = g'_x(0,0,0)$, $b_2 = g'_y(0,0,0)$, and $c = g'_u(0,0,0)$ are numbers, and the smooth nonlinearities $a(x,y)$ and $b(x,y,u)$ satisfy the relations

$$a(x,y) = o(|x| + |y|) \quad \text{as} \ \ |x| + |y| \to 0,$$
$$b(x,y,u) = o(|x| + |y| + |u|) \quad \text{as} \ \ |x| + |y| + |u| \to 0\,.$$

The nonlinear discrete system equivalent to system (17) has the first approximation system (12) with

$$A(h) = \begin{bmatrix} e^{a_1 h} & \dfrac{b_1}{a_1}(e^{a_1 h} - 1) \\ a_2 e^{a_1 h} & \dfrac{a_2 b_1}{a_1}(e^{a_1 h} - 1) + b_2 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ c \end{bmatrix}. \tag{18}$$

Then the following result is true.

Theorem 5. *If the hybrid system* (1) *contains scalar equations, and* $a_1 \neq 0$, $b_1 \neq 0$, *and* $c \neq 0$ *in the equivalent system* (17), *system* (1) *will be stabilizable for any* $h > 0$.

In this case, the stabilizing control law is found by the algorithm demonstrated in Example 1.

5. STABILIZATION OF SYSTEM (1) WITH VECTOR CONTROL

Let $n \geqslant 1$, $m > 1$, and $q > 1$ in the hybrid system (1), i.e., the discrete system (11) serves as the first approximation system for the equivalent discrete system (8). Assume that system (11) satisfies the complete reachability condition for $h = h_0 > 0$:

$$\operatorname{rank}[B, A(h_0)B, A^2(h_0)B, \ldots, A^{n+m-1}(h_0)B] = n + m. \tag{19}$$

Theorem 6. *If system* (11) *satisfies condition* (19), *there will exist a transformation*

$$z_k^* = S z_k, \quad u_{k,l}^* = \alpha_l z_k + u_{k,l}, \quad \det S \neq 0, \quad l = 1, \ldots, q, \tag{20}$$

reducing system (11) *to a set of* q *independent subsystems, each having the Brunovský canonical form.*

By the proof of this theorem (see the Appendix), the original system (11) decomposes into q independent subsystems of the form (15) whose dimensions are $s_1, \ldots, s_q$, with $s_1 + s_2 + \ldots + s_q = n + m$. Then we have the following result.

Theorem 7. *If the linear discrete system* (11) *satisfies condition* (19), *it will be stabilizable for* $h = h_0 > 0$.

The proof of Theorem 7 (see the Appendix) and equalities (20) lead to another fact.

Corollary 2. *The stabilizing control law for the discrete system* (11) *has the form*

$$u_k = S^* z_k,$$

where $u_{k,l} = s_l^* z_k$, $s_l^* = pS_l - \alpha_l$, $u_{k,l}^* = p z_{kl}^*$, S_l *is part of the matrix* S *corresponding to the* l*th subsystem* $(l = 1, \ldots, q)$, p *is a vector of dimension* s_l, z_{kl}^* *is the vector containing* s_l *components of the vector* z_k^* *corresponding to the* l*th subsystem, and* s_l^* *are the rows of the matrix* S^*. *In addition, all eigenvalues of the matrix* $(A(h_0) + BS^*)$ *are smaller than 1 by their magnitude.*

Theorem 8. *If the linear discrete system* (11) *satisfies condition* (19), *the corresponding nonlinear hybrid system* (1) *will be stabilizable for* $h = h_0 > 0$.

Example 2. It is required to construct a stabilizing control law for a hybrid system of the form

$$\begin{cases} x'(t) = -2x(t) + 2y_{k,1} - 2y_{k,2} + a(x(t), y_k), \quad t_k \leqslant t < t_{k+1} \\ y_{k+1,1} = -x_{k+1} + u_{k,1} - u_{k,2} + b_1(x_{k+1}, y_k, u_k) \\ y_{k+1,2} = x_{k+1} - y_{k,1} + y_{k,2} + u_{k,1} + 2u_{k,2} + b_2(x_{k+1}, y_k, u_k), \quad k = 0, 1, 2, \ldots, \end{cases}$$

where $x \in R^1$, $y \in R^2$, $u \in R^2$, $a(x, y)$, $b_1(x, y, u)$, and $b_2(x, y, u)$ are smooth nonlinearities in accordance with system (5), by reducing the corresponding discrete first approximation system to a set of independent subsystems written in the Brunovský canonical form.

For this system, the corresponding discrete system (11) has the matrices

$$A(h) = \begin{bmatrix} e^{-2h} & 1 - e^{-2h} & e^{-2h} - 1 \\ -e^{-2h} & e^{-2h} - 1 & 1 - e^{-2h} \\ e^{-2h} & -e^{-2h} & e^{-2h} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

Then we obtain

$$[B, A(h)B, A^2(h)B] = \begin{bmatrix} 0 & 0 & 0 & 3(e^{-2h} - 1) & 0 & (e^{-2h} - 1)(9e^{-2h} - 3) \\ 1 & -1 & 0 & 3(1 - e^{-2h}) & 0 & (e^{-2h} - 1)(3 - 9e^{-2h}) \\ 1 & 2 & 0 & 3e^{-2h} & 0 & e^{-2h}(9e^{-2h} - 6) \end{bmatrix},$$

and rank $[B, A(h)B, A^2(h)B] = 3$ for any $h > 0$. Hence, the corresponding linear discrete system (11) is controllable for any $h > 0$.

Let $h_0 = ln3$, and let b_1 and b_2 be the columns of the matrix B. Then

$$A_0 = A(h_0) = \begin{bmatrix} 1/9 & 8/9 & -8/9 \\ -1/9 & -8/9 & 8/9 \\ 1/9 & -1/9 & 1/9 \end{bmatrix}.$$

From the column sequence $(b_1, b_2, A_0 b_1, A_0 b_2, A_0^2 b_1, A_0^2 b_2)$ we compile the matrix F_0 of order 3 and rank 3. Note that the matrix $A_0 b_1$ is a zero column, so the columns $A_0 b_1$, and $A_0^2 b_1$ are excluded from consideration.

We have $A_0b_2 = [-8/3 \;\; 8/3 \;\; 1/3]^T$; therefore, $s_1 = 1$, $s_2 = 2$, and $F_0 = [b_1, b_2, A_0b_2]$, i.e.,

$$F_0 = \begin{bmatrix} 0 & 0 & -8/3 \\ 1 & -1 & 8/3 \\ 1 & 2 & 1/3 \end{bmatrix}, \quad F_0^{-1} = \begin{bmatrix} 17/24 & 2/3 & 1/3 \\ -7/24 & -1/3 & 1/3 \\ -3/8 & 0 & 0 \end{bmatrix},$$

$k = 1, 2,$ since $\operatorname{rank} B = 2$.

$k = 1: \; \sum\limits_{j=1}^{k} s_j = s_1 = 1,$ hence, f_1 is the first row of the matrix F_0^{-1}.

$k = 2: \; \sum\limits_{j=1}^{k} s_j = s_1 + s_2 = 3,$ hence, f_2 is the third row of the matrix F_0^{-1}.

$$S = \begin{bmatrix} f_1 \\ f_2 \\ f_2A_0 \end{bmatrix}, \text{ i.e., } S = \begin{bmatrix} 17/24 & 2/3 & 1/3 \\ -3/8 & 0 & 0 \\ -1/24 & -1/3 & 1/3 \end{bmatrix}.$$

$$z_k^* = Sz_k, \text{ then } \begin{cases} z_{k,1}^* = \dfrac{17}{24}x_k + \dfrac{2}{3}y_{k,1} + \dfrac{1}{3}y_{k,2} \\ z_{k,2}^* = -\dfrac{3}{8}x_k \\ z_{k,3}^* = -\dfrac{1}{24}x_k - \dfrac{1}{3}y_{k,1} + \dfrac{1}{3}y_{k,2}, \quad k = 0, 1, 2, \ldots, \end{cases}$$

$z_{k+1,1}^* = u_{k,1}^*,$ where $u_{k,1}^* = f_1A_0^{s_1}z_k + u_{k,1} = f_1A_0z_k + u_{k,1} = \dfrac{1}{24}x_k + u_{k,1};$

$z_{k+1,2}^* = z_{k,3}^*;\; z_{k+1,3}^* = u_{k,2}^*,$ where $u_{k,2}^* = f_2A_0^{s_2}z_k + u_{k,2} = \dfrac{5}{72}x_k + \dfrac{2}{9}y_{k,1} - \dfrac{2}{9}y_{k,2} + u_{k,2}.$

Thus, for $h_0 = ln3$, the set of the Brunovský canonical forms for the linear discrete system corresponding to the original system has the form

$$\begin{cases} z_{k+1,1}^* = u_{k,1}^* \\ z_{k+1,2}^* = z_{k,3}^* \\ z_{k+1,3}^* = u_{k,2}^*. \end{cases}$$

Now we find a stabilizing control law for the corresponding linear discrete system for $h_0 = ln3$.

1) $u_{k,1}^* = pz_{k1}^*$, where $p = p_1$, $z_{k1}^* = z_{k,1}^*$, since $s_1 = 1$, $z_{k+1,1}^* = pz_{k,1}^*$, and then $\lambda = p$. Letting $\lambda = \frac{1}{3}$ yields $p = \frac{1}{3}$, $\alpha_1 = f_1A_0 = [\frac{1}{24} \;\; 0 \;\; 0]$, $S_1 = f_1$, and, hence, $s_1^* = pS_1 - \alpha_1 = [\frac{7}{36} \;\; \frac{2}{9} \;\; \frac{1}{9}]$, $u_{k,1} = s_1^*z_k$, i.e., $u_{k,1} = \frac{7}{36}x_k + \frac{2}{9}y_{k,1} + \frac{1}{9}y_{k,2}$;
2) $u_{k,2}^* = pz_{k2}^*$, where $p = [p_1 \;\; p_2]$, $z_{k2}^* = [z_{k,2}^* \;\; z_{k,3}^*]^T$, since $s_2 = 2$, $z_{k+1,3}^* = p_1z_{k,2}^* + p_2z_{k,3}^*$, and then $\lambda^2 - p_2\lambda - p_1 = 0$.

Letting $\lambda_{1,2} = \frac{1}{3}$ yields $\lambda^2 - \frac{2}{3}\lambda + \frac{1}{9} = 0$, and $p = [-\frac{1}{9} \;\; \frac{2}{3}]$;

$$\alpha_2 = f_2A_0^2 = [5/72 \;\; 2/9 \;\; -2/9], \quad S_2 = \begin{bmatrix} -3/8 & 0 & 0 \\ -1/24 & -1/3 & 1/3 \end{bmatrix},$$

$$s_2^* = pS_2 - \alpha_2 = [-1/18 \;\; -4/9 \;\; 4/9], \text{ i.e., } u_{k,2} = -\frac{1}{18}x_k - \frac{4}{9}y_{k,1} + \frac{4}{9}y_{k,2}.$$

Thus, the stabilizing control law has the form

$$u_k = \begin{bmatrix} \dfrac{7}{36}x_k + \dfrac{2}{9}y_{k,1} + \dfrac{1}{9}y_{k,2} \\ -\dfrac{1}{18}x_k - \dfrac{4}{9}y_{k,1} + \dfrac{4}{9}y_{k,2} \end{bmatrix}.$$

This law also stabilizes the original nonlinear hybrid system because

$$S^* = \begin{bmatrix} 7/36 & 2/9 & 1/9 \\ -1/18 & -4/9 & 4/9 \end{bmatrix}, \quad A_0 + BS^* = \begin{bmatrix} 1/9 & 8/9 & -8/9 \\ 5/36 & -2/9 & 5/9 \\ 7/36 & -7/9 & 10/9 \end{bmatrix},$$

and the matrix $(A_0 + BS^*)$ has the eigenvalues $\lambda_{1,2,3} = \frac{1}{3}$, i.e., $|\lambda_{1,2,3}| < 1$.

6. CONCLUSIONS

This paper has established a general sufficient condition for stabilizing nonlinear continuous–discrete systems of the form (1) with both one-dimensional (scalar) and multidimensional (vector) control. This condition is based on the complete reachability of the first approximation systems of the corresponding equivalent nonlinear discrete dynamic systems.

A general approach to stabilizing the nonlinear hybrid systems (1) with control of different dimensions has been presented, and stabilizing control laws for such systems have been designed. The approach can be used to ensure stable operation modes for real technical, mechanical, and other control systems with a heterogeneous structure as well as to construct admissible and optimal control laws for such systems.

APPENDIX

Proof of Theorem 1. The desired result follows from the equivalence of systems (1) and (8), which considers the relation of their solutions, and from the definitions of controllable systems.

The proof of Theorem 1 is complete.

Proof of Theorem 2. Since the linear discrete system (12) satisfies condition (13), the matrix

$$F(h_0) = \left[b,\ A(h_0)b,\ A^2(h_0)b,\ \ldots,\ A^{n+m-1}(h_0)b\right]$$

is nonsingular, i.e., there exists the inverse $F^{-1}(h_0)$.

Let the vector $f(h_0) \in R^{n+m}$ be composed of the elements of the last row of the matrix $F^{-1}(h_0)$ [27]. In view of the definition of an inverse, we obtain

$$f(h_0)b = f(h_0)A(h_0)b = \ldots = f(h_0)A^{n+m-2}(h_0)b = 0, \tag{A.1}$$

$$f(h_0)A^{n+m-1}(h_0)b = 1. \tag{A.2}$$

Letting

$$\begin{cases} z^*_{k,1} = f(h_0)z_k \\ z^*_{k,2} = f(h_0)A(h_0)z_k \\ z^*_{k,3} = f(h_0)A^2(h_0)z_k \\ \vdots \\ z^*_{k,n+m} = f(h_0)A^{n+m-1}(h_0)z_k \end{cases} \tag{A.3}$$

yields $z^*_k = Sz_k$, where $S = \begin{bmatrix} f(h_0) \\ f(h_0)A(h_0) \\ f(h_0)A^2(h_0) \\ \vdots \\ f(h_0)A^{n+m-1}(h_0) \end{bmatrix}$, and $\det S \neq 0$ since the discrete system (12) satisfies condition (13).

For $h = h_0 > 0$, system (12) takes the form

$$z_{k+1} = A(h_0)z_k + bu_k, \quad k = 0, 1, 2, \ldots;$$

due to (A.1) and (A.2), from equalities (A.3) it follows that

$$\begin{cases} z^*_{k+1,1} = f(h_0)z_{k+1} = z^*_{k,2} \\ z^*_{k+1,2} = f(h_0)A(h_0)z_{k+1} = z^*_{k,3} \\ \vdots \\ z^*_{k+1,n+m-1} = f(h_0)A^{n+m-2}(h_0)z_{k+1} = z^*_{k,n+m} \\ z^*_{k+1,n+m} = f(h_0)A^{n+m-1}(h_0)z_{k+1} = \alpha z_k + u_k, \end{cases} \tag{A.4}$$

where $\alpha = f(h_0)A^{n+m}(h_0)$.

Therefore, system (A.4) is given by (15) with $u^*_k = \alpha z_k + u_k$ and is equivalent to the discrete system (12) for $h = h_0 > 0$.

The proof of Theorem 2 is complete.

Proof of Theorem 3. As the discrete system (12) satisfies condition (13), by Theorem 2 it can be represented in the form (15).

Let $u^*_k = pz^*_k$, where $p = [p_1 \ \ p_2 \ \ \ldots \ \ p_{n+m}]$; then system (15) takes the form

$$\begin{cases} z^*_{k+1,1} = z^*_{k,2} \\ z^*_{k+1,2} = z^*_{k,3} \\ \vdots \\ z^*_{k+1,n+m-1} = z^*_{k,n+m} \\ z^*_{k+1,n+m} = p_1 z^*_{k,1} + p_2 z^*_{k,2} + \ldots + p_{n+m} z^*_{k,n+m}. \end{cases} \tag{A.5}$$

In addition, let $z^*_{k,1} = \lambda^0 = 1$, $z^*_{k,2} = z^*_{k+1,1} = \lambda$, $z^*_{k,3} = z^*_{k+1,2} = z^*_{k+2,1} = \lambda^2, \ldots, z^*_{k,n+m} = \lambda^{n+m-1}$, and $z^*_{k+1,n+m} = \lambda^{n+m}$. Then system (A.5) is reduced to the equation

$$\lambda^{n+m} - p_{n+m}\lambda^{n+m-1} - p_{n+m-1}\lambda^{n+m-2} - \ldots - p_2\lambda - p_1 = 0. \tag{A.6}$$

Equation (A.6) is characteristic for the matrix of system (A.5). Since the choice of the coefficients of equation (A.6) is arbitrary, we assign them so that $|\lambda_i| < 1$, $i = 1, \ldots, n+m$. Then system (15) with the control law $u^*_k = pz^*_k$ has the asymptotically stable solution $z^*_k = 0$. Hence, the discrete system (12) represented in the form (15) is stabilizable for $h = h_0 > 0$, and the proof of Theorem 3 is complete.

Proof of Theorem 4. The desired result is immediate from Corollary 1, the relation (10), and the sufficient condition for the asymptotic stability of the trivial equilibrium of the nonlinear continuous–discrete system (1) obtained in [23]. The proof of Theorem 4 is complete.

Proof of Theorem 5. It can shown that if $a_1 \neq 0$, $b_1 \neq 0$, and $c \neq 0$ in the nonlinear hybrid system (17), the corresponding discrete system (12) with the matrices (18) will satisfy condition (13). In this case, the desired result is true based on Theorem 4. The proof of Theorem 5 is complete.

Proof of Theorem 6. Without losing generality, we suppose that $\operatorname{rank} B = q$, $q > 1$. Let $b_1, \ldots, b_q$ be the columns of the matrix B, $A_0 = A(h_0)$, and the discrete system (11) satisfy condition (19).

From the column sequence (see [27])

$$(b_1, b_2, \ldots, b_q, A_0b_1, A_0b_2, \ldots, A_0b_q, A_0^2b_1, A_0^2b_2, \ldots, A_0^2b_q, \ldots, A_0^{n+m-1}b_1, \ldots, A_0^{n+m-1}b_q)$$

we compile the matrix $F_0 = F(h_0)$ of the same order and rank $(n+m)$:

$$F_0 = [b_1, A_0 b_1, A_0^2 b_1, \ldots, A_0^{s_1-1} b_1, b_2, A_0 b_2, A_0^2 b_2, \ldots, \\ A_0^{s_2-1} b_2, \ldots, b_q, A_0 b_q, A_0^2 b_q, \ldots, A_0^{s_q-1} b_q]. \tag{A.7}$$

For each column b_k, the matrix (A.7) should include all columns $b_k, A_0 b_k, A_0^2 b_k, \ldots, A_0^{s_k-1} b_k$, where $s_k = 1, \ldots, n+m$, $k = 1, \ldots, q$, with each column $A_0^j b_k$ $(j = 0, 1, \ldots, n+m-1)$ being included if it forms a linearly independent system with all preceding columns in this matrix. Otherwise, the columns $A_0^j b_k$, $A_0^{j+1} b_k, \ldots, A_0^{n+m-1} b_k$ are excluded from consideration.

Since $\operatorname{rank} F_0 = n+m$, there exists the inverse F_0^{-1}. Let f_k $(k = 1, \ldots, q)$ denote the row of the matrix F_0^{-1} numbered by $\sum\limits_{j=1}^{k} s_j$. According to the definition of an inverse, we obtain the conditions

$$f_k A_0^{j-1} b_i = 0, \quad (k \neq i) \vee (j \neq s_k), \tag{A.8}$$

$$f_k A_0^{s_k-1} b_k = 1. \tag{A.9}$$

Let

$$\begin{cases} z^*_{k,1} = f_1 z_k \\ z^*_{k,2} = f_1 A_0 z_k \\ \vdots \\ z^*_{k,s_1} = f_1 A_0^{s_1-1} z_k \\ z^*_{k,s_1+1} = f_2 z_k \\ z^*_{k,s_1+2} = f_2 A_0 z_k \\ \vdots \\ z^*_{k,s_1+s_2} = f_2 A_0^{s_2-1} z_k \\ \vdots \\ z^*_{k,s_1+s_2+\ldots+s_q} = f_q A_0^{s_q-1} z_k, \end{cases} \tag{A.10}$$

where $s_1 + s_2 + \ldots + s_q = n+m$. Then $z^*_k = S z_k$, where $S = \begin{bmatrix} f_1 \\ f_1 A_0 \\ \vdots \\ f_1 A_0^{s_1-1} \\ f_2 \\ f_2 A_0 \\ \vdots \\ f_2 A_0^{s_2-1} \\ \vdots \\ f_q \\ \vdots \\ f_q A_0^{s_q-1} \end{bmatrix}$, S is a matrix of order $(n+m)$ with $\det S \neq 0$ because the discrete system (11) satisfies condition (19).

Then $z^*_{k+1} = S z_{k+1}$; for $h = h_0 > 0$, the discrete system (11) takes the form

$$z_{k+1} = A_0 z_k + B u_k.$$

Considering (A.8)–(A.10), we obtain

$$\begin{cases} z^*_{k+1,1} = f_1 A_0 z_k + f_1 B u_k = f_1 A_0 z_k = z^*_{k,2} \\ z^*_{k+1,2} = f_1 A_0^2 z_k + f_1 A_0 B u_k = f_1 A_0^2 z_k = z^*_{k,3} \\ \vdots \\ z^*_{k+1,s_1} = f_1 A_0^{s_1} z_k + f_1 A_0^{s_1-1} B u_k = f_1 A_0^{s_1} z_k + u_{k,1} = u^*_{k,1} \\ z^*_{k+1,s_1+1} = f_2 A_0 z_k + f_2 B u_k = z^*_{k,s_1+2} \\ \vdots \\ z^*_{k+1,s_1+s_2} = f_2 A_0^{s_2} z_k + f_2 A_0^{s_2-1} B u_k = f_2 A_0^{s_2} z_k + u_{k,2} = u^*_{k,2} \\ \vdots \\ z^*_{k+1,s_1+s_2+\ldots+s_q} = f_q A_0^{s_q} z_k + f_q A_0^{s_q-1} B u_k = f_q A_0^{s_q} z_k + u_{k,q} = u^*_{k,q}\,. \end{cases}$$

Thus, for $h = h_0 > 0$, the transformation (20) reduces the discrete system (11) to a set of q independent subsystems in the Brunovský canonical form:

$$\begin{cases} z^*_{k+1,1} = z^*_{k,2} \\ z^*_{k+1,2} = z^*_{k,3} \\ \vdots \\ z^*_{k+1,s_1} = u^*_{k,1} \\ z^*_{k+1,s_1+1} = z^*_{k,s_1+2} \\ \vdots \\ z^*_{k+1,s_1+s_2} = u^*_{k,2} \\ \vdots \\ z^*_{k+1,s_1+s_2+\ldots+s_q-1} = z^*_{k,s_1+s_2+\ldots+s_q} \\ z^*_{k+1,n+m} = u^*_{k,q}\,. \end{cases} \tag{A.11}$$

The dimensions of these subsystems (the number of equations figuring in them) are $s_1, s_2, \ldots, s_q$, respectively. In addition, $u^*_{k,l} = \alpha_l z_k + u_{k,l}$ $(l = 1, \ldots, q)$ are the components of the control vector, and $\alpha_l = f_l A_0^{s_l}$. The discrete system (11) is equivalent to system (A.11) for $h = h_0 > 0$.

The proof of Theorem 6 is complete.

Proof of Theorem 7. By the proof of Theorem 3, each of the q subsystems in (A.11) has an asymptotically stable solution $z^* = 0$ under appropriate control laws. In this case, system (A.11) has the asymptotically stable trivial solution. Hence, the equivalent system (11) is stabilizable for $h = h_0 > 0$.

The proof of Theorem 7 is complete.

Proof of Theorem 8. It follows from Theorem 7 that the discrete system (11) is stabilizable for $h = h_0 > 0$ and, according to Corollary 2, all eigenvalues of the matrix $(A_0 + BS^*)$ are smaller than 1 by their magnitude. Then the desired result is immediate from the relation (10) and the sufficient condition for the asymptotic stability of the trivial equilibrium of the nonlinear continuous–discrete system (1) obtained in [23].

The proof of Theorem 8 is complete.

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NONLINEAR SYSTEMS

Stability of Regular Precessions of a Body with a Fixed Point Bounded by the Ellipsoid of Revolution in a Flow of Particles

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Abstract—Stability of regular precessions of a rigid body bounded by the ellipsoid of revolution in a free molecular flow of particles is studied. Conditions of stability of regular precessions of the body are obtained and the Poincare–Chetaev bifurcation diagrams are constructed.

Keywords: rigid body with a fixed point, free molecular flow of particles, regular precessions, stability of steady motions

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1. INTRODUCTION

The problem of motion of a rigid body with a fixed point under the action of external forces and moments is a generalization of the classical problem of motion of a heavy rigid body with a fixed point. One of the cases of integrability of this problem was discovered by Lagrange [1]. Subsequently many papers devoted to study the dynamics of the Lagrange top were published. Regular precessions of the Lagrange top were firstly considered in [2, 3], and their stability were studied in [2, 4, 5].

The mathematical model of interaction of a free molecular flow of particles with a rigid body was proposed by Karymov [6, 7] and Beletsky [8, 9]. In their papers the dynamics of satellites moving in the upper layers of the atmosphere or moving under the action of solar radiation pressure is considered. For the first time the corresponding model of interaction of a free molecular flow with a rigid body was applied to the problem of motion of a rigid body with a fixed point by Burov and Karapetyan [10]. In the paper by Burov and Karapetyan [10] the equations of motion of the body were obtained, and the integrable case, similar to the Lagrange case in the classical problem of motion of a heavy rigid body with a fixed point, was found. The problem of stability of steady motions of a satellite under the action of solar radiation pressure was considered, for example, by Sidorenko [11].

In this paper we study the stability of regular precessions of a dynamically symmetric body, bounded by the ellipsoid of revolution, in a flow of particles. Using the Routh theory of analyzing the stability of steady motions of mechanical systems with known first integrals, we obtain stability conditions for regular precessions of a rigid body with a fixed point located in a flow of particles.

2. PROBLEM FORMULATION

Let us consider the problem of motion of a rigid body with a fixed point O in a free molecular flow of particles of constant density ρ (see Fig. 1). The particles in the flow move with a constant velocity

$$-\boldsymbol{v} = v_0\boldsymbol{\gamma},$$

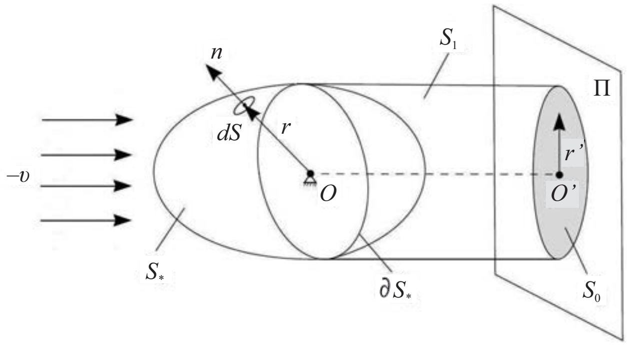

Fig. 1. Rigid body with a fixed point in a flow of particles.

where $\boldsymbol{\gamma}$ is a unit vector, fixed in inertial space and directed along the oncoming flow. Thus, particles move in a fixed direction, their thermal motion is neglected. The following model of particle—body interaction is chosen: a particle makes an inelastic collision with the body, transferring all its energy to the body and not being reflected. We will also consider sufficiently slow rotations of the body, i.e., we will assume that the flow velocity v_0 considerably exceeds the product of characteristic value of the angular velocity of the body and the characteristic length from any point of the body to a fixed point. Using the method, proposed by Beletsky [8, 9], we can write equations of motion of the body in the following form (see [10, 19–21]):

$$\begin{gathered}\mathbb{J}_0\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega} \times \mathbb{J}_0\boldsymbol{\omega}] = -\rho v_0^2 S(\boldsymbol{\gamma})\,[\boldsymbol{\gamma} \times \boldsymbol{c}(\boldsymbol{\gamma})]\,,\\ \dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega} \times \boldsymbol{\gamma}] = 0,\end{gathered} \tag{1}$$

where $\mathbb{J}_0 = \operatorname{diag}(A_1, A_2, A_3)$ is the inertia matrix of the body relative to a fixed point O, written in the coordinate system $Oxyz$, which is rigidly connected with the body. The origin of this coordinate system is located at the fixed point O and its axes are directed along the principal axes of inertia of the body at O. We denote the unit vectors of this coordinate system by $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$. Let $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_x + \omega_2\boldsymbol{e}_y + \omega_3\boldsymbol{e}_z$ be the absolute angular velocity of the body in the same coordinate system. The expression $S(\boldsymbol{\gamma})$ is the area of the figure S_0—the orthogonal projection of the surface of the body onto the plane Π, perpendicular to the direction of the oncoming particle flow $\boldsymbol{\gamma}$. We can say that S_0 is the shadow of the body on the plane Π, i.e., the projection of the body on the Π plane along the $\boldsymbol{\gamma}$ direction. Vector $\boldsymbol{c}(\boldsymbol{\gamma})$ is the vector drawn from the projection O' of the fixed point O to the centroid of the shadow—coinciding with the center of mass of a homogeneous plate occupying the region S_0 (see Fig. 1).

3. REGULAR PRECESSIONS OF THE BODY

Let us consider a dynamically symmetric rigid body ($A_1 = A_2$) with a fixed point lying on the axis of dynamical symmetry. Assume that the body is bounded by the ellipsoid of revolution with semiaxes $a_1 = a_2 = a$, $a_3 = b$, the axis of symmetry of which coincides with the axis of dynamical symmetry of the body. In this case, the equations of motion of the body (1) admit the quadratic and two linear in generalized velocities first integrals [19–22]:

$$U_0 = \frac{A_1}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{A_3}{2}\omega_3^2 - f\int_0^{\gamma_3} S(\gamma_3)\, c_3(\gamma_3)\, d\gamma_3 = k_0 = \text{const}, \tag{2}$$

$$U_1 = A_1(\omega_1\gamma_1 + \omega_2\gamma_2) + A_3\omega_3\gamma_3 = k_1 = \text{const}, \tag{3}$$

$$U_2 = \omega_3 = k_2 = \text{const}, \tag{4}$$

where $f = \rho v_0^2$. In this case, to study the steady motions of the system, one can use the Routh theory for holonomic systems with explicitly known first integrals [12–15, 18]. The effective potential in the case, when the body is bounded by the elongated ellipsoid of revolution is explicitly written as follows:

$$W(\theta) = \frac{(k_1 - A_3 k_2 \cos\theta)^2}{2A_1 \sin^2\theta} - \frac{f\pi a^2 bl}{2}\cos\theta\sqrt{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}}$$
$$- \frac{f\pi bl}{2\sqrt{\dfrac{1}{a^2} - \dfrac{1}{b^2}}}\arctan\left(\frac{\sqrt{\dfrac{1}{a^2} - \dfrac{1}{b^2}}\cos\theta}{\sqrt{\dfrac{\sin^2\theta}{a^2} + \dfrac{\cos^2\theta}{b^2}}}\right). \tag{5}$$

Here l is the distance from the fixed point O to the center of the ellipsoid of revolution, bounding the rigid body and θ is the angle between the axis of dynamical symmetry of the body and the direction of the flow of particles $\boldsymbol{\gamma}$, measured so that at $\theta = 0$ the body is oriented along the flow of particles. For any values of contants of the first integrals, the system of equations (1) admits a two-parametric family of particular solutions of the following form:

$$\begin{gathered} \omega_1 = \omega\gamma_1, \quad \omega_2 = \omega\gamma_2, \quad \omega_3 = \omega\cos\theta + \Omega, \\ \gamma_1^2 + \gamma_2^2 = 1 - \gamma_3^2 = \sin^2\theta, \quad \gamma_3 = \cos\theta, \end{gathered} \tag{6}$$

where ω and Ω are constants connected with the constants k_1 and k_2 of the first integrals (3) and (4) by the equations

$$\omega = \frac{k_1 - A_3 k_2 \cos\theta}{A_1 \sin^2\theta}, \quad \Omega = \frac{(A_1 \sin^2\theta + A_3 \cos^2\theta)\, k_2 - k_1 \cos\theta}{A_1 \sin^2\theta},$$

and the constant angle θ is determined from the condition of existence of regular precessions

$$\frac{dW(\theta)}{d\theta} = 0,$$

which is explicitly written as follows:

$$\frac{(k_1 - A_3 k_2 \cos\theta)(A_3 k_2 - k_1 \cos\theta)}{2A_1 \sin^3\theta} + f\pi a^2 bl \sin\theta\sqrt{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}} = 0. \tag{7}$$

Let us write the equation (7) in dimensionless form. For this purpose, we introduce dimensionless constants of the first integrals

$$p_1 = \frac{k_1}{\sqrt{A_3 f\pi a^2 l}}, \quad p_2 = k_2\sqrt{\frac{A_3}{f\pi a^2 l}} \tag{8}$$

and dimensionless parameters

$$y = \frac{A_1}{A_3} \in \left[\frac{1}{2}, +\infty\right), \quad z = \frac{b^2}{a^2}.$$

Then the condition for the existence of regular precessions (7) is dimensionless form is rewritten as follows:

$$\frac{(p_1 - p_2\cos\theta)(p_2 - p_1\cos\theta)}{y\sin^3\theta} + \sin\theta\sqrt{z\sin^2\theta + \cos^2\theta} = 0,$$

or, multiplying by $-y\sin^3\theta$,

$$a_{11}p_1^2 + 2a_{12}p_1p_2 + a_{22}p_2^2 + a_1 = 0, \tag{9}$$
$$a_{11} = \cos\theta, \quad a_{12} = -\frac{1+\cos^2\theta}{2}, \quad a_{22} = \cos\theta,$$
$$a_1 = -y\sin^4\theta\sqrt{z\sin^2\theta + \cos^2\theta}.$$

It is easy to see that for each fixed value of θ the equation (9) defines the second order curve. Let us transform it to its canonical form. To do this, we introduce new variables x_1 and y_1 according to the formulae:

$$x_1 = p_1 - p_2, \qquad y_1 = p_1 + p_2. \tag{10}$$

In order to understand the physical meaning of the variables x_1 and y_1, let us recall the definition of the constants k_1 and k_2 (see (3), (4)). For a dynamically symmetric body we can write

$$k_1 = A_1(\omega_1\gamma_1 + \omega_2\gamma_2) + A_3\omega_3\gamma_3 = (\mathbb{J}_O\boldsymbol{\omega},\, \boldsymbol{\gamma}), \quad k_2 = \omega_3 = \frac{(\mathbb{J}_O\boldsymbol{\omega},\, \boldsymbol{e}_z)}{A_3}.$$

Substituting these expressions into (8), we obtain the following equations:

$$p_1 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, \boldsymbol{\gamma}), \quad p_2 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, \boldsymbol{e}_z).$$

According to the formula (10) we can conclude that x_1 is, up to a constant factor, the projection of the angular momentum of the body onto the vector $\boldsymbol{\gamma} - \boldsymbol{e}_z$:

$$x_1 = p_1 - p_2 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, (\boldsymbol{\gamma} - \boldsymbol{e}_z)).$$

Similarly, y_1 is the projection of the angular momentum of the body onto the vector $\boldsymbol{\gamma} + \boldsymbol{e}_z$

$$y_1 = p_1 + p_2 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, (\boldsymbol{\gamma} + \boldsymbol{e}_z)).$$

Vectors $\boldsymbol{\gamma} - \boldsymbol{e}_z$ and $\boldsymbol{\gamma} + \boldsymbol{e}_z$ are orthogonal

$$((\boldsymbol{\gamma} - \boldsymbol{e}_z),\, (\boldsymbol{\gamma} + \boldsymbol{e}_z)) = \boldsymbol{\gamma}^2 - \boldsymbol{e}_z^2 = 0.$$

The third orthogonal vector is perpendicular to $\boldsymbol{\gamma}$ and $\boldsymbol{e}_z$ and therefore it is directed along the nodal line.

$$[(\boldsymbol{\gamma} - \boldsymbol{e}_z) \times (\boldsymbol{\gamma} + \boldsymbol{e}_z)] = 2\,[\boldsymbol{\gamma} \times \boldsymbol{e}_z]\,.$$

Thus, $\boldsymbol{\gamma} - \boldsymbol{e}_z$ and $\boldsymbol{\gamma} + \boldsymbol{e}_z$ are orthogonal vectors in the plane, perpendicular to the nodal line, and x_1 and y_1, up to a constant factor, are projections of the angular momentum of the body onto these vectors.

Let us write the equation (9) in the variables x_1, y_1

$$\frac{1}{4}(1+\cos\theta)^2 x_1^2 - \frac{1}{4}(1-\cos\theta)^2 y_1^2 + a_1 = 0. \tag{11}$$

The given equation, up to a constant factor, is the canonical equation of a second-order curve. Let us determine the type of this curve. For $\theta \neq \pi n$ we have $a_1 < 0$, and the coefficients in the quadratic part are nonzero. This means (see, for example, [16, 17]), that in each section by planes

$\theta \neq \pi n$ of the surface, defined by the equation (11), we have a hyperbola. For $\theta = \pi n$ we have $a_1 = 0$, as well as one of the coefficients in the quadratic part will be equal to zero. Thus, in the section by planes $\theta = \pi n$ of the surface, defined by the equation (11) we have a straight line of the form

$$p_1 = (-1)^n p_2.$$

These straight lines correspond to a single parametric family of solutions of equations of motion of the body in a flow of particles, which correspond to permanent rotations of the body about its axis of dynamical symmetry, coinciding with the direction of the flow of particles. The stability of these permanent rotations has been studied in [10, 22].

The condition of stability of steady motions (6) in the case when a rigid body with a fixed point, located in the flow of particles, is bounded by the elongated ellipsoid of revolution, has the form

$$\frac{d^2 W(\theta)}{d\theta^2} \geqslant 0$$

of, if we write it explicitly,

$$\frac{(1+2\cos^2\theta)}{y\sin^4\theta}p_1^2 - \frac{(5+\cos^2\theta)\cos\theta}{y\sin^4\theta}p_1p_2 + \frac{(1+2\cos^2\theta)}{y\sin^4\theta}p_2^2 + \frac{(2z\sin^2\theta + 2\cos^2\theta - 1)\cos\theta}{\sqrt{z\sin^2\theta+\cos^2\theta}} \geqslant 0.$$

After multiplying by the positive factor $y\sin^4\theta$, this inequality takes the form

$$b_{11}p_1^2 + 2b_{12}p_1p_2 + b_{22}p_2^2 + b_1 \geqslant 0, \tag{12}$$

$$b_{11} = 1 + 2\cos^2\theta, \quad b_{12} = -\frac{(5+\cos^2\theta)\cos\theta}{2}, \quad b_{22} = 1 + 2\cos^2\theta,$$

$$b_1 = \frac{y\,(2z\sin^2\theta + 2\cos^2\theta - 1)\sin^4\theta\cos\theta}{\sqrt{z\sin^2\theta+\cos^2\theta}}.$$

Let us write the inequality (12) using the variables x_1 and y_1:

$$\frac{1}{4}(1+\cos\theta)^2(2+\cos\theta)\,x_1^2 + \frac{1}{4}(1-\cos\theta)^2(2-\cos\theta)\,y_1^2 + b_1 \geqslant 0. \tag{13}$$

For each fixed θ, the boundary of the stability region (i.e., the curve, corresponding to the equal sign in the inequality (13)) will also be the second-order curve. The type of this curve depends on the sign of b_1. For $b_1 < 0$ the boundary of the stability region is an ellipse with the center at $x_1 = 0$, $y_1 = 0$. On the plane of dimensionless variables x_1, y_1 for each fixed θ, outside the corresponding ellipse there will be a stability region of regular precessions (6), and inside it—an instability region. For $b_1 > 0$ the boundary of the stability region degenerates into an imaginary ellipse, and for $b_1 = 0$ the boundary is a straight line. We can note here, that for $b_1 \geqslant 0$ the inequality (13) is satisfied for any values of x_1, y_1, and therefore the regular precessions of the body will be stable.

Considering simultaneously the condition of existence of regular precessions (11) and the condition of stability of regular precessions (13) of a body with a fixed point in the flow of particles, we can draw a conclusion about stability of such motions. Namely, if for some fixed θ the hyperbola (11) and the ellipse (13) do not intersect, then the steady motion (6), corresponding to this value of θ, is stable (Figs. 2–4).

Let us study the problem of stability of regular precessions in more details. From the condition of existence of regular precessions (11) we express y_1^2:

$$y_1^2 = \frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}x_1^2 + \frac{4a_1}{(1-\cos\theta)^2}. \tag{14}$$

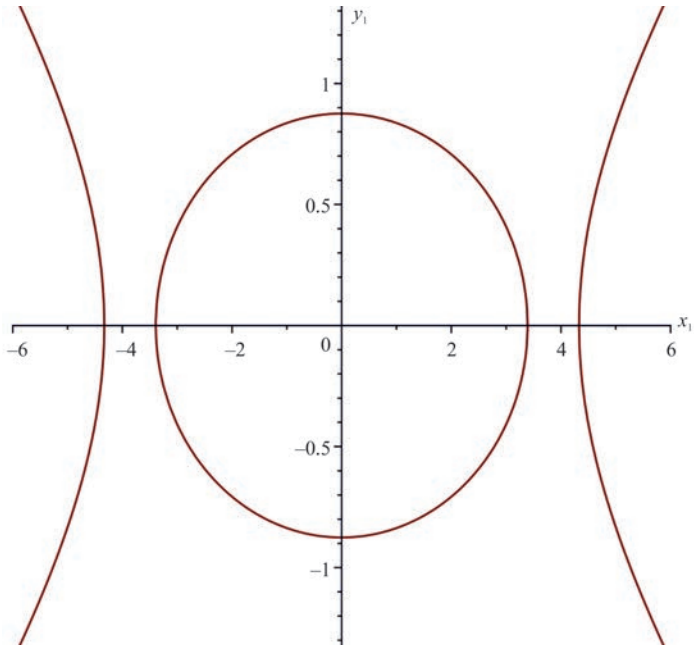

Fig. 2. Relative position of the hyperbola and the ellipse for $y = \frac{5}{6}$, $z = 8$, $\theta = \frac{2\pi}{3}$.

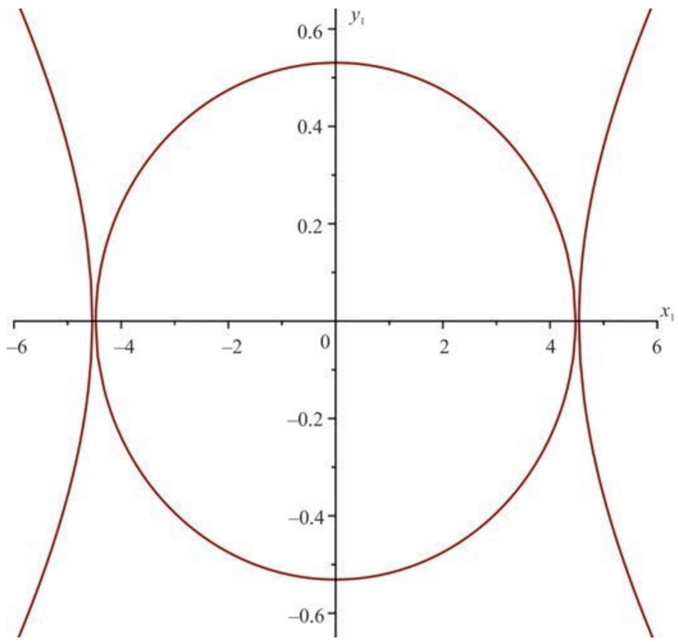

Fig. 3. Relative position of the hyperbola and the ellipse for $y = \frac{5}{6}$, $z = 8$, $\theta = \frac{3\pi}{4}$.

Since $y_1^2 \geqslant 0$, then from (14) we obtain that x_1 should satisfy the condition:

$$x_1^2 \geqslant \frac{4y\sin^4\theta}{(1+\cos\theta)^2}\sqrt{z\sin^2\theta+\cos^2\theta} = (x_1)_*^2. \tag{15}$$

Let us substitute the obtained expression (14) for y_1^2 into the stability condition (13). We obtain the following inequality

$$(1+\cos\theta)^2 x_1^2 + (2-\cos\theta)\, a_1 + b_1 \geqslant 0. \tag{16}$$

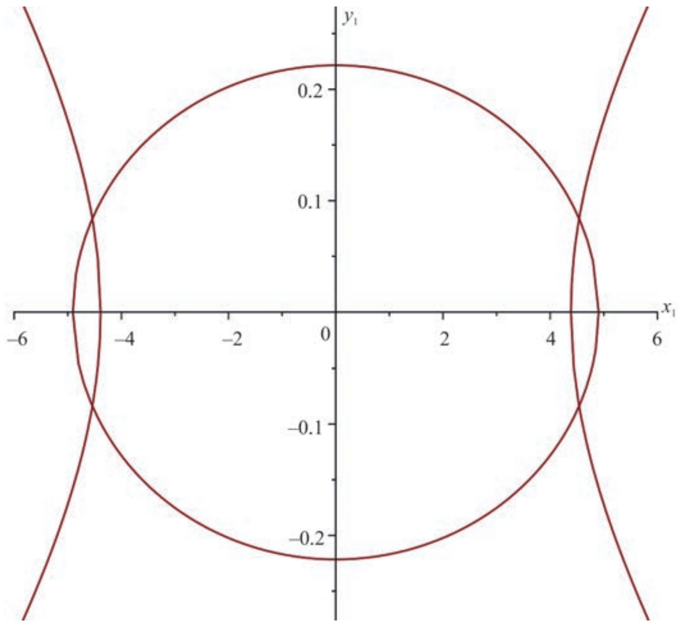

Fig. 4. Relative position of the hyperbola and the ellipse for $y = \frac{5}{6}$, $z = 8$, $\theta = \frac{5\pi}{6}$.

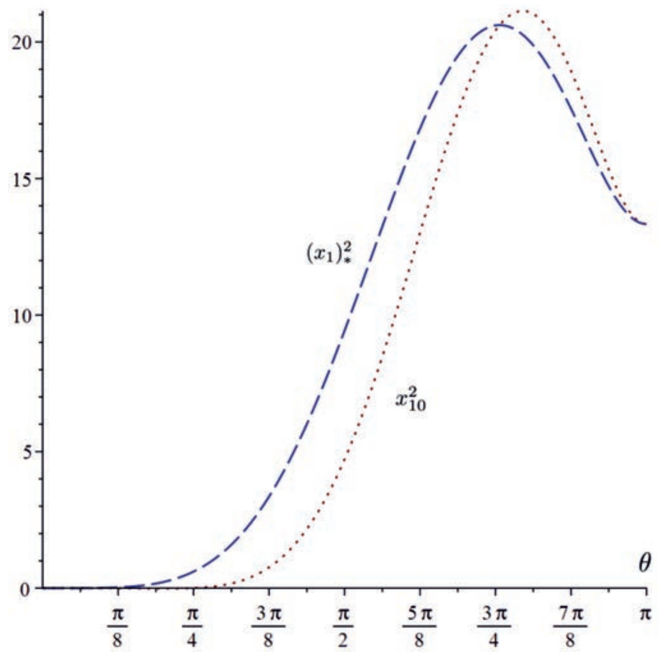

Fig. 5. Relative position of x^2_{10} and $(x_1)^2_*$ for $y = \frac{5}{6}$, $z = 8$.

The left hand side of defines a parabola whose branches are directed upwards. Let x_{10} be the abscissa of the intersection point of the parabola with the positive semiaxis Ox_1. Let us find x_{10} explicitly

$$x^2_{10} = -\frac{(2 - \cos\theta)\, a_1 + b_1}{(1 + \cos\theta)^2}.$$

After substitution the values of a_1 and b_1, we write the expression for x^2_{10} in explicit form:

$$x^2_{10} = \frac{y \sin^4\theta \sqrt{z \sin^2\theta + \cos^2\theta}}{(1 + \cos\theta)^2} \left(2 - 3\cos\theta + \frac{\cos\theta}{z \sin^2\theta + \cos^2\theta}\right). \tag{17}$$

For $x_1^2 \geqslant x_{10}^2$ the stability condition (3) will be satisfied. Thus, regular precessions exist when the condition (15) is satisfied, and they will be stable if the condition (3) is satisfied. From this fact we can draw the following conclusions (Fig. 5):

1. For $x_{10}^2 < (x_1)_*^2$, regular precessions of the body are stable for any admissible values of x_1, y_1 (and, accordingly, for any values of constants p_1 and p_2 of the first integrals).

2. For $x_{10}^2 > (x_1)_*^2$, regular precessions are stable only if the condition $x_1^2 > x_{10}^2$ holds.

3.1. The Case $z > 1$

First of all let us consider the case of elongated ellipsoid of revolution, $z > 1$. We will find the conditions under which $x_{10}^2 > (x_1)_*^2$. We obtain

$$3\,(1-z)\cos^3\theta + 2\,(1-z)\cos^2\theta + (3z-1)\cos\theta + 2z < 0.$$

By factoring the left hand side, we obtain the following inequality:

$$3\,(z-1)\,(\cos\theta+1)\left(\cos\theta - \frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right)\left(\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right) > 0. \qquad (18)$$

The first three factors in the left hand side of inequality (18) are positive. It can be shown that for $z > 1$ the following condition is valid:

$$\frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}} > 1.$$

Thus, the factor

$$\cos\theta - \frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}$$

in the left hand side of (18) is always negative. Therefore the condition (18) is equivalent to the condition

$$\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}} < 0. \qquad (19)$$

Let us consider the latter condition. The function

$$\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}$$

increases monotonically for $z > 1$. It takes the value -1 for $z = 2$ and then it asymptotically tends to $-\frac{2}{3}$. Thus for $1 < z \leqslant 2$ we obtain

$$\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}} \geqslant 0,$$

and therefore the condition (19) is not satisfied. From this fact we conclude that for $1 < z \leqslant 2$ regular precessions of the rigid body are stable for any value of θ and for any admissible values of the parameters x_1 and y_1 (and accordingly, for any admissible values of the constants p_1 and p_2 of the first integrals).

Figure 6 presents the Poincare–Chetaev bifurcation diagram for $y = 2$, $z = \frac{12}{11}$ at the level $y_1 = 0$. On this figure the straight line $\theta = \pi$ corresponds to permanent rotations of the body about its axis

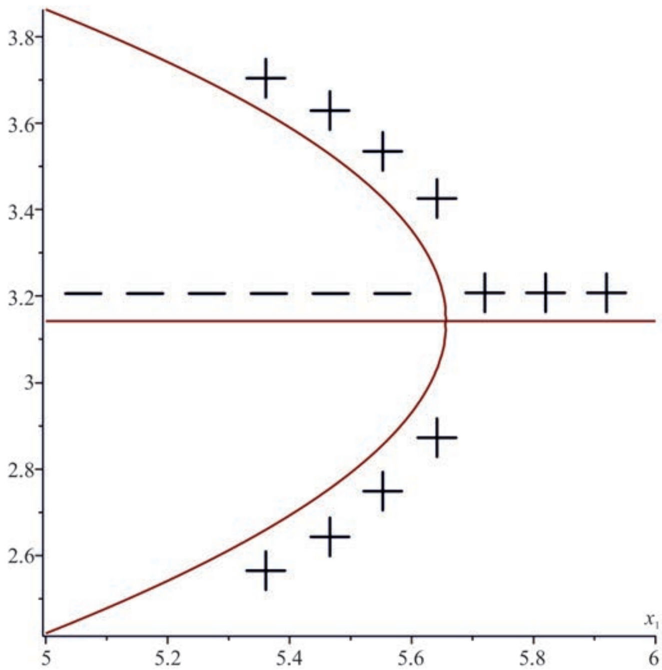

Fig. 6. Section of the surface (11) by the plane $y_1 = 0$ for $y = 2$, $z = \frac{12}{11}$.

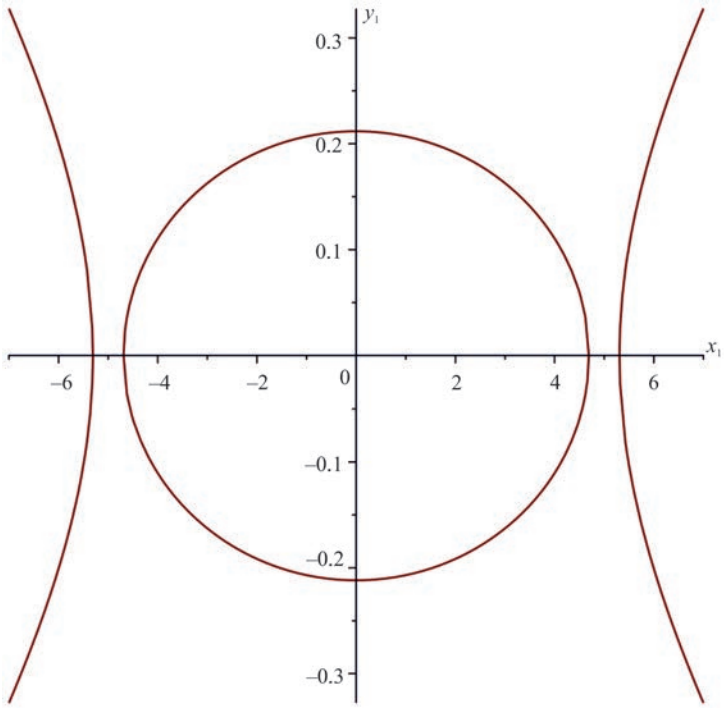

Fig. 7. Relative position of the hyperbola and the ellipse for $y = 2$, $z = \frac{12}{11}$, $\theta = \frac{5\pi}{6}$.

of dynamical symmetry. It is easy to see the stable regular precessions of the body which branch off from this straight line. In addition, on the plane of dimensionless variables x_1 and y_1 for the fixed values of θ the corresponding hyperbola (11) and ellipse (13) were constructed (see Fig. 7). It is easy to see, that they do not intersect in the considered case.

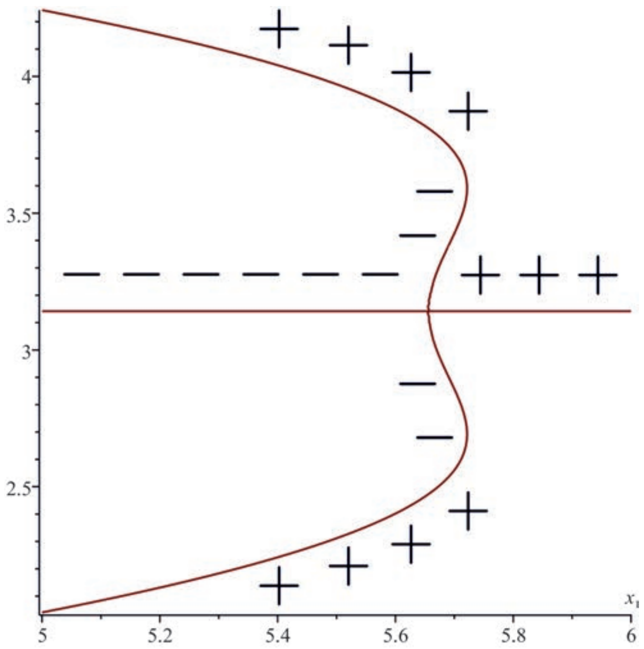

Fig. 8. Section of the surface (11) by the plane $y_1 = 0$ for $y = 2$, $z = \frac{5}{2}$.

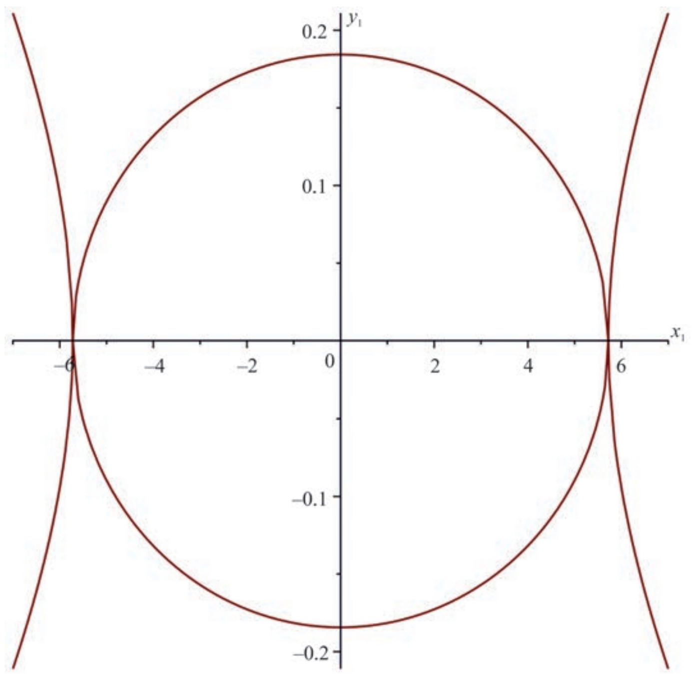

Fig. 9. Relative position of the hyperbola and the ellipse for $y = 2$, $z = \frac{5}{2}$, $\theta = \theta_2$.

Let us introduce the new value $\theta = \theta_2$:

$$\theta_2 = \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right). \tag{20}$$

We can conclude that for $z > 2$ the inequality (19) holds for $\cos\theta < \cos\theta_2$ and does not hold for $\cos\theta \geqslant \cos\theta_2$.

Figure 8 presents the Poincare–Chetaev bifurcation diagram for $y = 2$, $z = \frac{5}{2}$ at the level $y_1 = 0$. It is easy to see, that the regular precessions of the body, branching off the permanent rotations, are unstable near the value $\theta = \pi$, and then, passing through the inflection point, become stable. The inflection point corresponds to $\theta = \theta_2$, where θ_2 is determined by (20).

In addition Fig. 9 shows the hyperbola (11) and the ellipse (13) on the plane of dimensionless variables x_1 and y_1 for $\theta = \theta_2$, where θ_2 is determined by (20). It is clear that for $\theta = \theta_2$ the hyperbola and the ellipse touch but do not intersect.

Taking into account all the obtained results we can conclude that the condition (18) is valid for $z > 2$ and $\cos\theta < \cos\theta_2$. And it is not satisfied for any other values of the parameters. Thus, regular precessions of the rigid body located in the flow of particles are stable for any values of x_1, y_1 (and, accordingly, for any values of dimensionless constants p_1 and p_2 of the first integrals), if $1 < z \leqslant 2$ or if $z > 2$ and $\cos\theta > \cos\theta_2$. If $z > 2$ and $\cos\theta < \cos\theta_2$, then the regular precessions of the body are stable under condition $x_1^2 > x_{10}^2$.

3.2. The Case $z < 1$

Now we study the stability of regular precessions of the body bounded by the oblate ellipsoid of revolution, $z < 1$. All previously find expressions for x_{10}^2 and $(x_1)_*^2$ will have the same form (15), (17) for the oblate ellipsoid. The conditions, under which $x_{10}^2 > (x_1)_*^2$, will be the same (18). Simplifying them, we obtain the following condition:

$$\left(\cos\theta - \frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\left(\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right) < 0. \tag{21}$$

It is easy to see, that for $z \in \left[\frac{1}{25},\, 1\right)$ the condition (21) does not hold. Thus, we can conclude, that for $z \in \left[\frac{1}{25},\, 1\right)$ we have $x_{10}^2 \leqslant (x_1)_*^2$ and therefore regular precessions of the body in the flow of particles will be stable for any values of x_1, y_1.

3.3. The Case $z < \frac{1}{25}$

Finally, let us consider the case $z < \frac{1}{25}$. The condition (21) will be satisfied for the following values of θ:

$$\theta \in \left(\arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right),\, \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\right). \tag{22}$$

Let us introduce the value $\theta = \theta_1$ such that

$$\theta_1 = \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right).$$

It is easy to obtain that for $\theta \in (0,\, \theta_1) \cup (\theta_2,\, \pi)$ the condition (21) will not satisfied, and therefore, regular precessions of the body will be stable for any values of x_1, y_1. If $\theta \in (\theta_1,\, \theta_2)$, then regular precessions will be stable only for $x_1^2 > x_{10}^2$.

Figure 10 presents the Poincare–Chetaev bifurcation diagram for $y = \frac{101}{200}$, $z = \frac{1}{100}$ at the level $y_1 = 0$. It is easy to see, that the regular precessions of the body, branching off from the permanent rotations, are stable near $\theta = \pi$, and its stability is preserved as the angle θ decreases to θ_2. Further, for $\theta \in (\theta_1,\, \theta_2)$, the regular precessions of the body become instable, and for $\theta < \theta_1$ they become stable again.

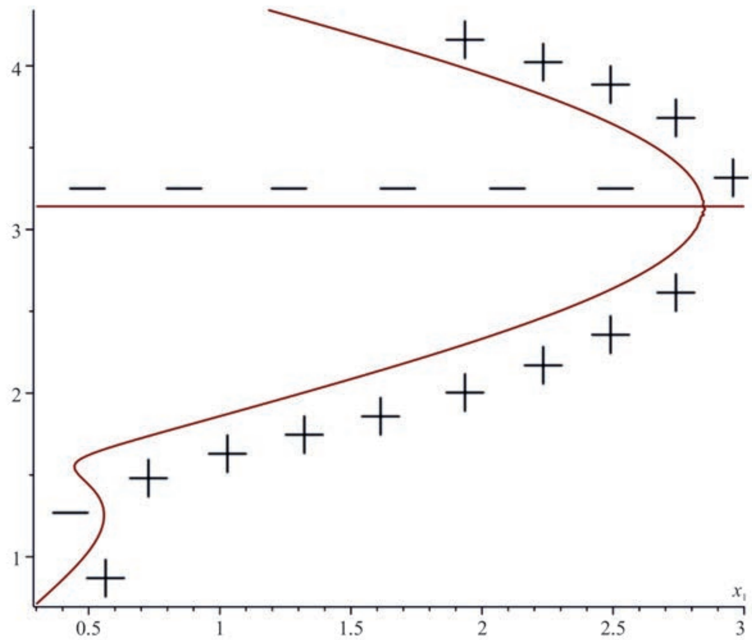

Fig. 10. Section of the surface (11) by the plane $y_1 = 0$ for $y = \frac{101}{200}, z = \frac{1}{100}$.

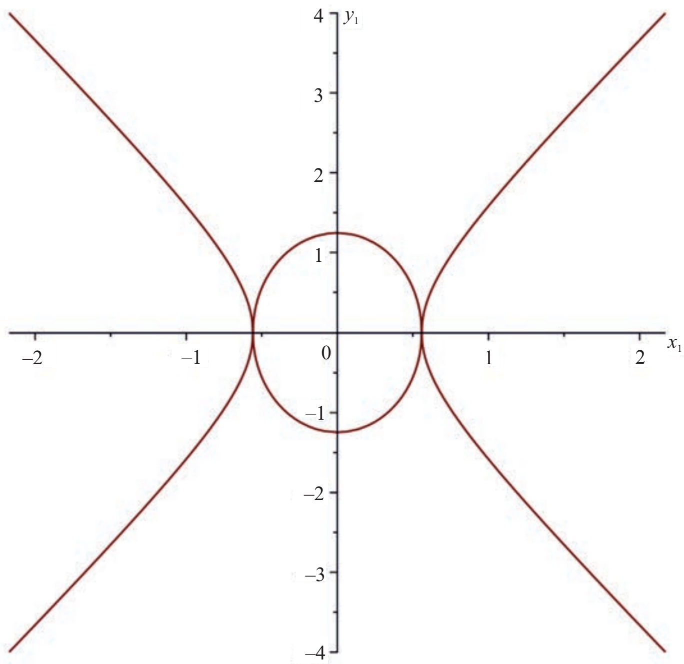

Fig. 11. Relative position of the hyperbola and the ellipse for $y = \frac{101}{200}$, $z = \frac{1}{100}$, $\theta = \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)$.

Figures 11 and 12 shows the hyperbola (11) and the ellipse (13) on the plane of dimensionless variables x_1 and y_1 for

$$z = \frac{1}{100}, \quad \theta = \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right) \text{ and } \quad \theta = \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right).$$

It is clear that for these values of θ the hyperbola (11) and the ellipse (13) touch, but do not intersect.

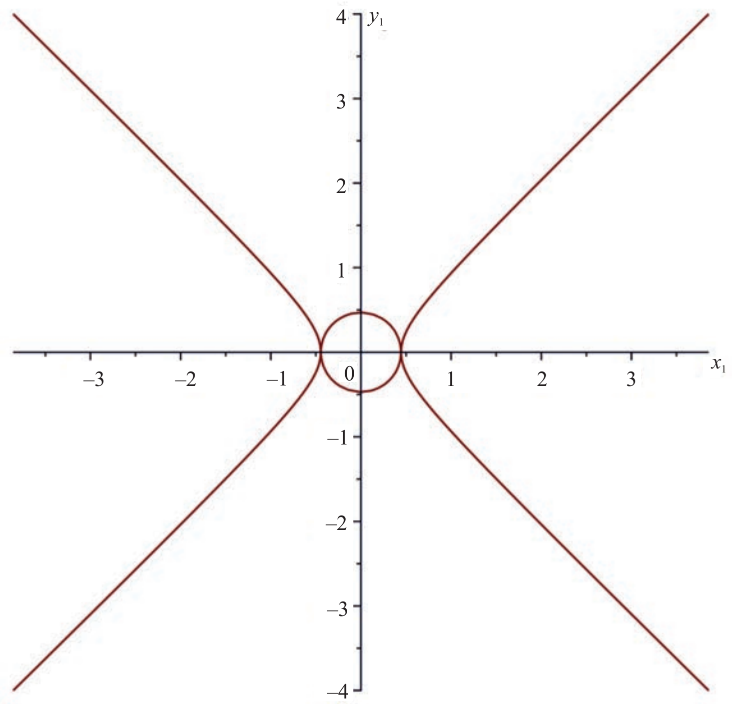

Fig. 12. Relative position of the hyperbola and the ellipse for $y = \frac{101}{200}$, $z = \frac{1}{100}$, $\theta = \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)$.

Thus we can finally conclude that all the regular precessions (6) of a dynamically symmetric rigid body with a fixed point, bounded by the ellipsoid of revolution and located in a flow of particles, will be stable for any values of x_1, y_1 (and, accordingly, for any values of the dimensionless constants p_1 and p_2 of the first integrals), if the ratio of the squares of the semiaxes of the ellipsoid is satisfied to the condition

$$\frac{1}{25} \leqslant z \leqslant 2.$$

For $z > 2$ the regular precessions of the body will be stable for any values of x_1 and y_1 if θ lies in the interval

$$\theta \in \left[0, \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right)\right).$$

For values of θ, belonging to the interval

$$\theta \in \left[\arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right), \pi\right)$$

the regular precessions are stable if the following condition is valid

$$x_1^2 > \frac{y\sin^4\theta\sqrt{z\sin^2\theta + \cos^2\theta}}{(1+\cos\theta)^2}\left(2 - 3\cos\theta + \frac{\cos\theta}{z\sin^2\theta + \cos^2\theta}\right).$$

For

$$0 < z < \frac{1}{25}$$

the regular precessions of the body will be stable for any values of x_1 and y_1, if θ belongs to the set

$$\theta \in \left(0, \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\right) \bigcup \left(\arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right), \pi\right).$$

For any θ, belonging to the interval

$$\theta \in \left[\arccos\left(\frac{1}{6}+\frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right), \arccos\left(\frac{1}{6}-\frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\right]$$

the regular precessions will be stable if the following condition is valid

$$x_1^2 > \frac{y\sin^4\theta\sqrt{z\sin^2\theta+\cos^2\theta}}{(1+\cos\theta)^2}\left(2-3\cos\theta+\frac{\cos\theta}{z\sin^2\theta+\cos^2\theta}\right).$$

These are the main results of study the stability of regular precessions of a dynamically symmetric body with a fixed point, bounded by the ellipsoid of revolution, in a flow of particles.

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NONLINEAR SYSTEMS

An Adaptive Stabilization Scheme for Autonomous System Oscillations[1]

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Abstract—A smooth autonomous system of general form is considered. A global family of nondegenerate periodic solutions by the parameter h is constructed; the period varies monotonically on this family. The problem of stabilizing the oscillations of the reduced controlled system is solved. A smooth autonomous control law with a parameter depending on h is applied, and an attracting cycle is constructed. The results are concretized for an nth-order differential equation. The relation of these results with the conclusions obtained for the reversible mechanical system is established. An adaptive control scheme for the reduced conservative system is proposed to stabilize any oscillation of the family. Some applications are presented.

Keywords: autonomous system, nondegenerate periodic solution, global family, Lyapunov center theorem, adaptive scheme, attracting cycle, natural stabilization

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1. INTRODUCTION

Consider the problem of controlling the oscillations (periodic solutions) of an autonomous system of general form. In the nondegenerate case, the following alternative is valid: either a cycle (isolated oscillation) or a family of oscillations. On a family of nondegenerate oscillations, the period monotonically depends on the family parameter h. Therefore, when stabilizing a family of oscillations, it is natural to look for a control law with a parameter depending on h. In an adaptive control system, the "controller automatically changes its structure or *its parameters depending on the changes of plant parameters* or disturbance properties" [1, p. 108].

In this paper, we apply an *autonomous* control law in which the controller parameter is found depending on the parameter h of the stabilized oscillation of the family: the controller has *adaptivity*. In this sense, the control scheme is said to be adaptive.

In the example

$$\ddot{x} + \omega^2 x = \varepsilon u, \quad u = (1 - Kx^2)\dot{x}, \tag{1}$$

the control law u contains the parameter K; ω is the frequency of a linear oscillator. For $\varepsilon = 0$, equation (1) admits the family of oscillations $x = A\cos\varphi$ with amplitude A and energy $h = \omega^2 A^2/2$. Let $K = 2\omega^2/h^*$ in (1) to stabilize the selected oscillation family with energy $h = h^*$. Then an attracting cycle close to the oscillation of the linear oscillator with energy h^* is implemented in (1). The formula for K is valid for any oscillation of the family. An adaptive stabilization scheme is applied.

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Equation (1) is called a van der Pol oscillator. The van der Pol equation [2] is obtained from (1) for $K = 1$: in a regerative receiver, ω is tuned to the frequency of the received signal.

A linear oscillator admits a family of isochronous oscillations. In a mathematical pendulum, the oscillation period is a monotonic function of the family parameter. Such a family of oscillations is called nondegenerate [3].

When studying oscillations in a multidimensional autonomous system, it is necessary to describe the entire set of nondegenerate oscillations. The problem of the global family of oscillations arises: existence, construction, and properties. Besides the interest for the theory of ordinary differential equations (ODEs), the knowledge of the global family ensures the complete solution of the control problem.

The main point about the global family is the possibility of continuing any local nondegenerate family of periodic solutions to the global family on which the family parameter will take all possible values for the solutions of the family. The global family for a particular case was constructed in [3]. An adaptive stabilization scheme for the oscillations of a reduced system on the plane was presented in [3]: an attracting cycle was constructed.

A global family is described by a reduced system whose dimension coincides with the dimension k of this family. In the general case (a system of n differential equations), we have $1 < k \leqslant n$. The existence of the global family, the isolation of the reduced system, and the design of an adaptive oscillation stabilization scheme for this system are considered below. The results are concretized for an nth-order differential equation. Also, we relate these conclusions to the results for systems possessing the properties of reversibility and conservativeness.

Some applications are presented. The three-body problem was introduced by L. Euler in 1764 when studying the motion of the Moon; it was published later in [4]. We demonstrate the construction of the global families of periodic orbits in this problem and apply the adaptive scheme for orbital stabilization. V.V. Beletskii's problem [5] describes the oscillations of a satellite in the elliptical orbit plane. In a particular case (a circular orbit), we stabilize any oscillation of the satellite using the adaptive control scheme.

2. GLOBAL FAMILY THEOREM

Consider a smooth autonomous equation of the form

$$\dot{z} = Z(z), \quad z \in \mathrm{R}^n. \tag{2}$$

Let $z(z_1^0, \ldots, z_n^0, t)$ denote the general solution of this equation, where $z^0 = (z_1^0, \ldots, z_n^0)$ is an initial point (for $t = 0$). A T-periodic solution of system (2) exists under the necessary and sufficient condition

$$f \equiv z(z_1^0, \ldots, z_n^0, T) - z^0 = 0. \tag{3}$$

Assume that equation (3) has a solution $z^0 = z^*$, $T = T^*$, not coinciding with the trivial equilibrium: $Z(z^*) \neq 0$. Then equation (3) admits a family of solutions in the parameter γ :

$$z^0 = z^*(\gamma), \quad T = T^*, \tag{4}$$

where γ is the shift of the initial time instant along the trajectory. In addition, the rank Ra of the functional (Jacobi) matrix A_f for the function f with the parameter T satisfies the inequality $\mathrm{Ra} \leqslant n - 1$ at the point z^* for $T = T^*$.

The case $\mathrm{Ra} = n - 1$ is said to be nondegenerate for a periodic solution; for details, see [3]. The case $\mathrm{Ra} = n - 1$ will be considered below.

In the nondegenerate case, the matrix A_f has a simple zero eigenvalue or a kth-order Jordan block of zero eigenvalues, $1 < k \leqslant n$. In the first situation, equation (3) admits a unique solution of the form (4). Therefore, the autonomous equation (2) admits an isolated periodic solution, i.e., a cycle with the period T^*.

The second situation is called the case of a family of periodic solutions.

In the neighborhood of the solution of equation (4), we introduce the linear system

$$\begin{gathered}\xi_s \equiv \frac{\partial f_s}{\partial z_1^0} dz_1^0 + \ldots + \frac{\partial f_s}{\partial z_n^0} dz_n^0 + \frac{\partial f_s}{\partial T} dT = 0, \quad s = 1, \ldots, n, \\ f = (f_1, \ldots, f_n),\end{gathered} \tag{5}$$

which is valid for arbitrary γ. The matrix A_f in (5) has rank $\mathrm{Ra} = n - 1$. The cycle is obtained from the simple zero eigenvalue: in (5) we have the solution $dz_1(\gamma) = \ldots = dz_n(\gamma) = 0$, $dT = 0$.

The family case is implemented under $dT \neq 0$. The derivatives

$$\frac{df_s}{dT} = Z_s(z^*(\gamma)), \quad s = 1, \ldots, n,$$

are calculated at the point (4) and ensure that the rank of the augmented matrix in (5) is equal to the number $(n-1)$. The values $dz_1^0(\gamma), \ldots, dz_k^0(\gamma)$ are computed from (5) as linear functions of dT. Varying dT yields a family of solutions. The period T changes from solution to solution, i.e., it is a function of the scalar parameter $h : z^0 = z^0(\gamma, h)$, $T = T(h)$. Thus, the local h-family is found from system (5). The derivative is $T'(h^*) \neq 0$ for the parameter value $h = h^*$. Therefore, on the family, the period $T(h)$ changes monotonically together with the family parameter h. In this sense, the h-family is nondegenerate [3].

Definition 1. A family of periodic solutions of equation (2) is said to be nondegenerate if the period $T(h)$ on this family monotonically depends on the parameter h.

According to the presentation, the point $z^0 = z^*$ of a periodic solution of equation (2) has the property of leading to a nondegenerate family of periodic solutions. The reference periodic solution also belongs to the family. In this sense, it is said to be nondegenerate. Any periodic solution of a nondegenerate family is also nondegenerate.

Definition 2. A solution belonging to a nondegenerate family of periodic solutions called a nondegenerate solution.

A nondegenerate periodic solution can be continued in the period T or, equivalently, in the family parameter h. This is called *local extensibility property*. A nondegenerate periodic solution is continued simultaneously toward increasing and decreasing the period.

The concept of a global family of periodic solutions was introduced in [3].

Definition 3. A nondegenerate family of periodic solutions on which the parameter h takes all possible values for the family solutions is called a global family.

When reducing the matrix A_f to the canonical form, system (5) decomposes into two subsystems; one subsystem, with a zero kth-order Jordan block, leads to a family of periodic solutions, whereas the other subsystem has a zero solution in (5). Therefore, in the new variables, a family of periodic solutions is described by k variables. In the phase space, a global family is represented by a connected k-dimensional set of points.

In the case $\mathrm{Ra} = n - 1$ (nondegenerate for a periodic solution), we have the following result.

Theorem 1. *Assume that equation* (2) *admits a nondegenerate periodic solution. Then it extends in the period T to a global family Σ. On Σ the period $T(h)$ monotonically depends on the family* *parameter h. The family Σ fills the global domain $\hat{\Sigma}$; Σ is described by a reduced system of order k. For the points of the domain $\hat{\Sigma}$, the rank is* $\mathrm{Ra} = n-1$; *on its boundary $\partial\hat{\Sigma}$, the condition* $\mathrm{Ra} = n-1$ *fails.*

Proof. A nondegenerate periodic solution has local extensibility. This property is independent of the dimension k of the Jordan block with zero eigenvalues of the matrix A_f. Therefore, the proof of Theorem 1 coincides, within the dimension of the zero kth-order Jordan block and minor editorial changes, with that of [3, Theorem 1] for the case $k = 2$. The reduced system is described by a kth-order system.

Remark 1. When approaching the boundary $\partial\hat{\Sigma}$, the derivative $T'(h)$ may tend to zero, infinity, or cease to exist.

Remark 2. System (2) may simultaneously have several global families of periodic solutions with the same k. Also, there are no obstacles to the simultaneous existence of global families with different k.

3. ADAPTIVE CONTROL SCHEME FOR THE REDUCED SYSTEM

According to Theorem 1, the global family of periodic solutions $\Sigma = \{\varphi_s(h,t)\}$ can be described by a reduced system in R^{k} :

$$\dot{x}_s = X_s(x_1, \ldots, x_k), \quad s = 1, \ldots, k. \tag{6}$$

The problem is to stabilize any $T(h)$-periodic oscillation φ from the family Σ chosen by the value of the parameter h. We apply a control law containing the parameter K, assigning the value of K depending on $h : K = K(h)$. The control law is defined by a smooth function F of the variables $x_1, \ldots, x_k$ that acts with a small gain ε of the controller signal. In the controlled system

$$\dot{x}_s = X_s(x_1, \ldots, x_k) + \varepsilon F_s, \quad s = 1, \ldots, k, \tag{7}$$

and the stabilized oscillation will be ε-close to the oscillation of system (7). The parameter K in the control law ensures the existence of a periodic solution in (7) identically in h. Then the oscillation with the parameter $h = h^*$ is stabilized by substituting into the control function F the number $K = K(h^*)$ for which $dK(h^*)/dh \neq 0$. In the special case $k = 2$, the adaptive control scheme was implemented in [3].

The stabilization problem is posed in small. It is solved by constructing an attracting cycle. Therefore, we will find existence conditions for such a cycle and calculate its characteristic exponent (CE). These problems are solved in the neighborhood of the reference (basic) oscillation $x = \varphi(h^*, t)$ by a linearized system. The control value F_s is computed on the reference oscillation. The deviations from the reference oscillation and the number ε are considered of the same order.

Letting $\Delta_s = x_s - \varphi_s$, $s = 1, \ldots, k$, in system (7), we write equations for the variables Δ_s. In the linear approximation with respect to Δ_s, the resulting equations contain the variations of δx_s. Then it is necessary to analyze the periodic solution of the controlled system

$$\delta\dot{x}_s = p_{s1}\delta x_1 + \ldots + p_{sk}\delta x_k + \varepsilon F_s, \quad s = 1, \ldots, k. \tag{8}$$

System (8) coincides with the system derived from (7) for the increments Δ_s within the nonlinear terms. The cycle of system (8) is obtained from the linearized system (7) in the neighborhood of the reference oscillation.

On the manifold $\hat{\Sigma}$ the variables

$$\delta x_j = \frac{\partial x_j}{\partial h}, \quad j = 1, \ldots, k,$$

are functions of h and t. We denote by $\{y_1, \ldots, y_k\}$ the solution of the adjoint system. Then the expression

$$y_1\delta x_1 + \ldots + y_k\delta x_k = \text{const} \tag{9}$$

is the first integral of the linear homogeneous system in (8). The set of these integrals are used to reduce the homogeneous linear system with periodic coefficients in (8) to a system with constant coefficients. In the case of the zero kth-order Jordan block, we obtain the equations

$$\dot{u}_1 = 0, \quad \dot{u}_2 = -u_1, \ldots, \dot{u}_k = -u_{k-1}. \tag{10}$$

All solutions of the adjoint system have the form

$$\begin{aligned} &y_{s1} = \psi_{s1}, \\ &y_{s2} = t\psi_{s1} + \psi_{s2}, \\ &\ldots \\ &y_{sk} = \frac{t^{k-1}\psi_{s1}}{k-1} + \ldots + t\psi_{s,k-1} + \psi_{sk}, \end{aligned} \tag{11}$$

where ψ_{sj} are functions with the period $T(h)$. Then the Lyapunov transform is given by

$$u_j = \psi_{1j}\delta x_1 + \ldots + \psi_{kj}\delta x_k, \qquad j = 1, \ldots, k, \tag{12}$$

with a nonsingular periodic matrix $||\psi_{sj}(h,t)||$. Hence, the corresponding derivatives are

$$\dot{u}_j = \psi_{1j}\frac{d\delta x_1}{dt} + \ldots + \psi_{kj}\frac{d\delta x_k}{dt} + \dot{\psi}_{1j}\delta x_1 + \ldots + \dot{\psi}_{kj}\delta x_k, \quad j = 1, \ldots, k.$$

As a result, the controlled system (8) in the variables u_j differs from (10) by the inhomogeneous terms:

$$\begin{aligned} &\dot{u}_1 = \varepsilon(\psi_{11}F_1 + \ldots + \psi_{k1}F_k) = \varepsilon\hat{F}, \\ &\dot{u}_j = -u_{j-1} + \varepsilon(\psi_{1j}F_1 + \ldots + \psi_{kj}F_k), \quad j = 2, \ldots, k. \end{aligned} \tag{13}$$

For $T = T^*$, system (13) defines a mapping $t : 0 \to T^*$ on the manifold $\hat{\Sigma}$. For $\varepsilon = 0$, the mapping admits a zero fixed point. Given $\varepsilon \neq 0$, a fixed point exists under the necessary conditions in the form of the amplitude equation

$$I(h) \equiv \int_0^{T^*} \hat{F}dt = \int_0^{T^*} \sum_{s=1}^{k} \psi_{s1}F_s dt = 0; \tag{14}$$

for details, see [6, Ch. VI, § 8, p. 413, §9, p. 417]). Equation (14) is with respect to the unknown h, with the function $\hat{F}$ containing the parameter $K = K(h^*)$ along with h. A particular form of (14) is provided in the example below for (21). The simple root $h = h^*$ of the amplitude equation (14) corresponds to an isolated periodic solution of system (13), hence, that of system (8). System (8) describes the cycle of system (7) within the nonlinear terms (in the neighborhood of the oscillation under study). Therefore, the cycle of system (7) is derived from the amplitude equation (14). The inequality $dI(h^*)/dh \neq 0$ is a sufficient condition for the existence of this cycle.

The CE of the cycle are found from the equations in variations. In the case of the zero kth-order Jordan block, a single number α is calculated to determine the CE [6, Ch. 3, § 11]. It shows the change in the variation over the period. The variations are found as the derivatives of the functions u_s with respect to the parameter h.

The first equation of system (13) has the form

$$\dot{u}_1(h^*, t) = \varepsilon \left[\hat{F}_* + \sum_{s=1}^{k} \left(\frac{\partial \hat{F}}{\partial x_s} \frac{\partial x_s}{\partial h} \right)_* \Delta h + \ldots \right],$$

where the asterisk denotes the values calculated for $h = h^*$. Then the increment Δh satisfies

$$\frac{d(\Delta u_1)}{dt} = \varepsilon \left[\sum_{s=1}^{k} \left(\frac{\partial \hat{F}}{\partial x_s} \frac{\partial x_s}{\partial h} \right)_* \Delta h + \ldots \right],$$

and we obtain the following equation for the derivative:

$$\frac{d}{dt} \left(\frac{\partial u_1}{\partial h} \right)_* = \varepsilon \sum_{s=1}^{k} \left(\frac{\partial \hat{F}}{\partial x_s} \frac{\partial x_s}{\partial h} \right)_* = \varepsilon \frac{\partial \hat{F}(h^*, t)}{\partial h}.$$

The change in the derivative over the period leads to the CE of the cycle

$$\left(\frac{\partial u_1}{\partial h} \right)_* = \frac{\varepsilon}{T^*} \int_0^{T^*} \frac{\partial \left(\sum\limits_{s=1}^{k} \psi_s(h^*, t) F_s(h^*, t) \right)}{\partial h} dt.$$

Thus, the CE α of the cycle is given by

$$\alpha = \frac{\varepsilon}{T^*} \int_0^{T^*} \frac{\partial \hat{F}(h^*, t)}{\partial h} dt. \tag{15}$$

Theorem 2. *For the reduced controlled system* (7)*, the cycle stabilization problem is solved for any chosen parameter value $h = h^*$ by a smooth control function F acting with a small gain of the controller signal. A sufficient condition for cycle stabilization is the inequality $dI(h^*)/dh < 0$ imposed on the root of the amplitude equation* (14)*. The characteristic exponent of the cycle is calculated using formula* (15)*.*

Theorem 2 is applied to all oscillations of the global family. The control law contains the parameter h, is found from the amplitude equation holding identically in h, and is designed by generalizing the universal control from [7]. In the case $k = 2$, several particular control laws satisfying Theorem 2 were given in [3]. For the nth-order equation, the adaptive control law is presented in Section 4.

Remark 3. According to Theorem 2, the cycle stabilization problem is solved by the control function $\hat{F}$. Therefore, we can choose in system (8), e.g., the control function $\hat{F}$ with $F_s \equiv 0; s = 2, \ldots, k$. For the second-order equation ($k = 2$), the control function is applied with $F_1 \equiv 0,\ F_2 \neq 0$.

4. nTH-ORDER DIFFERENTIAL EQUATION

An nth-order Jordan block with the zero eigenvalues of the matrix A_f is the only case for a single nth-order differential equation

$$x^{(n)} = X(x, x', \ldots, x^{(n-1)}), \tag{16}$$

where $x^{(j)}$ denotes the jth derivative of x.

Indeed, equation (16) turns into the system

$$\begin{aligned} &\dot{x}_s = x_{s+1}, \quad s = 1, \ldots, n-1, \\ &\dot{x}_n = X(x_1, \ldots, x_n), \end{aligned} \tag{17}$$

where the functions in (2) are $X_s = x_{s+1}$, $s = 1, \ldots, n-1$. Therefore, due to equation (3), the matrix A_f represents an nth-order Jordan block with zero eigenvalues.

Thus, if system (17) has a nondegenerate periodic solution, it belongs to the global family of $T(h)$-periodic solutions on which the period monotonically depends on the parameter h (Theorem 1). The solutions are defined in the space $(x, x', \ldots, x^{(n-1)})$, and the periodic solution of equation (16) is denoted by $x = \varphi(h, t)$.

Let us formulate the following problem: it is required to stabilize the solution of equation (16) selected by a value $h = h^*$. For this purpose, we consider system (7) with a vector control function $F = (F_1, \ldots, F_n)$ satisfying the amplitude equation (14) with a simple root. Theorem 2 is applied with a scalar control function $\hat{F}$ formed from the coordinates of the vector F. In the particular case of system (7) (i.e., equations (17)), we take $F_1 = \ldots = F_{n-1} = 0$ and the explicit-form function F_n. As a result, the controlled system is described by

$$x^{(n)} = X(x, x', \ldots, x^{(n-1)}) + \sigma\varepsilon[1 - Kx^2]x', \tag{18}$$

where the control value F_n acts with a small gain and the number σ is 1 or (-1). The coefficient K is assigned depending on the value of the parameter $h : K = K(h)$. To stabilize the oscillation with $h = h^*$, we let $K = K(h^*)$.

Thus, the adaptive control scheme is designed.

The control law applied in (18) is an analog of the universal control proposed in [7]. It satisfies the amplitude equation (14), which has a simple root for almost all points of the family in the parameter h.

The function $K(h)$ is obtained from the amplitude equation (14) holding identically in h. This function is calculated using the formula

$$K(h) = \frac{\int\limits_0^{T(h)} \psi_{nn}(h, t)\varphi'(h, t)dt}{\int\limits_0^{T(h)} \varphi^2(h, t)\psi_{nn}(h, t)\varphi'(h, t)dt};$$

the function $\psi_{nn}(h, t)$ is taken from (12). Then we have

$$\frac{dI(h^*)}{dh} = \chi\nu(h^*), \quad \chi = \frac{dK(h^*)}{dh}, \quad \nu(h^*) = \int\limits_0^{T(h^*)} \psi_{nn}(h^*, t)\varphi'(h^*, t)dt \tag{19}$$

for the function $I(h)$ (14) at the point $h = h^*$.

Theorem 3. *If the nth-order differential equation admits a nondegenerate periodic solution, it belongs to the global family in the scalar parameter h. The solution with the parameter value* $h = h^*$, $\chi\nu \neq 0$, *is stabilized by the adaptive control scheme* (18) *with* $K = K(h^*)$: *the sign of the number* σ *ensures the attraction to the cycle.*

Remark 4. By writing equation (16) as (17) and applying the adaptive control system of Section 4 to (17), we establish the attraction to the cycle in the space $(x, x', \ldots, x^{(n-1)})$ in equation (18).

5. SYMMETRIC PERIODIC MOTIONS

An autonomous system of the general form (2) may have additional properties such as conservativeness or reversibility, be written in the Hamiltonian form, etc. In this case, Theorem 1 remains valid as well. However, for some systems, this theorem needs to be clarified.

In what follows, the concept of a global family of periodic solutions of ODEs will be concretized for symmetric periodic motions.

A reversible dynamic system with the phase vector z and a nondegenerate mapping G has spatiotemporal symmetry in the sense of invariance with respect to the transformation $(z, t) \to (Gz, -t)$. It describes models in various fields of knowledge; see the survey in [8]. In the case

$$G = \left\| \begin{matrix} I_l & 0 \\ 0 & -I_n \end{matrix} \right\|, \quad l \geqslant n$$

(I_j is an identity matrix of dimensions $(j \times j)$), we obtain a reversible mechanical system [9]. The phase space of this system is described by vectors u and v such that $\dim u = l$, $\dim v = n$, and the symmetry transformation is $(u, v, t) \to (u, -v, -t)$. In mechanics, u and v are usually taken to be the vectors of generalized coordinates (quasi-coordinates) and generalized velocities (quasi-velocities), respectively. The set $M = \{u, v : v = 0\}$ is called the fixed set of a reversible mechanical system.

The phase portrait of a reversible mechanical system is symmetric with respect to the set M. The trajectories intersecting M are called symmetric. The twofold intersection of the set by a trajectory leads to a symmetric periodic motion (SPM). On an SPM of period $T/2$, the trajectory intersects the set M at the time instants $t = 0, T/2$. An SPM of period $T/2$ exists under the necessary and sufficient conditions

$$v_s(u_1^0, \ldots, u_l^0, \tau) = 0, \quad \tau = 0, T/2; \quad s = 1, \ldots, n, \tag{20}$$

where $u^0 \in M$ is the value on the SPM. Let us introduce the matrix

$$A(u^0, T/2) = ||a_{sj}|| = \left\| \frac{\partial v_s(u_1^0, \ldots, u_l^0, T/2)}{\partial u_j^0} \right\|.$$

Definition 4. The case $\det A(u^0, T/2) \neq 0$ is said to be nondegenerate for an SPM, and the latter is called a nondegenerate SPM.

An SPM is a periodic solution. By Definition 4, the nondegenerate SPMs form a family on which the period varies monotonically: Definition 1 is valid for it. Inequality (20) also holds on the period. Conditions (3) written for an SPM are satisfied identically in the $(l-n)$ values u_j^0. Hence, the matrix A_f in (5) contains $(l-n)$ simple zero eigenvalues. Therefore, the nondegeneracy condition $\operatorname{rank} A_f = \mathrm{Ra} = l + n - 1$ introduced for the general-form system (2) is true for a reversible mechanical system only if $l = n$. Accordingly, Theorem 1 applies to an SPM only in the case $l = n$.

On the other hand, without the nondegeneracy condition $\mathrm{Ra} = l + n - 1$ for a reversible mechanical system, Definition 3 remains valid for an SPM in the general situation. Therefore, Theorem 1 on the global family is true in the following formulation.

Theorem 4. *A nondegenerate SPM of a reversible mechanical system always extends to a global family of nondegenerate SPMs described by a reduced reversible mechanical system with the vector* $u \in \mathrm{R}^{l-n}$ *and scalar* v *as the variables.*

The most complete proof of Theorem 4 was given in [10]. An adaptive oscillation stabilization scheme for a reduced reversible mechanical system was designed in [11].

Theorem 4 settles an important case of SPMs, which is degenerate for Theorem 1.

6. CONSERVATIVE SYSTEM

For definiteness, assume that a conservative system is given by the Lagrange equations of the second kind and subjected to the action of potential forces. Then it is described by a system of second-order equations. Let q and $\dot{q}$ denote the coordinate and velocity vectors, respectively. Then the dynamic equations are invariant with respect to the change of variables $(q, \dot{q}, t) \to (q, \dot{q}, -t)$. Therefore, a conservative system belongs to the class of reversible mechanical systems with the coinciding dimensions of the vectors q and $\dot{q}$. The equilibria of the system belong to its fixed set. The nondegenerate equilibria are divided into centers and saddles. According to Lyapunov's center theorem [12], a local family of nonlinear periodic motions (a Lyapunov family) adjoins to a center. For such a family, the zero Jordan block has dimension $k = 2$. Due to the energy integral present in the system, the period on the family will be a monotonic function of the constant energy h. Therefore, the Lyapunov family consists of nondegenerate periodic motions. By Theorem 1, it extends to a global family of nondegenerate periodic solutions, with the reduced system containing two first-order equations. The same conclusion follows from Theorem 4, but the latter specifies that the reduced system is a reversible mechanical system. Due to the conservativeness of the original system, we obtain a reduced conservative (RCC) system with one degree of freedom; see [13, Lemma A.1].

Thus, the Lyapunov families of a conservative system always extend to global families of symmetric periodic motions: the global Lyapunov center theorem is valid, first derived in [3].

A conservative system can be described by equations not belonging to the class of reversible mechanical systems. In this case, applying Theorem 1 to the system also yields an RCC system: the energy integral is preserved.

For an RCC system, a controlled system stabilizing almost all oscillations was constructed in [7, Theorem 1]. As it turns out, all oscillations of the family are stabilized in the reduced conservative system.

Consider a controlled RCC system of the form

$$\ddot{x} + f(x) = \varepsilon\sigma(1 - Kx^2)\dot{x} \tag{21}$$

that contains a parameter K and admits, for $\varepsilon = 0$, the energy integral

$$\frac{\dot{x}^2}{2} + \int f(x)dx = h(\text{const}).$$

(Here, σ takes value 1 or (-1).) By assumption, for $\varepsilon = 0$ system (21) admits a family of nondegenerate periodic motions $x = \varphi(h, t)$.

An attracting cycle close to an oscillation with a system parameter $h = h^*$ exists under the necessary and sufficient conditions

$$I(h) \equiv \int\limits_0^{T(h^*)} [1 - K(h^*)\varphi^2(h,t)]\dot{\varphi}(h,t)dt = 0$$

(the simple root of the amplitude equation). The function $K(h)$ is calculated using the formula

$$K(h) = \frac{\int\limits_0^{T(h)} \dot{\varphi}^2(h,t)dt}{\int\limits_0^{T(h)} \varphi^2(h,t)\dot{\varphi}^2(h,t)dt}, \tag{22}$$

and the inequality $dK(h^*)/dh \neq 0$ ensures the simple root of the amplitude equation.

Theorem 5. *System* (21) *with the parameter* K *stabilizes any oscillation close to that of a conservative system with one degree of freedom.*

Proof. With the new time variable $\tau = t/T(h)$, the oscillation period becomes independent of h and equal to 1, whereas formula (22) is written as

$$K(h) = \frac{\int\limits_0^1 z^2(\tau)d\tau}{\int\limits_0^1 \varphi^2(h, T(h)\tau)z^2(\tau)d\tau}.$$

On the family of oscillations, the function $\varphi^2(h,t)$ with fixed t monotonically depends on h. This is true for any family, nondegenerate or degenerate. Hence, the function $K(h)$ is monotonic on the family of oscillations of a conservative system.

Thus, the sufficient condition in [7, Theorem 1] always holds for any family (nondegenerate and degenerate), and the proof of Theorem 5 is complete.

Remark 5. For a degenerate family, the function $K(h)$ is calculated in explicit form; for details, see [13].

Remark 6. According to Theorem 5, the adaptive scheme provides a complete solution of the stabilization problem for any oscillation from the family of an RCC system.

Remark 7. The result remains valid for a conservative system with an arbitrary number of degrees of freedom.

7. APPLICATIONS

1. The bounded planar three-body problem is described by the equations [14]

$$\begin{gathered}\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y},\\ \Omega = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_0} + \frac{\mu}{r_1}, \quad \Omega(x,y) = \Omega(x,-y),\\ r_0^2 = (x+\mu)^2 + y^2, \quad r_1^2 = (x-1+\mu)^2 + y^2.\end{gathered} \tag{23}$$

They contain a single dimensionless mass parameter μ.

System (23) admits an energy integral. Due to its invariance with respect to the transformation

$$\{x, y, \dot{x}, \dot{y}, t\} \to \{x, -y, -\dot{x}, \dot{y}, -t\},$$

system (23) also belongs to the class of reversible mechanical systems.

The relative equilibria (libration points) are found from the equations

$$\frac{\partial\Omega}{\partial x} = 0, \quad \frac{\partial\Omega}{\partial y} = 0.$$

The problem admits five libration points L_i, $i = 1, \ldots, 5$. In addition, L_1, L_2, and L_3 lie on the abscissa axis x, whereas the points L_4 and L_5 are located symmetrically with respect to the axis x and form equilateral triangles with the main bodies on the axis x.

The points L_1, L_2, and L_3 belong to the equilibria of a reversible mechanical system. They are adjoined by a symmetric Lyapunov family for which the matrix A_f (Section 2) has a zero second-order Jordan block. Therefore, Theorem 4 can be applied here. The global family is described by an conservative system with one degree of freedom and symmetric orbits. By Theorem 5, the orbital stabilization problem of any orbit is solved by the adaptive control scheme.

Theorem 1 can be applied to the points L_4 and L_5. System (23) is conservative, so the global family is described by an RCC system. Any orbit of the global family is stabilized using Theorem 5.

The bounded three-body problem is basic in the theory of orbits [15]. In the theory of controlled orbital motion, the problem of orbital stabilization is solved. The corresponding results will be considered in detail in a separate paper.

2. Stabilization of satellite oscillations. Under gravitational forces, the motion of a satellite in an orbital plane is described by V.V. Beletskii's equation [5]. Consider a particular case of a circular orbit. The corresponding Beletskii equation takes the form

$$\ddot{\alpha} + \mu \sin\alpha \cos\alpha = 0, \quad \dot{\alpha} = \frac{d\alpha}{dv}, \tag{24}$$

where μ is the inertial parameter ($||\mu| \leqslant 3$); α is the angle between the radius vector of the center of mass and the main central axis of inertia of the satellite in the orbital plane; v is the true anomaly chosen as the independent variable. As a result, we obtain the mathematical pendulum equation

$$\ddot{y} + \mu \sin y = 0, \quad \mu > 0, \quad y = 2\alpha,$$

or

$$\ddot{y} + |\mu| \sin y = 0, \quad \mu < 0, \quad y = 2\alpha + \pi.$$

The satellite oscillations form a family from an initial deviation in the angle y, and the period $T(h)$ increases on this family.

A mechatronic oscillation stabilization scheme with a van der Pol oscillator was proposed in [13]. The mechatronic scheme is adaptive in the sense of this paper.

According to Theorem 5, any oscillation is locally stabilized using the adaptive scheme

$$\ddot{x} + |\mu| \sin x = \varepsilon\sigma(1 - Kx^2)\dot{x},$$

where $x = 2\alpha$ and $\mu > 0$ or $x = 2\alpha + \pi$ and $\mu < 0$, and $\sigma = 1$. Here, the van der Pol oscillator is not applied.

Note that for the problem of rotational motion of a satellite, an equilibrium stabilization scheme was proposed, e.g., in [16].

8. CONCLUSIONS

In the nondegenerate case, the periodic solution of an autonomous system of general form can be a cycle or belong to a family. For a cycle, the Jacobi matrix has one zero eigenvalue, whereas a family corresponds to a zero kth-order Jordan block. For $k = 2$, the case usually considered, a global family of nondegenerate periodic solutions was constructed in [3]; this family is described by a reduced second-order system. The results are valid for the general case of dimension $1 < k \leqslant n$, where n denotes the dimension of the autonomous system. An adaptive oscillation stabilization scheme is designed for the reduced system of order k. An autonomous control with the parameter $K(h)$ depending on the global family parameter h is applied. It represents a generalization of the universal control [7] to a system of arbitrary order; the value of h separates an oscillation in the global family. Stabilization is achieved by implementing an attracting cycle.

For an nth-order differential equation in the variable x, the reduced system coincides with the original one. The periodic solution is stabilized by a control law that represents nonlinear dissipation, being linear in velocity, acting in the neighborhood of the cycle, containing a parameter, and ensuring attraction to the cycle in the space $(x, x', \ldots, x^{(n-1)})$.

Similar results are known for symmetric periodic motions of reversible mechanical systems [10]. In the spatiotemporal symmetry case, the Jacobi matrix admits zero eigenvalues: simple ones and those of a single 2nd-order Jordan block. This block corresponds to a conservative system. The stabilization problem of family oscillations finds an exhaustive solution for the conservative system with one degree of freedom: the stabilization conditions of the selected oscillation are automatically satisfied in the controlled system.

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On Reliability of Repairable Hot Double Redundant System with Arbitrarily Distributed Life- and Repair Times of Its Elements

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Abstract—The article introduces the concept of Marked Markov processes, which are used to study a repairable hot double redundant system with a single repair facility. It is assumed that components' life- and repair times follow arbitrary distributions. The proposed approach allows for calculating the system's reliability function, and its mean lifetime, as well as conducting the sensitivity analysis to the shape of input distributions. The new method was validated with numerical examples by comparing it with previously obtained analytical results and showed high accuracy.

Keywords: Marked Markov Processes, hot double redundant system, arbitrarily distributed life- and repair times, reliability, sensitivity analysis

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1. INTRODUCTION

Ensuring the reliability of systems, objects, and processes is one of the primary goals during their creation and subsequent operation. One of the methods for enhancing the structural reliability of a system is redundancy. It involves duplicating its critically important components or implementing higher-order redundancy. A system consisting of a primary element and one duplicate element is the simplest type of redundant system, and its study is intriguing from both theoretical and practical perspectives.

Redundant systems with various types of redundancy, as well as with the capability to restore failed elements, have been well-studied by many authors. Key findings regarding Markovian redundant systems, both repairable and non-repairable, can be found in classical works on mathematical reliability theory (see, for example, [1, 2], and others).

The extensive practical applications of redundant systems have led to the development of new mathematical models for more complex systems. For instance, in [3, 4], the reliability of a redundant system with heterogeneous elements, which have different lifetime distributions, is examined. The papers [5, 6] focus on systems with preventive and priority servicing of failed elements. The studies [7, 8] are dedicated to redundant systems with various types of system repair, where the failures of elements adhere to the Marshall-Olkin model. In the context of implementing redundancy in systems, it is also essential to consider optimization issues related to redundancy based on different criteria. Numerous studies have addressed this problem as well (see, for example, [9, 10]). Another challenge is determining the fundamental parameters of the system, specifically the shape of life- and repair times distributions of system elements.

Often in practice, the shape of life- and repair times distributions of system elements is unknown. In the earliest research on the reliability of redundant systems, assumptions were made regarding the exponential distribution of the corresponding random variables[1]. However, the emergence of new mathematical methods has enabled the examination of systems where one time follows the exponential distribution while the other one follows an arbitrary distribution. There exists a number of studies dedicated to the application of methods such as the introduction of supplementary variables, semi-Markov processes, and others to address these issues (see, for example, [11–14]). An extension of these studies is the sensitivity analysis of reliability metrics to the shape of initial distributions.

In [15], the study of a k-out-of-n system with exponentially distributed lifetime and arbitrary repair time distribution utilized the theory of decomposable semi-regenerative processes, which was previously proposed by V.V. Rykov as a generalization and development of Smith's regeneration idea [16]. Later, this approach was applied to investigate the reliability of cold redundant systems [17, 18], featuring arbitrary initial distributions for elements' life- and repair times. However, it turned out that this approach is not suitable for analyzing similar systems with hot redundancy and other more complex models.

In this article, a novel approach is proposed for the study of a repairable hot double redundant system, based on the new concept of Marked Markov processes (MMP).

The concept of MMP is introduced as an extension of the notion and theory of point processes [19]. In queuing theory, a point process is defined as an increasing sequence of non-negative random variables that describe the moments of occurrence of certain events along the positive half-axis of time [20]. Specific instances of point processes include the Poisson process and the Cox point process, where the times of event occurrences are governed by the exponential distribution. Another generalization of the Poisson process is the renewal process, in which the intervals between the occurrences of events follow a distribution that is different from the exponential one [21].

To describe a point process that possesses the Markov property, the concept of Markov point processes was introduced in 1977 [22]. Later, it was generalized to the concept of a Marked Markov point process, which is defined as a pair of random variables, where the first component is the time of occurrence of a random event, and the second is a mark, also a random variable, that additionally describes the corresponding event.

Since the introduction of the concept of point processes, the corresponding theory has been hotly developed, as it has found applications in various fields of science and engineering. The paper [23] is devoted to constructing a mathematical model for controlling automobile traffic at intersections using marked point processes. In [24], a multivariate point process is investigated for the problem of monitoring the state of a telecommunication connection. The applicability of spatial Poisson processes for modeling interactions between wireless devices is discussed in [25].

Introduced in this investigation, MMP differs from known concepts in that it consists of two components: the first represents the number of failed elements of the system, while the second one is a set of random marks that define their residual life- and repair times. Using the proposed approach, it is possible to compute the reliability function and the mean lifetime of a hot double redundant system with arbitrary life- and repair time distributions of its elements.

The article is organized as follows. The problem statement, key notations, and assumptions are provided in the next Section 2. Section 3 presents the transformation of marks over a separate regeneration period. In Section 4, the main characteristics of the model are obtained in terms of marks, and Section 5 proposes an algorithm for their computation. The validation of the proposed algorithm and examples of numerical analysis are presented in Section 6. Finally, the main results of the work are formulated, and directions for future research are outlined.

[1] Following the classical works by A.Y. Khinchin, B.V. Gnedenko, and others, from now on, whenever distributions of times are mentioned, it is understood that these refer to distributions of the corresponding random variables.

2. PROBLEM STATEMENT. MMP

2.1. Problem Statement, Notations, and Assumptions

Let's consider a repairable hot double redundant system with a single recovery device for failing elements, denoted as $< GI_2|GI|1 >$. Here, the angle brackets $<>$ indicate a closed system, and the symbols GI refer to a recurrent failure flow and a recurrent servicing mechanism with arbitrary distributions of life- and repair times at the first and second positions, respectively. The subscript index on the first symbol indicates the total number of elements in the system. The last position denotes the number of repair units. Introduce the following notations and assumptions:

- $A_i:$ $(i = 1, 2, \dots)$: lifetimes of system elements, which are assumed to be independent identically distributed (i.i.d.) random variables (r.v.) with common cumulative distribution function (c.d.f.) $A(t) = \mathbb{P}\{A_i \leqslant t\}$, probability density function (p.d.f.) $a(t) = A'(t)$, and finite mean time $\mu_a = \mathbb{E}[A_i] < \infty$ and variance $\sigma_a^2 = \mathbb{D}[A_i] < \infty$.
- $B_i:$ $(i = 1, 2, \dots)$: repair times of system elements, which are assumed to be i.i.d. r.v. with common c.d.f. $B(t) = \mathbb{P}\{B_i \leqslant t\}$, p.d.f. $b(t) = B'(t)$, and finite mean $\mu_b = \mathbb{E}[B_i] < \infty$ and variance $\sigma_b^2 = \mathbb{D}[B_i] < \infty$.

2.2. Definition of MMP

To study such a model, we introduce the concept of MMP with a discrete parameter,

$$Z(n) = \{(J(n), \mathbf{X}(n)),\ n = 0, 1, \dots\},$$

the first component of which is a conditional Markov chain $J(n) := J$ with at most countable state space $\mathcal{J}$, and the second is a set of random marks $\mathbf{X}(n) := \mathbf{X}_i(n)$, where $i \in \mathcal{J}$, with values in a measurable space $(E_i, \mathcal{E}_i)$. Such a process is determined by:

- transition probabilities $p_{ij}(\mathbf{X}_i)$ of the process J, which depend on the content of the mark $\mathbf{X}_i$ in state i;
- marks transformation operators $\Phi_{ij}(\mathbf{X}_i)$ for the transition from state i to state j of Markov chain and their distributions.

Remark 1. This process can be considered as a special type of Markov sequence with a common state space and specified using a transition kernel (process generator). However, for the study of various applications, it is more convenient to start from the concept of MMP.

Remark 2. Based on such a model, it is possible to study a wide class of processes, such as semi-Markov, semi-regenerative processes, Markov renewal processes, etc.

2.3. MMP Construction

Consider MMP construction on an example of $< GI_2|GI|1 >$ system. In this case, $J(n)$ represents the number of failed elements. As the marks $\mathbf{X}_i(n)$, we choose multidimensional r.v.'s whose contents are

— residual lifetime $X_0^{(1)}(n)$ of one element and residual repair time $X_0^{(2)}(n)$ of another element in state $i = 0$,
— residual lifetime $X_1^{(1)}(n)$ of one element and newly assigned repair time $X_1^{(2)}(n)$ of other one in state $i = 1$,
— residual repair time $X_2(n)$ of the element being under repair in state $i = 2$.

In this representation, the upper index indicates the serial number of the mark, the lower index indicates the system state as it was introduced above, and the variable in brackets stands for the step number.

Suppose that at the initial time (at step $n = 0$) both elements of the system are operational, $J(0) = 0$. Therefore, the initial content of the marks corresponds to the time to failure for each of

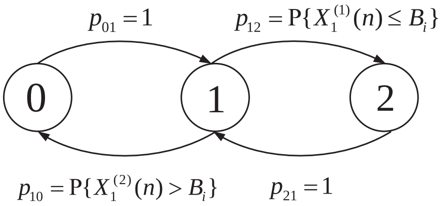

Fig. 1. Transition graph of the chain $J(n)$.

the system elements, $X_0^{(1)}(0)=A_0^{(1)}$, $X_0^{(2)}(0)=A_0^{(2)}$, where both $A_0^{(1)}$ and $A_0^{(2)}$ are i.i.d. r.v.'s with cdf $A(\cdot)$. Subsequently, the process finds itself in state $i=0$ with a certain set of marks, the values of which will be calculated further.

The possible transitions of the considered chain and the marks transformations, describing the behavior of the system under consideration, are shown in Fig. 1.

From state 0, the process moves to state 1 regardless of the value of the marks with probability $p_{01}(\mathbf{X}(n))=1$. In this case, at the first step, the marks in state 1 are transformed as follows,

$$\mathbf{X}_1(1)=\Phi_{01}[\mathbf{X}_0(0)]: \quad X_1^{(1)}(1)=A_0^{(1)}\vee A_0^{(2)}-A_0^{(1)}\wedge A_0^{(2)}, \quad X_1^{(2)}(1)=B_1. \tag{1}$$

From state 1, transitions can occur to state 0 if the failed element is restored, or to state 2 if the second element fails. The probability of transitioning to state 0 in the first step is,

$$p_{10}(\mathbf{X}_1(1))=\mathbb{P}\{X_1^{(1)}(1)>X_1^{(2)}(1)\}.$$

At that, the marks in state 0 are transformed as follows,

$$\mathbf{X}_0(1)=\Phi_{10}[\mathbf{X}_1(0)]: \quad X_0^{(1)}(1)=(X_1^{(1)}(1)-B_1)1_{\{X_1^{(1)}(1)>B_1\}}, \quad X_0^{(2)}(1)=A_1. \tag{2}$$

Similarly, when transitioning to state 2, the corresponding transition probability takes the form,

$$p_{12}(\mathbf{X}_1(1))=\mathbb{P}\{X_1^{(1)}(1)\leqslant X_1^{(2)}(1)\},$$

then, the mark in state 2 is transformed as follows,

$$\mathbf{X}_2(1)=\Phi_{12}[\mathbf{X}_1(1)]: \ X_2(1)=(B_1-X_1^{(1)}(1))1_{\{X_1^{(1)}(1)\leqslant B_1\}}. \tag{3}$$

Finally, from state 2, there is only one possible transition to state 1, so $p_{21}(\mathbf{X}_2(1))=1$. The marks transformation in the new state (at the second step) has the form,

$$\mathbf{X}_1(2)=\Phi_{21}[\mathbf{X}_2(1)]: \ X_1^{(2)}(2)=A_1, \quad X_1^{(1)}(2)=B_2. \tag{4}$$

Thus, it is clear that the transitions from state 2 to state 1 signify the regeneration of the system. Consequently, it suffices to focus on the system's behavior during a separate regeneration period.

3. BEHAVIOR OF THE PROCESS IN A SEPARATE REGENERATION PERIOD

Taking into account the regenerative nature of the system's behavior, consider the behavior of the process $Z(n)$ during a separate regeneration period. Similarly to the previous section, we will outline the transformation of marks during the regeneration period of the system. Denote $W(l)$ the residual lifetime of the system after failure of one of its elements at lth step,

$$W(l)=X_0^{(1)}(l)\vee X_0^{(2)}(l)-X_0^{(1)}(l)\wedge X_0^{(2)}(l) \ \ (l=0,1,2,\dots).$$

Then, taking into account transformations of the marks (1)–(4) the following lemma is valid.

Lemma 1. *Between transitions from state 2 to state 1 (i.e. during the regeneration period), the marks are transformed as follows:*

$$
\begin{aligned}
&X_0^{(1)}(l) = (X_1^{(1)}(l-1) - X_1^{(2)}(l-1))1_{\{X_1^{(1)}(l-1) > X_1^{(2)}(l-1)\}}, \quad X_0^{(2)} = A_l \in A(\cdot),\\
&X_1^{(1)}(l) = W(l-1), \quad X_1^{(2)} = B_l \in B(\cdot),\\
&X_2(l) = (X_1^{(1)}(l) - X_1^{(2)}(l))1_{\{X_1^{(1)}(l) \leqslant X_1^{(2)}(l)\}},
\end{aligned} \tag{5}
$$

where the initial contents of the marks are

$$X_0^{(1)}(0) = X_0^{(2)}(0) = 0, \quad X_1^{(1)}(0) = A_0, \quad X_1^{(2)}(0) = B_0, \quad X_2(0) = 0.$$

Proof. Indeed, with the initial marks defined as $X_1^{(1)}(0) = A_0$ and $X_1^{(2)}(0) = B_0$, the process goes directly to state 2 from state 1 if the failure of the operational elements occurs before the restoration of the repaired one, specifically when $A_0 \leqslant B_0$. In this scenario, the process will find itself in state 2 with the following mark:

$$X_2(1) = (X_1^{(1)}(0) - X_1^{(2)}(0))1_{\{X_1^{(1)}(0) \leqslant X_1^{(2)}(0)\}}.$$

From state 1, the process will move to state 0 if the repair of the failing element occurs before the failure of the working one, i.e., when $A_0 > B_0$; thus, the marks are transformed as follows:

$$X_0^{(1)}(1) = (X_1^{(1)}(0) - X_1^{(2)}(0))1_{\{X_1^{(1)}(0) > X_1^{(2)}(0)\}}, \quad X_0^{(2)}(1) = A_1.$$

Then the process will return to state 1 with probability 1 and the following marks,

$$X_1^{(1)}(1) = X_0^{(1)}(1) \vee A_1 - X_0^{(1)}(1) \wedge A_1 \equiv W(1), \quad X_1^{(2)}(1) = B_1.$$

From state 1 the process can go to state 2 with probability

$$\mathbb{P}\{X_1^{(1)}(1) \leqslant X_1^{(2)}(1)\} \equiv \mathbb{P}\{W(1) \leqslant B_1\}$$

and residual repair time (the mark)

$$X_2(2) = (X_1^{(2)}(1) - X_1^{(1)}(1))1_{\{X_1^{(1)}(1) \leqslant X_1^{(2)}(1)\}},$$

after which it will return to state 1 with probability 1. This will end the regeneration period.

Otherwise, the regeneration period will continue and the process will go to state 0 with probability

$$\mathbb{P}\{X_1^{(1)}(1) > X_1^{(2)}(1)\} \equiv \mathbb{P}\{W(1) > B_1\}$$

and marks

$$X_0^{(1)}(2) = (X_1^{(1)}(1) - X_1^{(2)}(1))1_{\{X_1^{(1)}(1) > X_1^{(2)}(1)\}}, \quad X_0^{(2)}(2) = A_2.$$

Similarly, the transformation of the marks at the lth step occurs. From state 1, the process will return to state 0 l times if, during the $l-1$th visit to state 1 from state 0, the repair of the element completes before the residual operational time of the working element expires,

$$X_1^{(2)}(l-1) = B_{l-1} < W(l-1) = X_1^{(1)}(l-1).$$

In this case, the residual time to failure will decrease by $X_1^{(2)}(l-1) = B_{l-1}$, and the content of the mark in state 0 transforms as

$$X_0^{(1)}(l) = (X_1^{(1)}(l-1) - X_1^{(2)}(l-1))1_{\{X_1^{(1)}(l-1) > X_1^{(2)}(l-1)\}} \equiv (W(l-1) - B_{l-1})1_{\{W(l-1)>B_{l-1}\}}.$$

The repaired element initiates a new time to failure given by $X_0^{(2)}(l) = A_l$. Thus, considering the contents of the marks, this transition confirms the first of the formulas stated in (5).

From state 0, the process will return to state 1 with probability 1. Then, the residual time to failure of the system $W(l)$ is defined as the difference between the maximum and minimum of the r.v.'s $X_0^{(1)}(l)$ and A_l, and the repair of the second element begins, i.e., the second mark takes the value B_l,

$$X_1^{(1)}(l) = X_0^{(1)}(l) \vee A_l - X_0^{(1)}(l) \wedge A_l \equiv W(l), \quad X_1^{(2)}(l) = B_l,$$

that proves the second of the formulas (5).

Finally, from state 1, in case of a failure of the working element, i.e.,

$$X_1^{(2)}(l) = B_l \geqslant W(l) = X_1^{(1)}(l)$$

the process goes to state 2 with the mark

$$X_2(l) = (X_l^{(1)}(l) - X_1^{(2)}(l))1_{\{X_1^{(1)}(l) \leqslant X_l^{(2)}(l)\}} \equiv (B_l - W(l))1_{\{W(l) \leqslant B_l\}},$$

that proves the last of the formulas (5) and completes the proof of the lemma. □

In the next section, the obtained expressions are used to calculate the main characteristics of the system reliability.

4. CALCULATION OF THE MAIN RELIABILITY CHARACTERISTICS

The main reliability characteristics of the considered system during the regeneration period with the initial value of the marks

$$X_0^{(1)}(0) = X_0^{(2)}(0) = 0, \quad X_1^{(1)}(0) = A_0, \quad X_1^{(2)}(0) = B_0, \quad X_2(0) = 0,$$

are:

- the distribution $\pi_l = \mathbb{P}\{\nu = l\}$ of the number of steps ν until the system failure;
- transition durations $T_{ij}(l)$, $(i, j = 0, 1, 2,\ l = 1, 2, \dots)$ for moving from state i to state j at the lth step and their characteristics;
- the time R from the moment the process returns from state 2 to state 1 until the next moment of transition to state 2, the corresponding distribution $R(t)$, i.e. the reliability function of the system, as well as the corresponding mean system lifetime μ_R;
- the distribution of the regeneration period of the system $\Pi = R + T_{21}(\nu)$.

Calculate further the main characteristics of the model in terms of the marks. Denote by

$$T(l) = T_{10}(l-1) + T_{01}(l), \quad l = 1, 2, \dots,$$

the time until returning to state 1 at the lth step of the regeneration period. Denote by Ω_l and $\hat{\Omega}_l$ the sets of elementary events for which the following relations are satisfied,

$$\Omega_l = \{\omega : W(0) > B_0, W(1) > B_1, \dots, W(l) > B_l\} \quad (l = 1, 2, \dots),$$
$$\hat{\Omega}_l = \Omega_{l-1} \cap \{W(l) \leqslant B_l\}, \ \hat{\Omega}_0 = \{A_0 \leqslant B_0\},$$

as well as the corresponding probability

$$p_0 = \mathbb{P}\{A_0 > B_0\}, \;\; p_i = \mathbb{P}\{W(i) > B_i \,|\, W(i-1) > B_{i-1}\}. \tag{6}$$

Thus, the following holds.

Theorem 1. *The distribution π_l of the number of steps ν during the regeneration period, transition durations $T_{ij}(l)$ in state i before transition to state j at the lth step, the system operating time R before its failure, its regeneration period Π in terms of the marks are expressed as follows:*

$$\begin{aligned}
\pi_l &= \mathbb{P}\{\nu = l\} = \mathbb{P}(\hat{\Omega}_l), && (7)\\
T_{10}(l) &= X_1^{(2)}(l)1_{\{X_1^{(1)}(l) > X_1^{(2)}(l)\}}, \quad T_{01}(l) = X_0^{(1)}(l) \wedge X_0^{(2)}(l), && (8)\\
T(l) &= T_{10}(l-1) + T_{01}(l), && (9)\\
T_{12}(l) &= X_1^{(1)}(l)1_{\{X_1^{(1)}(l) \leqslant X_1^{(2)}(l)\}}, \quad T_{21}(l) = X_2(l), && (10)\\
R &= \sum_{l \geqslant 0} [T(l) + T_{12}(l)]1_{\{\hat{\Omega}_l\}}, && (11)\\
\Pi &= R + T_{21}(l)1_{\{\hat{\Omega}_l\}}. && (12)
\end{aligned}$$

At that, the condition for the finiteness of the regeneration period mean time is the convergence of the series,

$$\mathbb{P}\{\nu \geqslant l\} = \sum \mathbb{P}(\hat{\Omega}_l) = \sum_{l \geqslant 0} (1 - p_l) \prod_{i=0}^{l-1} p_i < \infty. \tag{13}$$

Proof. Indeed, according to lemma 1, starting from state 1 with the marks $X_1^{(1)}(0) = A_0$, $X_1^{(2)}(0) = B_0$, the process immediately goes to state 2 if the failure of the working element occurs before the restoration of the one being repaired, i.e., when $X_1^{(1)}(0) \leqslant X_1^{(2)}(0)$, having spent time in this state equal to

$$T_{12}(1) = X_1^{(1)}(0) \wedge X_1^{(2)}(0) = X_1^{(1)}(0)1_{\{X_1^{(1)}(0) \leqslant X_1^{(2)}(0)\}}.$$

In this case, the process will be in state 2 with the mark $X_2(1) = (B_0 - A_0)1_{\{A_0 \leqslant B_0\}}$ and will spend a random time in it equal to this value,

$$T_{21}(1) = (B_0 - A_0)1_{\{A_0 \leqslant B_0\}}.$$

Thus,

$$\pi_0 = \mathbb{P}\{A_0 \leqslant B_0\}, \;\; R = A_0, \;\; \Pi = B_0,$$

which corresponds to formulas (7), (10)–(12) for $l = 0$.

The number of returns to state 1 will be equal to 1 if the element being repaired is restored before the working element fails, i.e., when $A_0 > B_0$; in this case, the process will spend a random time in state 1 before transitioning to state 0, equaled to

$$T_{10}(1) = X_1^{(1)}(0) \wedge X_1^{(2)}(0)1_{\{X_1^{(1)}(0) > B_0\}} \equiv A_0 \wedge B_0 1_{\{A_0 > B_0\}}$$

and will go to state 0 with marks

$$X_0^{(1)}(1) = (A_0 - B_0)1_{\{A_0 > B_0\}}, \quad X_0^{(2)}(1) = A_1.$$

Then the process will return to state 1 with probability 1 and the marks

$$X_1^{(1)}(1) = X_0^{(1)}(1) \vee A_1 - X_0^{(1)}(1) \wedge A_1 \equiv W(1), \quad X_1^{(2)}(1) = B_1$$

during random time $T_{01}(1) = X_0^{(1)}(l) \wedge X_0^{(2)}(l)$.

Then, from state 1 the process will go to state 2 with probability $\mathbb{P}\{W(1) \leqslant B_1\}$ and the mark

$$X_2(1) = (B_1 - W(1))1_{\{W(1)\leqslant B_1\}},$$

spending time $T_{12}(1) = W(1)1_{\{W(1)\leqslant B_1\}}$ in it, after which with probability 1 during time $T_{21}(1) = (B_1 - W(1))1_{\{W(1)\leqslant B_1\}}$ it will return to state 1. This will end the cycle. Thus, we have

$$\begin{aligned}\pi_1 &= \mathbb{P}\{\nu = 1\} = \mathbb{P}(\hat{\Omega}_1),\\ R &= T_{10}(1) + T_{01}(1) + T_{12}(1),\\ \Pi &= R + T_{21}(1),\end{aligned}$$

which corresponds to the formulas (7)–(12) for $l = 1$.

Similarly, the number of returns to state 1 will be equal to l if the event $\hat{\Omega}_l$ occurs, i.e., if the process $l-1$ times goes from state 1 to 0 and back, and at the lth step goes to state 2. The probability of this event is,

$$\pi_l = \mathbb{P}\{\nu = l\} = \mathbb{P}(\hat{\Omega}_l),$$

which proves formula (7). In this case, the transformations of the marks are carried out by lemma 1, and transitions durations $T_{10}(l)$ from state 1 to 0 and back $T_{01}(l)$ at the lth step are equal to

$$\begin{aligned}T_{10}(l) &= X_1^{(2)}(l)1_{\{X_1^{(1)}(l)>X_1^{(2)}(l)\}}, \quad T_{01}(l) = X_0^{(1)}(l) \wedge X_0^{(2)}(l)1_{\{X_0^{(1)}(l)>X_0^{(2)}(l)\}},\\ T_{12}(l) &= X_1^{(1)}(l)1_{\{X_1^{(1)}(l)\leqslant X_1^{(2)}(l)\}}, \quad T_{21}(l) = X_2(l).\end{aligned}$$

The idea that the system's operating time R before failure consists of sequences of repair times and residual operating times before failure of elements, i.e., transitions durations between states 1 and 0 before transition to state 2, combined with the regeneration period Π that encompasses the time $T_{21}(l)$ spent in state 2, substantiates the validity of equations (11), (12).

Find the condition for the finiteness of the regeneration period mean time. Since the time of return from state 2 to state 1 is finite with probability 1 for any step of the cycle, it is sufficient that the system operates until failure with probability 1,

$$\mathbb{P}\{R < \infty\} = 1.$$

First, consider the condition of the finiteness of the number of steps in the regeneration period, $\mathbb{P}\{\nu < \infty\} = 1$. Noting that

$$\mathbb{P}\{\nu < \infty\} = \sum_{l\geqslant 0} \mathbb{P}\{\nu = l\} = \sum_{l\geqslant 0} \mathbb{P}(\hat{\Omega}_l),$$

calculate the probability of the event occurrence $\hat{\Omega}_l$ using the Markov dependence of the marks,

$$\begin{aligned}\mathbb{P}(\hat{\Omega}_l) &= \mathbb{P}\{W(l) \leqslant B_l \,|\, \Omega_{l-1}\}\mathbb{P}(\Omega_{l-1}) = \mathbb{P}\{W(l) \leqslant B_l \,|\, W(l-1) > B_{l-1}\}\mathbb{P}(\Omega_{l-1})\\ &= (1-p_l)\mathbb{P}(\Omega_{l-1}) = (1-p_l)\mathbb{P}(\Omega_{l-1}|\Omega_{l-2})\mathbb{P}(\Omega_{l-2}) = \cdots = (1-p_l)p_{l-1}\cdots p_0,\end{aligned}$$

where the notations (6) are used. Summing up the obtained expressions, we find

$$\sum_{l\geqslant 0} \pi_l = 1 - p_0 + p_0(1-p_1) + \ldots p_0 \cdots p_{l-1}(1-p_l) + \cdots = 1 - \prod_{l\geqslant 0} p_l.$$

Thus, the condition for the finiteness of the number of steps in the regeneration period is the divergence of the product $\prod_{l\geqslant 0} p_l$ or, equivalently, the divergence of the series $\sum\limits_{l\geqslant 0} \ln p_l$. Moreover, for the finiteness of the regeneration period mean time $\mathbb{E}[R] < \infty$ it is sufficient

$$\mathbb{E}[R] = \sum \mathbb{P}\{\nu > l\} = \sum \mathbb{P}(\hat{\Omega}_l) = \sum_{l\geqslant 0}(1-p_l)\prod_{i=0}^{l-1} p_i < \infty$$

which coincides with the formula (13) and completes the proof of the theorem. $\square$

The obtained expressions show that the main characteristics of the model are expressed in terms of the marks. However, analytical expressions of their distributions and numerical indicators are quite cumbersome and require the introduction of special transformations. At the same time, the computational procedures for presenting the final results are quite problematic. Therefore, the next section proposes an algorithm for calculating the main characteristics of the model by simulation modeling, based directly on the proposed approach.

5. ALGORITHM FOR CALCULATING THE MAIN RELIABILITY CHARACTERISTICS

This section presents an algorithm for calculating the reliability characteristics of a repairable hot double redundant system based on the proposed method.

Algorithm 1.

Initial data:

$A(\cdot)$, $B(\cdot)$ – the distributions of r.v.'s A_i, B_i, corresponding means (μ_a, μ_b) and coefficients of variation (v_a, v_b),

N – total number of model realizations,

n – the current number of realizations,

ν_l – the array of the number of cycles length l during the regeneration period,

l – regeneration period length counter,

R – the array of times to system failure,

Π – system regeneration period array.

Input: $A(\cdot)$, $B(\cdot)$, N, $n = 1$, $\nu_l = [0] * \max(l)$, $R = [0] * N$, $\Pi = [0] * N$.

Beginning. Put: $l = 0$, $X_1^{(1)}(0) = A_0 \in A(\cdot)$, $X_1^{(2)}(0) = B_0 \in B(\cdot)$, $X_0^{(1)}(0) = X_0^{(2)}(0) = X_2(0) = 0$, $T(0) = 0$.

Step 1. If $n < N$, go to Step 2, if no, go to Step 4.

Step 2. While $X_1^{(1)}(l) > X_1^{(2)}(l)\ \forall\, l = 0, 1, \ldots$, repeat:

$$l := l + 1$$

$$T_{10}(l) = X_1^{(2)}(l-1)$$

$$X_0^{(1)}(l) = X_1^{(1)}(l-1) - X_1^{(2)}(l-1), \quad X_0^{(2)}(l) = A_l \in A(\cdot)$$

$$T_{01}(l) = X_0^{(1)}(l) \wedge X_0^{(2)}(l)$$

$$T(l) := T(l-1) + T_{10}(l) + T_{01}(l)$$

$$X_1^{(1)}(l) = X_0^{(1)}(l) \vee X_0^{(2)}(l) - X_0^{(1)}(l) \wedge X_0^{(2)}(l), \quad X_1^{(2)}(l) = B_l \in B(\cdot),$$

in another case $X_1^{(1)}(l) < X_1^{(2)}(l)$

$$T_{12}(l) = X_1^{(1)}(l)$$

$$T_{21}(l) = X_2(l) = X_1^{(2)}(l) - X_1^{(1)}(l).$$

Go to Step 3.

Step 3. Collect statistics:

– Filling the array $\nu_l := \nu_l + 1$,

– Filling the array $R_n := T(l) + T_{12}(l)$,

– Filling the array $\Pi_n := R_n + T_{21}(l)$.

Put $n := n + 1$ and go to the Beginning.

Step 4. Processing statistics:

– calculating of the estimate of the distribution of steps number ν during the regeneration period,

$$\hat{\pi} = \frac{\nu_l}{\sum_{l>0} \nu_l},$$

– calculating of the empirical reliability function R,

$$\hat{R}(t) = 1 - \frac{1}{N} R_n, \quad R_n \leqslant t \leqslant R_{n+1},$$

– calculating of the system mean lifetime,

$$\hat{\mu}_R = \frac{1}{N} \sum_{n=1}^{N} R_n,$$

– Results printing.

STOP.

6. EXAMPLES

In this section, we will consider several examples of numerical analysis, and compare the results obtained using the proposed approach and methods of Markov process theory. The choice of initial data does not refer to any example from a practical problem and is only demonstrative.

6.1. $< M_2|M|1 >$ *System*

Let life- and repair times of elements follow an exponential distribution with parameters α and β, respectively, $A(t) \sim Exp(\alpha)$, $B(t) \sim Exp(\beta)$. The reliability characteristics of such a system can be calculated by a direct method from the birth and death process. Thus, the reliability function of $< M_2|M|1 >$ system is calculated as

$$R(t) = \frac{1}{2r} e^{-\frac{1}{2}(3\alpha+\beta+r)t} \left(e^{rt}(\alpha + \beta + r) - (\alpha + \beta - r) \right), \tag{14}$$

where $r = \sqrt{\alpha^2 + 6\alpha\beta + \beta^2}$, from which we obtain the mean system lifetime μ_R,

$$\mu_R = \int_0^\infty R(t)dt = \frac{1 + 2\alpha\beta^{-1}}{2\alpha^2\beta^{-1}}. \tag{15}$$

To illustrate the example, suppose that $\alpha = \beta = 1$. In this case, the mean life- and repair times are $\mu_a = \mu_b = 1$. For the experiment, the number $N = 10^5$ of algorithm realizations is chosen experimentally to achieve high accuracy of the algorithm results. The direct implementation of the algorithm is performed using Python programming language. To assess the accuracy of the results

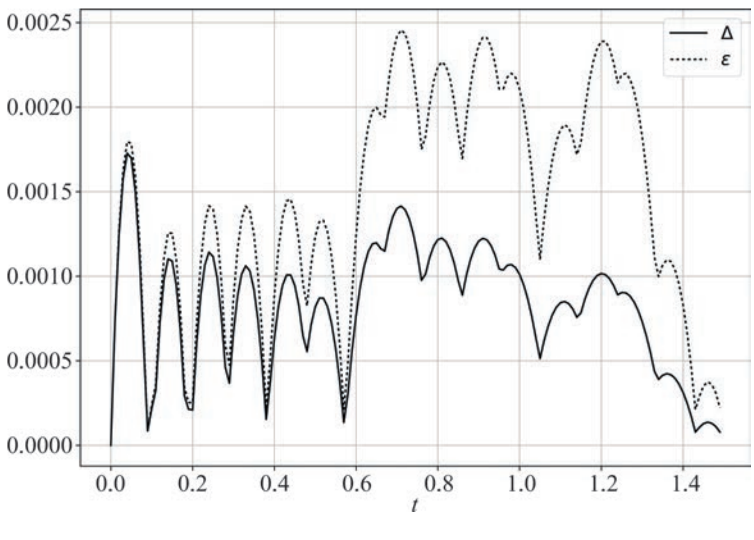

Fig. 2. Absolute and relative errors of the reliability function estimation for $< M_2|M|1 >$ system.

obtained using the proposed algorithm, we consider the absolute Δ and relative ε errors,

$$\Delta = \left|R_{th}(t) - \hat{R}(t)\right|, \quad \varepsilon = \frac{\Delta}{R_{th}(t)},$$

where $R_{th}(t)$ denotes the values of the reliability function, calculated by the formula (14), and $\hat{R}(t)$ presents the values obtained using the algorithm. Figure 2 shows the graphical results of calculating these estimates on the interval $t = [0, 1.5]$, where $t = 1.5$ corresponds to the mean lifetime of the system μ_R, calculated by the formula (15). It is evident from the graph that on the considered interval t the maximum value of the absolute error is $\Delta \approx 0.0017$, and the maximum value of the relative error is $\max(\varepsilon) \approx 0.0025$. At the same time, mean values of estimates on the entire interval t are equal to $\mathbb{E}[\Delta] = 0.0008$ and $\mathbb{E}[\varepsilon] = 0.0014$, respectively.

The values of the mean system lifetime μ_R, calculated analytically and using the algorithm, are equal to $\mu_R = 1.5$ and $\hat{\mu}_R = 1.502127$, respectively, and emphasize the comparability of the obtained results.

6.2. $< M_2|GI|1 >$ System

Consider further the case when lifetime of elements has an exponential distribution and repair time follows an arbitrary distribution, i.e., consider $< M_2|GI|1 >$ system.

To calculate the analytical expression for the reliability function of such a system, the method of introducing supplementary variables is used [26], which leads to the need to solve the system of Kolmogorov partial differential equations,

$$\begin{aligned}
\frac{d}{dt}\pi_0(t) &= -2\alpha\pi_0(t) + \int_0^t \beta(x)\pi_1(t,x)dx, \\
\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\pi_1(t;x) &= -(\alpha + \beta(x))\pi_1(t,x), \\
\frac{d}{dt}\pi_2(t) &= \alpha \int_0^t \pi_1(t,x)dx,
\end{aligned}$$

jointly with initial $\pi_0(0) = 0$, $\pi_1(0) = 1$, $\pi_2(0) = 0$ and boundary conditions $\pi_1(t, 0) = 2\alpha\pi_0(t)$ using the method of characteristics. By solving this system in terms of Laplace transform (LT), LT of the reliability function is obtained, which has the following form in terms of LT of components repair time $\tilde{b}(s)$,

$$\tilde{R}(s) = \frac{s + \alpha(2 - \tilde{b}(s+\alpha))}{(s+\alpha)(s + 2\alpha(1 - \tilde{b}(s+\alpha)))}, \tag{16}$$

from where we calculate the mean system lifetime,

$$\mu_R = \tilde{R}(0) = \frac{2 - \tilde{b}(\alpha)}{2\alpha(1 - \tilde{b}(\alpha))}.$$

To compare the analytical results and their estimates obtained using the new algorithm, we assume that the repair time of the elements has a Gamma distribution ($\Gamma = \Gamma(l, \theta)$) with the following characteristics, i.e., p.d.f. $b(t) = \theta^l t^{l-1} e^{-\theta t}\Gamma(l)^{-1}$, $t > 0$, mean $\mu_b = l\theta^{-1} = 1$ and coefficient of variation $v_b = \sqrt{l}/l = 0.5$. For the remaining parameters, we again use $\alpha = 1$, $N = 10^5$. To evaluate the obtained results, we also consider the absolute Δ and relative ε errors (see Fig. 3). To calculate these estimates, the inverse LT of the reliability function (16), calculated numerically using the built-in function of Python programming language, is taken as the theoretical value $R_{th}(t)$. The argument t is considered on the interval $t = [0, 1.35]$, where $\max(t) = 1.35$ corresponds to μ_R for the given parameters.

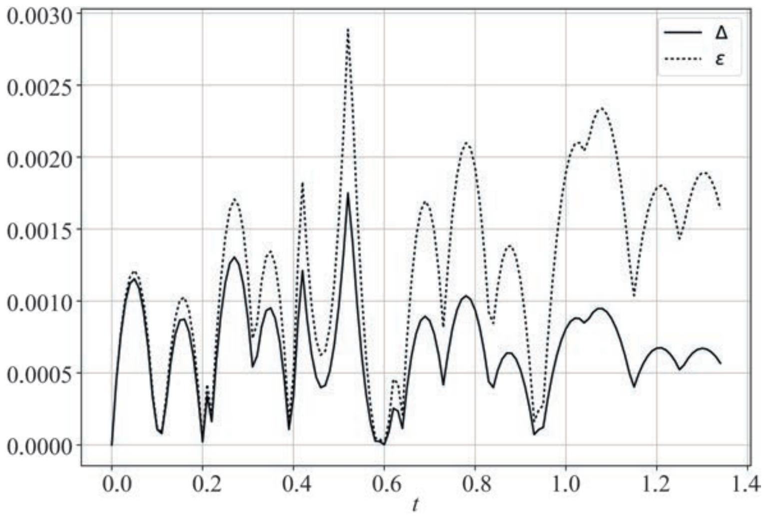

Fig. 3. Absolute and relative errors of the reliability function estimation for $< M_2|\Gamma|1 >$ system.

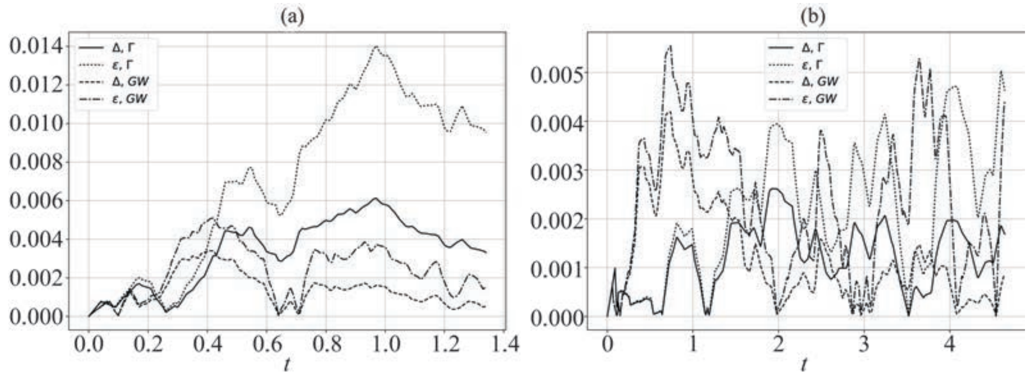

Fig. 4. Absolute and relative errors of the reliability function estimation for $< M_2|GI|1 >$ system. (a) — $v_b = 0.5$. (b) — $v_b = 5$.

According to the results of the experiment, the maximum value for Δ is $\max(\Delta) \approx 0.0017$, and for ε it is $\max(\varepsilon) \approx 0.0028$. The mean values over the entire interval t are $\mathbb{E}[\Delta] = 0.0007$ and $\mathbb{E}[\varepsilon] = 0.0012$. In this case, the mean system lifetime is $\mu_R = 1.3469$ for the analytical result and $\hat{\mu}_R = 1.3478$ for the algorithm. The results of the algorithm evaluation again demonstrate high accuracy relative to the analytical results.

For additional verification, we consider the simulation modeling of a repairable hot double redundant system using the discrete-event modeling (DEM) method [27]. The method is implemented using Python programming language, for which the total simulation time is taken as $T = 10^5$. As an example, consider Γ and Gnedenko-Weibull (GW) distributions of the repair time of elements, while fixing the mean $\mu_b = 1$ and the coefficient of variation $v_b = 0.5$, 5 for both distributions. The remaining parameters are the same as previously.

Figures 4a and 4b show the absolute and relative errors of estimating the reliability function of $< M_2|GI|1 >$ system.

Parameter t is chosen as the maximum value of the mean system lifetime for $v_b = 0.5$ and $v_b = 5$, respectively, for each figure. The results of the reliability function assessment using DEM method are taken as $R_{th}(t)$. The obtained results demonstrate that among the considered repair time distributions and the corresponding coefficients of variation, the maximum errors were $\max(\Delta) \approx 0.006$ and $\max(\varepsilon) \approx 0.014$.

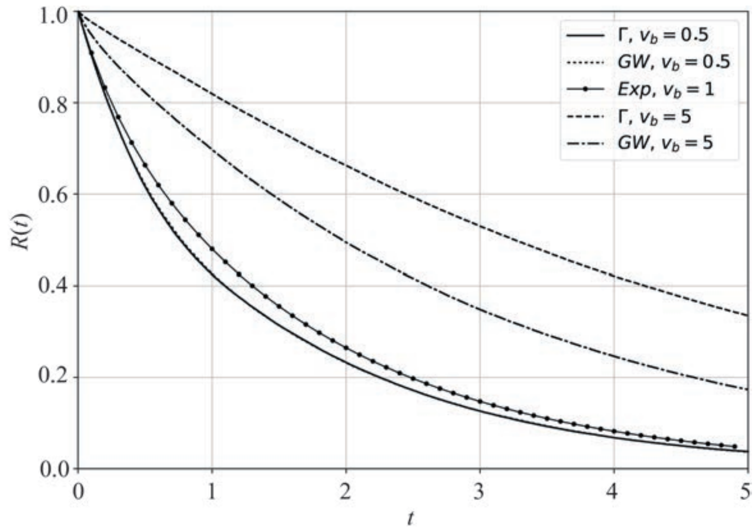

Fig. 5. The reliability function of $< M_2|GI|1 >$ system.

The mean values of these errors for all cases are presented in Table 1. The results of this experiment confirm the accuracy of the reliability characteristics obtained using the proposed approach.

Below is the graph of the reliability function of $< M_2|GI|1 >$ system obtained using the proposed approach (see Fig. 5). The legend of the figure defines the line type for the selected repair time distributions and the corresponding coefficients of variation. To identify the influence of the coefficient of variation of repair time on the reliability function, the figure also shows the case of the Markov model (Exp).

The results of this example show that the shape of repair time distribution and coefficient of variation significantly affect the reliability function. However, it is worth noting that the reliability function curves as $v_b = 0.5$ and $v_b = 1$ are located quite close to each other. Moreover, as $v_b = 0.5$, despite the different repair time distributions, these curves merge on the considered interval t. In the case of $v_b = 5$, on the contrary, the curves of the reliability function for the same repair time distributions differ significantly from each other. Thus, we can conclude that the reliability function is insensitive to the shape of repair time distribution for fixed mean μ_b and coefficient of variation v_b as $v_b < 1$ and, conversely, about its sensitivity as $v_b > 1$.

Table 2 presents the mean system lifetime for this example and also demonstrates the corresponding values obtained using DEM method. The results show the insensitivity of the mean μ_R to the shape of repair time distribution for fixed μ_b and v_b as $v_b < 1$. In addition, the mean system lifetime turns out to be sensitive to the shape of repair time distribution and coefficient of variation as $v_b > 1$.

Table 1. Mean errors Δ and ε

$B(\cdot)$, v_b	$\mathbb{E}[\Delta]$	$\mathbb{E}[\varepsilon]$
Γ, $v_b = 0.5$	0.0034	0.0070
Γ, $v_b = 5$	0.0006	0.0007
GW, $v_b = 0.5$	0.0014	0.0024
GW, $v_b = 5$	0.0023	0.0031

Table 2. Mean lifetime of $< M_2|GI|1 >$ system

$B(\cdot)$, v_b	DEM	Algorithm
Γ, $v_b = 0.5$	1.34619	1.34457
Γ, $v_b = 5$	4.63125	4.64059
GW, $v_b = 0.5$	1.34111	1.352305
GW, $v_b = 5$	2.80304	2.82017

6.3. $< GI_2|GI|1 >$ *System*

In the last example, we consider a system whose life- and repair time of elements have arbitrary distributions. We again take Γ and GW as these distributions. Denote v_a the coefficient of variation of components' lifetime and put $v_a = 0.5$. The remaining initial parameters are retained from the

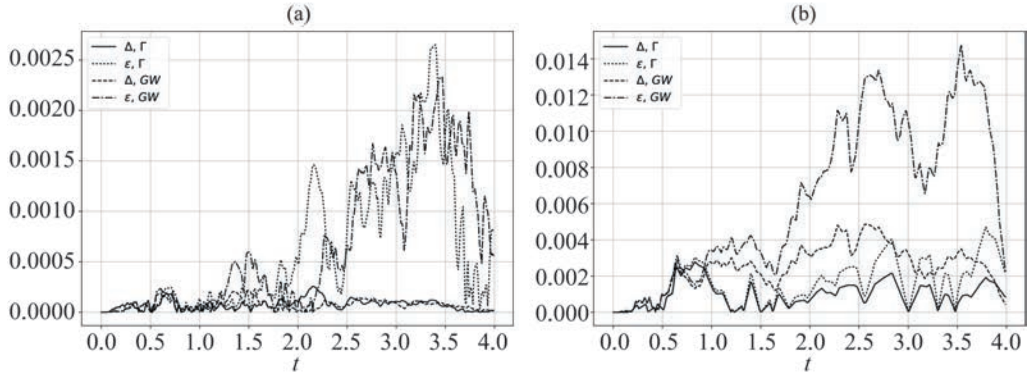

Fig. 6. Absolute and relative errors of the reliability function estimation for $< \Gamma_2|GI|1 >$ system. (a) — $v_b = 0.5$. (b) — $v_b = 5$.

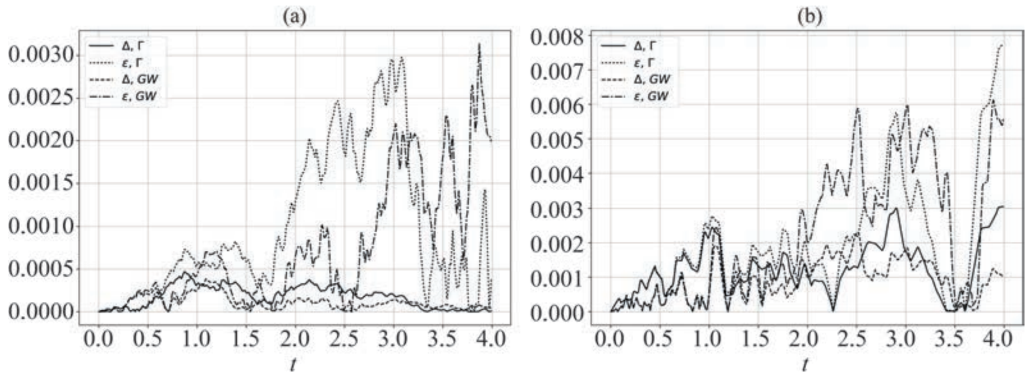

Fig. 7. Absolute and relative errors of the reliability function estimation for $< GW_2|GI|1 >$ system. (a) — $v_b = 0.5$. (b) — $v_b = 5$.

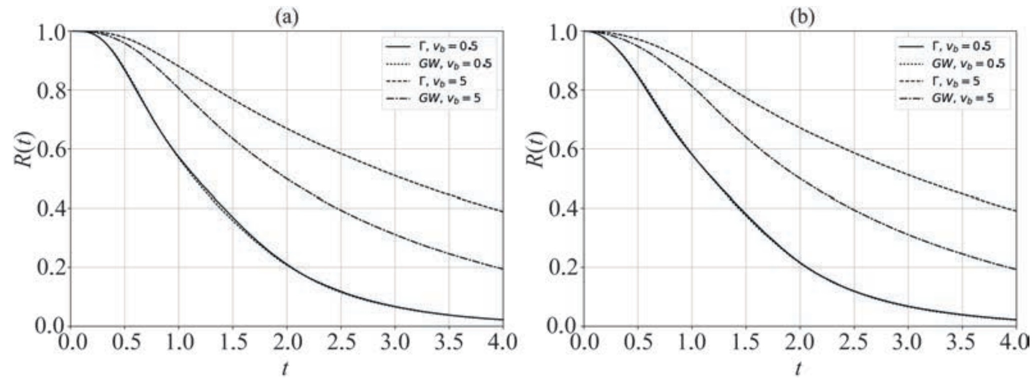

Fig. 8. The reliability function of $< GI_2|GI|1 >$ system. (a) — $A \sim \Gamma$, $v_a = 0.5$. (b) — $A \sim GW$, $v_a = 0.5$.

previous example. Figures 6 and 7 demonstrate the estimates of the absolute Δ and relative ε errors for various initial distributions and coefficients of variation. The legend of the figure determines the line type for errors with Γ and GW repair time distributions. As $R_{th}(t)$, we again consider the estimate of the reliability function obtained by DEM method.

According to the obtained graphical results, the maximum absolute error among the considered cases is $\max(\Delta) < 0.005$. The maximum value of the relative error varies in the interval is $\max(\varepsilon) \approx [0.003, 0.014]$. Given the given values $N = 10^5$ and $T = 10^5$, these results are objectively satisfactory.

Figure 8 shows the graphs of the reliability function for Γ and GW distributions of elements' lifetime. The experiment demonstrates results similar to those in the previous example. The value of the coefficient of variation of repair time $v_b = 5$ leads to an increase in the values of the reliability function on the considered interval t. In addition, the shape of repair time distribution affects the behavior of this curve as $v_b > 1$. It is obvious that the reliability function curves are quite close to each other for different initial distributions, but fixed values of the parameters μ_a and v_a.

Table 3 presents the mean system lifetime, including estimates obtained by DEM method. Tables 3a and 3b demonstrate close results for different components' lifetime distributions. The mean system lifetime μ_R differs only at the second or third decimal places. Thus, we can conclude that the mean system lifetime is insensitive to the shape of lifetime distribution for fixed mean μ_a and coefficient of variation v_a.

Table 3. Mean lifetime of $< GI_2|GI|1 >$ system

$B(\cdot)$, v_b	DEM	Algorithm
Γ, $v_b = 0.5$	1.4059	1.4003
Γ, $v_b = 5$	4.1305	4.2208
GW, $v_b = 0.5$	1.3900	1.3877
GW, $v_b = 5$	2.6242	2.6349

(a) $A \sim \Gamma,\ v_a = 0.5$

$B(\cdot)$, v_b	DEM	Algorithm
Γ, $v_b = 0.5$	1.4061	1.4044
Γ, $v_b = 5$	4.1637	4.2347
GW, $v_b = 0.5$	1.3983	1.3985
GW, $v_b = 5$	2.6329	2.6433

(b) $A \sim GW,\ v_a = 0.5$

Additionally, comparing the values in Tables 2 and 3, we can conclude that the mean lifetime of the system has low sensitivity to the shape of components' lifetime distribution for fixed mean μ_a and coefficient of variation $v_a \leqslant 1$.

7. CONCLUSION

The article introduces the concept of the Marked Markov process, which is used to study a repairable hot double redundant system with arbitrary initial distributions. A method and algorithm for calculating the reliability function of such a system are proposed. The results obtained using the new approach were validated by comparing them with the results of analytical calculations and discrete-event modeling. The new approach showed high accuracy in comparison with known methods.

The main advantage of the proposed modeling algorithm compared to discrete-event modeling is that the presented procedure is entirely based on a mathematically formulated approach and, in essence, consists of implementing marks' transformation that uniquely determines the system operation. Moreover, this algorithm eliminates the need to write additional code for the discrete-event modeling program, which significantly simplifies the process of studying a repairable hot double redundant system with arbitrary initial distributions.

In the future, it is planned to apply this approach to the study of other complex stochastic systems, including the study of the reliability characteristics of a k-out-of-n system with arbitrary initial distributions and an arbitrary number of repair units.

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Aggregate Estimates of Reflexive Collective Behavior Dynamics in a Cournot Oligopoly Model

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Abstract—This paper considers an oligopoly model with an arbitrary number of Cournot-reflexive agents under incomplete information in the classical case of linear cost and demand functions. Agents make decisions based on a collective behavior model. The problem of identifying convergence conditions to the model equilibrium is studied, with an emphasis placed on the trajectory of the sum of the action residuals of all agents. Aggregate estimates of this trajectory are obtained. They can be used to judge how the trajectory of each agent evolves towards the equilibrium.

Keywords: Cournot oligopoly, reflexive behavior, trajectory residuals, aggregate estimates, convergence conditions

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1. INTRODUCTION

In research areas dealing with behavioral models of rational agents under incomplete information, considerable attention is paid to the oligopoly market. Game theory was the first scientific method to analyze oligopoly [1]. Recent advances in oligopoly game theory were reviewed in [2].

Collective behavior theory models complement game-theoretic models by providing an opportunity to study the behavior dynamics of rational agents under rather weak assumptions about their awareness [3, 4]. The agent's dynamic decision process is based on reflexive thinking about his best choice considering the best responses of the other competitors. The dynamics are guided by the agents' choices. The determinative effect of reflexion is reaching an equilibrium [4, 5].

A significant number of studies of game-theoretic and collective behavior models in competitive markets are devoted to the problem of identifying conditions for the existence and uniqueness of an equilibrium as well as the convergence of agents' trajectories to it. For example, see [6–16] and many other related publications.

By accepting a certain hypothesis about the behavior of agents and their interaction, it is possible to calculate the trajectory of each agent (the first approach). However, an aggregate description of the behavior of the entire system, without characterizing the behavior of each agent in detail, seems more appropriate (the second approach) for several reasons. Obviously, the activity of individual agents at separate time instants cannot noticeably affect the convergence of trajectories. Besides, the growth of the number of agents in the market, their trajectories, and the computing time make the first approach less and less attractive. In some cases, the asymptotic convergence of the calculated trajectories can be judged only after a considerable time, particularly when the process evolves ambiguously or slowly, and the trend appears late. An intuitively aggregate description of the collective behavior of a system of agents on considerable time intervals can be no less accurate than a detailed one.

2. FORMAL PROBLEM STATEMENT

As the basic model, we consider the Cournot oligopoly of n agents [1] competing in the outputs of homogeneous products. By assumption, demand is determined by the function (the inverse demand function depending on the total output of agents)

$$p(Q) = a - bQ, \tag{1}$$

where $p(Q)$ is the uniform market price, $Q = \sum_{i \in N} q_i$ is the total output of all n agents; q_i is the output of agent i, $i \in N = \{1, \ldots, n\}$; a and b are parameters. The parameter a characterizes the maximum possible price of the product under which the demand will tend to zero; the parameter b characterizes the slope of the demand curve.

The total costs of agents have the form

$$\phi_i(q_i) = c_i q_i + d_i, \tag{2}$$

where c_i and d_i are the marginal and fixed costs of agent i, respectively. The goal functions of agents are given by

$$\Pi_i(p(Q), q_i) = p(Q)q_i - \phi_i(q_i) \to \max_{q_i}. \tag{3}$$

By assumption, everything produced is sold, no capacity constraints are imposed, and no coalitions are allowed. The market state at a time instant t ($t = 0, 1, 2, \ldots$) is described by the n-dimensional vector $q^t = (q_1^t, \ldots, q_i^t, \ldots, q_n^t)$.

Agents are said to be collectively competitive in a Cournot market if each agent i satisfies the marginal costs constraint

$$c_i < \frac{a + \sum_{j \in N \setminus \{i\}} c_j}{n}. \tag{4}$$

In this case, each agent is supposed to be competitive and, as in a normal form game, there exists a unique (static Nash) equilibrium [17] $q^* = (q_1^*, \ldots, q_i^*, \ldots, q_n^*)$ with $q_i^* > 0$ $\forall i \in N$ in the Cournot oligopoly model; for example, see [18].

Under game uncertainty (about the actions chosen by the competitive environment) and incomplete knowledge (of costs, goal functions, and other attributes of competitors), market equilibrium can usually be reached not by one-time decision-making by agents but as the result of an iterative reflexive process [3, 4, 18–20].

Consider a reflexive discrete process where the change of market states satisfies the axiom of indicator behavior [4] as follows: at each time instant $(t + 1)$, each agent observes the outputs of all agents chosen at the previous time instant t and adjusts his output by taking a step towards the current position of his goal, $x_i(q_{-i}^t)$, in the iterative procedure

$$q_i^{t+1} = q_i^t + \gamma_i^{t+1}(x_i(q_{-i}^t) - q_i^t), \quad i \in N. \tag{5}$$

Here, the parameter $\gamma_i^{t+1} \in [0, 1]$, independently chosen by each agent i, determines his step towards the current position of his goal. An agent can take a full step with $\gamma_i^{t+1} = 1$ (his best response), remain on the spot with $\gamma_i^{t+1} = 0$, or take a partial step with $\gamma_i^{t+1} \in (0, 1)$.

For agent i, the current position $x_i(q_{-i}^t)$ of his goal is an output maximizing his goal function provided that at the current time instant, the other agents choose the same output as at the previous time instant [4, 21]. Here, $q_{-i}^t = (q_1^t, \ldots, q_{i-1}^t, q_{i+1}^t, \ldots, q_n^t)$ is the opponents' output profile for agent i (the output vector of all agents except agent i) at the time instant t. According to [10, 18]),

$$x_i(q_{-i}^t) = \frac{h_i - \sum_{j \neq i} q_j^t}{2} = \frac{h_i - Q_{-i}^t}{2}, \tag{6}$$

where $h_i = \frac{a-c_i}{b}$ is the volume of a perfectly competitive market under the marginal cost pricing $p(Q) = c_i$ (the perfectly competitive output of firm i); $Q_{-i}^t = \sum_{j \neq i} q_j^t$ is the total output of the environment of agent i ($i, j \in N$).

Let us derive the variable $x_i(q_{-i}^t)$ to show its relation to q_i^t. Using (1)–(3) for time instant t, we obtain

$$\frac{\partial \prod_i^t}{\partial q_i^t} = a - bq_i^t - bQ_{-i}^t + \left(-b - b\frac{\partial Q_{-i}^t}{\partial q_i^t}\right) q_i^t - c_i = 0$$

and

$$q_i^t = \frac{(a-c_i)/b - Q_{-i}^t}{2 + \partial Q_{-i}^t / \partial q_i^t}.$$

Under the Cournot assumption, the output of the agent's environment output will not change if he changes his output. Therefore, $\frac{\partial Q_{-i}^t}{\partial q_i^t} = 0$ and the agent's optimal output q_i^t is $\frac{(a-c_i)/b - \sum_{j \in N \setminus \{i\}} q_j^t}{2}$. This optimal output will be the current position $x_i(q_{-i}^t)$ of the agent's goal in (6).

For model (5)–(6), the trajectory of agent i is understood as the sequence of outputs $q_i^0, q_i^1, \ldots,$ $q_i^t, \ldots$ *realized* within this model.

The problem of identifying convergence conditions of agents' trajectories (5)–(6) to equilibrium and its modifications in the classical Cournot oligopoly model (1)–(4) were considered by many researchers; for example, see [18–20, 22–26] and other publications on the subject.

The peculiarity and novelty of the study presented below are the development of an aggregate description of the entire system's behavior to judge how the trajectory of each agent evolves towards the equilibrium.

In this study, we sequentially solve the following tasks:

1) reduce agents' models to a homogeneous form;

2) investigate the effect of total residuals on the convergence of agents' trajectories;

3) describe in aggregate terms the transformation (recalculation) of the total residuals during the transition between time instants;

4) describe in aggregate terms (estimate) the dynamics of the total residuals over the set of time instants;

5) form convergence conditions of agents' trajectories to the equilibrium using the aggregate estimates of collective behavior model dynamics.

As intuition suggests, aggregates are preferably formed from homogeneous elements. Ideally, homogeneous agents may differ only by their choice of the parameter γ. Let us analyze the possibility of the ideal case for the market model (1)–(4) and the iterative process (5)–(6).

The next section begins with this task.

3. THE METHODS AND RESULTS OF THIS STUDY

Assume that within model (1)–(4), the process (5)–(6) converges under step parameters $\{\gamma_i^{t+1}\}$ ($i \in N; t = 0, 1, 2, \ldots$) and the marginal costs $c = (c_1, \ldots, c_i, \ldots c_n)$ of the agents.

Will this process converge under the same step parameters if the marginal costs of agents or market parameters change?

To answer this question, we introduce the change of variables

$$\varepsilon_i^t = q_i^* - q_i^t \quad (i \in N; t = 0, 1, 2, \ldots), \tag{7}$$

where q_i^* and q_i^t are the equilibrium and current outputs of agent i, respectively.

Using the well-known Cournot equilibrium relations $h_i = Q^* + q_i^*$, we transform (5)–(6) as follows:

$$q_i^* - q_i^{t+1} = q_i^* - q_i^t - \gamma_i^{t+1}\left(\frac{h_i - Q_{-i}^t}{2} - q_i^t\right) = q_i^* - q_i^t - \gamma_i^{t+1}\left(\frac{Q^* + q_i^* - Q_{-i}^t}{2} - q_i^t\right)$$
$$= q_i^* - q_i^t - \gamma_i^{t+1}\left(\frac{\sum_{j\neq i}(q_j^* - q_j^t) + 2q_i^*}{2} - q_i^t\right).$$

As a result,

$$\varepsilon_i^{t+1} = \varepsilon_i^t + \gamma_i^{t+1}\left(-\frac{\sum_{j\neq i}\varepsilon_j^t}{2} - \varepsilon_i^t\right). \tag{8}$$

Here, by analogy with (1)–(6), $\left(-\frac{\sum_{j\neq i}\varepsilon_j^t}{2}\right)$ is the current position of the goal of agent i, and (8) is the indicator behavior model, and the agent's goal function has the form

$$\Pi_i(\varepsilon) = -\left(\sum_{j\in N}\varepsilon_j\right)\varepsilon_i \to \max_{\varepsilon_i}, \tag{9}$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_i, \ldots, \varepsilon_n)$.

Let us demonstrate this fact. For time instant t, we find the optimal residuals of agent i using (9):

$$\frac{\partial \prod_i^t}{\partial \varepsilon_i^t} = -\varepsilon_i^t - \sum_{j\in N\setminus\{i\}}\varepsilon_j^t - \left(1 + \frac{\partial \sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{\partial \varepsilon_i^t}\right)\varepsilon_i^t.$$

Consequently,

$$\varepsilon_i^t = -\sum_{j\in N\setminus\{i\}}\varepsilon_j^t \bigg/ \left(2 + \frac{\partial \sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{\partial \varepsilon_i^t}\right).$$

By the Cournot assumption, the agent's environment will not change its output if he does so. Obviously, this assumption also applies to the residuals. Therefore, $\frac{\partial \sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{\partial \varepsilon_i^t} = 0$, and the current position of his goal (the optimal residuals at the current time instant) is $\left(-\frac{\sum_{j\in N\setminus\{i\}}\varepsilon_j^t}{2}\right)$.

For model (8), the trajectory of agent i will be understood as the sequence of residuals $\varepsilon_i^0, \varepsilon_i^1, \ldots, \varepsilon_i^t, \ldots$ realized within this model.

The convergence of (8) means that $\varepsilon_i^t \to \varepsilon_i^* = 0$ as $t \to \infty$.

Proposition 1. *The indicator behavior process* (8) *converges if and only if the process* (5)–(6) *converges within the model* (1)–(4).

The proof of this proposition is provided in the Appendix.

The next proposition shows that the convergence of the trajectory of the total residuals of all agents is sufficient for the convergence of the trajectory of each agent.

Proposition 2. *If* $\sum_{j\in N}\varepsilon_j^t \to 0$ *as* $t \to \infty$, *then* $\varepsilon_j^t \to \varepsilon_j^* = 0\ \forall j \in N$.

This fact is proved in the Appendix using mathematical induction.

Note. The equality $\sum_{j\in N}\varepsilon_j^t = 0$ for (8) implies the equalities $|\varepsilon_j^{t+1}| = (1 - \gamma_j^{t+1}/2)|\varepsilon_j^t|$ and the inequalities $|\varepsilon_j^{t+1}| < |\varepsilon_j^t|$ under $\gamma_j^{t+1} \neq 0\ \forall j \in N$. They indicate that, in the case under consideration, the trajectories of all agents at time instant $(t+1)$ will be closer to the equilibrium than at time instant t.

The final result for model (1)–(6), which essentially follows from Proposition 1, is presented below.

Theorem 1. *In the linear Cournot model* (1)–(4) *with competitive agents, the market parameters a and b and the agents' parameters c_i and d_i do not affect the steps $\{\gamma_i^t\}_{t=1,2,\ldots}$ ensuring the convergence of the indicator behavior model* (5)–(6).

Let us mention the advantages of studying the equilibrium problem based on model (8):

– The model is indifferent to the market and agents' cost parameters (a, b, c_i, d_i), which simplifies the analysis procedure.

– The agents and their trajectories differ merely in the choice of the parameter γ and initial data. Only the first factor matters here, since we are interested in convergence conditions for any initial data.

– In model (5)–(6), economic constraints may require that the agents' current outputs be nonnegative. In model (8), there is no basis for such nonnegativity requirements.

Now we proceed to the next problem, which is important for the convergence of the trajectory of each agent (see Proposition 2). Let us discuss conditions on the parameters γ under which $\sum_{j\in N}\varepsilon_j^t \to 0$.

Within model (8), we have $\sum\limits_{j\in N}\varepsilon_j^{t+1} = \left(1-\sum\limits_{j\in N}\gamma_j^{t+1}/2\right)\sum\limits_{j\in N}\varepsilon_j^{t+1} - \sum\limits_{j\in N}\varepsilon_j^t\gamma_j^{t+1}/2.$

For $\gamma_j^{t+1}\equiv 1\ \ \forall j\in N$, it follows that $\sum\limits_{j\in N}\varepsilon_j^{t+1} = (1-(1+n)/2)\sum\limits_{j\in N}\varepsilon_j^{t+1}.$

For $\gamma_j^{t+1}\equiv 0\ \ \forall j\in N$, the result is $\sum\limits_{j\in N}\varepsilon_j^{t+1} = \sum\limits_{j\in N}\varepsilon_j^{t}.$

Thus, if $\sum_{j\in N}\varepsilon_j^t \neq 0$, then there exists a value of the parameter $\tilde{\gamma}^{t+1}$ such that

$$\sum_{j\in N}\varepsilon_j^{t+1} = (1-\tilde{\gamma}^{t+1}(1+n)/2)\sum_{j\in N}\varepsilon_j^t. \tag{10}$$

The value of the parameter $\tilde{\gamma}^{t+1}$ is given by

$$\tilde{\gamma}^{t+1} = \left(\sum_{j\in N}\gamma_j^{t+1} + \sum_{j\in N}\gamma_j^{t+1}\varepsilon_j^t \Big/ \sum_{j\in N}\varepsilon_j^t\right)\Big/\left(1+n\right), \tag{11}$$

where $\tilde{\gamma}^{t+1}$ *is the arithmetic weighted mean* of the set of the parameters $\{\gamma_j^{t+1}\}_{j\in N}$ with weights $\{\omega_j^t\}_{j\in N}$, i.e., $\tilde{\gamma}^{t+1} = \frac{\sum_{j\in N}\omega_j^t\gamma_j^{t+1}}{\sum_{j\in N}\omega_j^t}$ and the weights are the real numbers $\omega_i^t = \sum_{j\in N}\varepsilon_j^t + \varepsilon_i^t$.

The negative contribution of an agent to the parameter $\tilde{\gamma}^{t+1}$ is possible if the sign of his residuals does not coincide with that of the total residuals $\sum_{j\in N}\varepsilon_j^t = Q^* - Q^t$ of all agents.

Let some time instants t_0 and τ, $\tau > t_0$, be fixed. Assuming that $\sum_{j\in N}\varepsilon_j^{t_0}\neq 0$ and $\tilde{\gamma}^t \neq \frac{2}{1+n}$, for $t_0+1\leqslant t\leqslant \tau$ we sequentially obtain

$$\sum_{j\in N}\varepsilon_j^{t_0+\tau} = \sum_{j\in N}\varepsilon_j^{t_0}\prod_{t=t_0+1}^{\tau}(1-\tilde{\gamma}^t(1+n)/2) \tag{12}$$

based on (10). (The cases in which $\tilde{\gamma}^t = \frac{2}{1+n}$ and, accordingly, $\sum_{j\in N}\varepsilon_j^t = 0$, have been discussed in the note to Proposition 2.)

Let us introduce the notations

$$T = \{t_0 + 1, \ldots, \tau\}, \quad T_+ = \{t \in T | 1 - \tilde{\gamma}^t(1+n)/2 > 0\},$$
$$T_- = \{t \in T | 1 - \tilde{\gamma}^t(1+n)/2 < 0\}.$$

Two cases of possible inequalities can occur at some time instants from set $T_+ : 1 \leqslant 1 - \tilde{\gamma}^t(1+n)/2$ (if $\tilde{\gamma}^t \leqslant 0$) and $0 < \tilde{\gamma}^t(1+n)/2 < 1$ (if $0 < \tilde{\gamma}^t < \frac{2}{1+n}$). The first case is "unfavorable" and the second "favorable" for the convergence of the process. Similarly, in the set T_-, there also exist favorable time instants when $-1 < 1 - \tilde{\gamma}^t(1+n)/2 < 0$ (if $\frac{2}{1+n} < \tilde{\gamma}^t < \frac{4}{1+n}$), and unfavorable ones when $1 - \tilde{\gamma}^t(1+n)/2 < -1$ (if $\frac{4}{1+n} \leqslant \tilde{\gamma}^t$).

Using the Cauchy inequality between the arithmetic mean and the geometric mean, we have

$$\prod_{t\in T_+} (1 - \tilde{\gamma}^t(1+n)/2) \leqslant \left[\frac{1}{t_+}\sum_{t\in T_+} (1 - \tilde{\gamma}^t(1+n)/2)\right]^{t_+} = \left[1 - \bar{\tilde{\gamma}}^\tau_+(1+n)/2\right]^{t_+},$$
$$\prod_{t\in T_-} (\tilde{\gamma}^t(1+n)/2 - 1) \leqslant \left[\frac{1}{t_-}\sum_{t\in T_-} (\tilde{\gamma}^t(1+n)/2 - 1)\right]^{t_-} = \left[\bar{\tilde{\gamma}}^\tau_-(1+n)/2 - 1\right]^{t_-},$$

where $\bar{\tilde{\gamma}}^\tau_+ = \frac{1}{t_+}\sum_{t\in T_+} \tilde{\gamma}^t$ and $\bar{\tilde{\gamma}}^\tau_- = \frac{1}{t_-}\sum_{t\in T_-} \tilde{\gamma}^t$ are the mean values of the weighted parameter $\tilde{\gamma}^t$ over the sets T_+ and T_-, respectively; $t_+(t_-)$ is the number of time instants in the set T_+ (T_-, respectively), and $\tau = t_+ + t_-$.

In view of (12), it follows that

$$\left|\sum_{j\in N} \varepsilon_j^{t_0+\tau}\right| \leqslant \left[1 - \bar{\tilde{\gamma}}^\tau_+(1+n)/2\right]^{t_+} \left|1 - \bar{\tilde{\gamma}}^\tau_-(1+n)/2\right|^{t_-} \left|\sum_{j\in N} \varepsilon_j^{t_0}\right|. \tag{13}$$

Inequality (13) and Proposition 2 lead to the following result regarding the convergence of the process (8).

Proposition 3. *Model* (8) *converges to the equilibrium if, for an arbitrary* t_0 *and* $\tau > t_0$, *the expression* $\left[1 - \bar{\tilde{\gamma}}^\tau_+(1+n)/2\right]^{t_+} \left|1 - \bar{\tilde{\gamma}}^\tau_-(1+n)/2\right|^{t_-}$ ($\tau = t_+ + t_-$) *vanishes as* $\tau \to \infty$.

Corollary 1. *If* $\exists t_0$ *such that* $0 < \bar{\tilde{\gamma}}^\tau_+$ *and* $\bar{\tilde{\gamma}}^\tau_- < \frac{4}{n+1}$ $\forall \tau > t_0$, *then model* (8) *converges to the equilibrium.*

Let us explain the validity of this corollary. If $t \in T_+$, then $\tilde{\gamma}^t < \frac{2}{1+n}$ and $\bar{\tilde{\gamma}}^\tau_+ < \frac{2}{1+n}$. If $t \in T_-$, then $\tilde{\gamma}^t > \frac{2}{1+n}$ and, consequently, $\bar{\tilde{\gamma}}^\tau_- > \frac{2}{1+n}$. Therefore, for the inequalities $\left[1 - \bar{\tilde{\gamma}}^\tau_+(1+n)/2\right] < 1$ and $\left|1 - \bar{\tilde{\gamma}}^\tau_-(1+n)/2\right| < 1$ to be true, it suffices to require $0 < \bar{\tilde{\gamma}}^\tau_+$ and $\bar{\tilde{\gamma}}^\tau_- < \frac{4}{1+n}$.

In addition, by Propositions 1 and 3, model (5)–(6) converges for the sets of parameters $\{\gamma_i^t\}$ whose averaged estimates satisfy the inequalities $0 < \bar{\tilde{\gamma}}^\tau_+$ and $\bar{\tilde{\gamma}}^\tau_- < \frac{4}{1+n}$.

Corollary 2. *Let the parameters* $\tilde{\gamma}^t$, $t = (1, 2, \ldots, \tau)$, *be random variables, and let* $\bar{\tilde{\gamma}}^\tau = \frac{1}{\tau}\sum_{t=1}^{\tau} \tilde{\gamma}^t$ *converge in probability to the overall mean* $\bar{\tilde{\gamma}}$. *Then model* (8) *converges in probability to the equilibrium as* $\tau \to \infty$ *if* $\bar{\tilde{\gamma}} < \frac{4}{1+n}$.

Numerical calculation: a brief example. Consider model (1)–(3) with $n = 4$, $a = 100$, $b = 0.1$, $c = (20; 25; 20; 30)$, and $q^0 = (250; 250; 100; 200)$. All agents bear the same fixed costs of 500. By the formula $h_i = \frac{a - c_i}{b}$ we find $h = (800; 750; 800; 700)$.

When the agents are completely informed, the static Nash equilibrium q^* is the solution of the system of linear algebraic equations $q_i + Q = h_i$ $(i = \overline{1,4})$. As a result, $q_i^* = h_i - \frac{1}{5}\sum_{j=1}^{4} h_j$ and $q^* = (190; 140; 190; 90)$.

One fragment of the dynamics for $n = 4$

Time instants	Residuals of agents' actions				Step parameters							
t	ε_1	ε_2	ε_3	ε_4	γ_1	γ_2	γ_3	γ_4	$\sum \varepsilon_j$	$\tilde{\gamma}$	$\bar{\bar{\gamma}}_+$	$\bar{\bar{\gamma}}_-$
1	2	3	4	5	6	7	8	9	10	11	12	13
0	−60.0	−110.0	90.0	−110.0					−190.0			
1	−10.0	−65.0	105.0	−72.5	0.40	0.30	0.30	0.25	−42.5	0.31	0.31	
2	0.5	−48.9	95.6	−61.0	0.40	0.30	0.30	0.20	−13.8	0.27	0.29	
3	3.2	−39.5	62.9	−51.7	0.40	0.30	0.80	0.25	−25.1	−0.33	0.08	
4	7.5	−29.8	57.2	−42.1	0.40	0.30	0.30	0.25	−7.1	0.29	0.13	
5	7.5	−22.4	39.7	−35.9	0.40	0.40	0.70	0.25	−11.2	−0.23	0.06	
6	8.2	−15.7	35.4	−30.0	0.40	0.40	0.30	0.25	−2.1	0.32	0.11	
7	6.1	−13.0	30.4	−26.0	0.70	0.30	0.30	0.25	−2.5	−0.08	0.08	
8	5.4	−9.9	26.2	−22.4	0.40	0.40	0.30	0.25	−0.7	0.28	0.10	
9	5.1	−8.3	22.4	−19.5	0.10	0.30	0.30	0.25	−0.3	0.24	0.12	
10	4.7	−7.0	19.1	−17.1	0.20	0.30	0.30	0.25	−0.3	−0.03	0.10	
11	3.4	−3.3	14.4	−10.1	0.60	1.00	0.50	0.80	4.3	5.61		5.61
12	−0.5	−3.8	5.0	−7.2	1.00	1.00	1.00	1.00	−6.5	1.00		3.31
13	3.0	1.3	5.4	−0.4	1.00	1.00	0.50	1.00	9.3	0.98		2.53
14	−3.2	−4.0	−2.0	−4.9	1.00	1.00	1.00	1.00	−14.0	1.00		2.15
15	5.4	5.0	6.0	4.6	1.00	1.00	1.00	1.00	21.0	1.00		1.92
16	−7.8	−8.0	−7.5	−8.2	1.00	1.00	1.00	1.00	−31.5	1.00		1.77
17	11.9	11.8	12.0	11.7	1.00	1.00	1.00	1.00	47.3	1.00		1.66
18	−17.7	−17.8	−17.6	−17.8	1.00	1.00	1.00	1.00	−71.0	1.00		1.57
19	26.6	26.6	26.7	26.6	1.00	1.00	1.00	1.00	106.5	1.00		1.51
20	−39.9	−6.7	−26.6	−39.9	1.00	0.50	0.80	1.00	−113.1	0.82		1.44
21	36.6	23.3	8.3	36.6	1.00	0.50	0.50	1.00	104.8	0.77		0.77
22	−34.1	−8.7	−19.9	−34.1	1.00	0.50	0.50	1.00	−96.9	0.77		0.77
23	31.4	17.7	9.3	31.4	1.00	0.50	0.50	1.00	89.7	0.77		0.77
24	−11.0	−9.2	−15.5	−29.2	0.70	0.50	0.50	1.00	−64.8	0.69		0.75
25	26.9	9.3	4.6	−5.7	1.00	0.50	0.50	0.50	35.2	0.62		0.72
26	−4.1	−1.8	−5.3	−20.4	1.00	0.50	0.50	1.00	−31.7	0.76		0.73
27	13.8	6.6	3.9	5.6	1.00	0.50	0.50	1.00	29.9	0.78		0.74
28	5.0	−2.5	−4.5	−12.1	0.40	0.50	0.50	1.00	−14.2	0.59		0.72
29	9.6	1.6	0.1	1.0	1.00	0.50	0.50	1.00	12.4	0.75		0.72
30	−1.4	−1.9	−3.0	−5.7	1.00	0.50	0.50	1.00	−12.0	0.79		0.73

Note: the notations in the table header are provided without the superscript "t," which is supposed to be the corresponding time instant (period or iteration) from column 1.

The transition to model (8) is performed using the formula for calculating the residuals $\varepsilon_i^t = q_i^* - q_i^t$. We have $\varepsilon^0 = (-60; -110; 90; -110)$.

For each time instant (column 1), the current values of the residuals ε_i^t (columns 2–5), and the current values of the parameters γ_i^t (columns 6–9), the table presents the *arithmetic weighted means* $\tilde{\gamma}^t$ of the sets of parameters $\{\gamma_j^t\}$ (column 11) calculated using formulas (8) and (11).

The convergence of the process depends on what steps γ the agents choose. As is known, if the agents take the maximum (unit) steps γ, the process converges only in the case $n = 2$. For $n = 4$, the process diverges if all agents take steps exceeding 0.8. Generally speaking, for $n = 4$ the question of convergence remains open if the agents act differently: some agents choose steps greater than 0.8 while the others smaller.

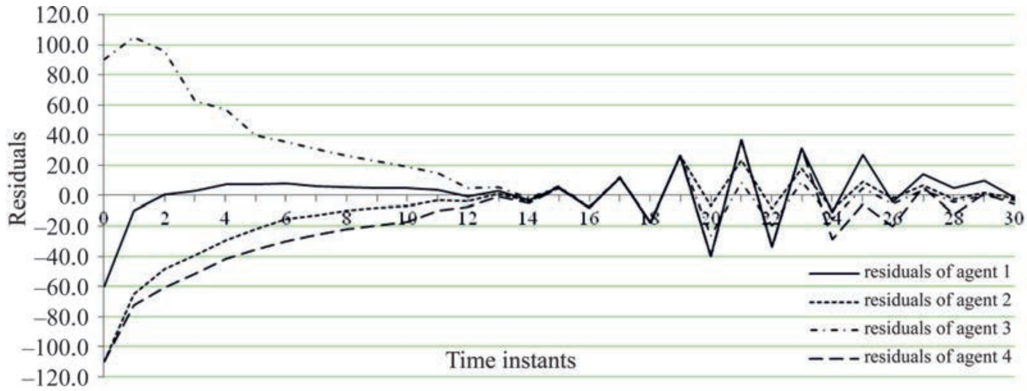

Fig. 1. The dynamics of action residuals of individual agents.

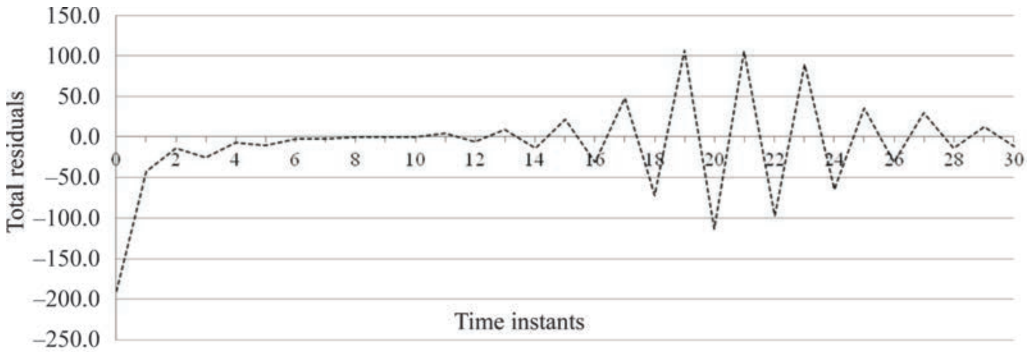

Fig. 2. The dynamics of total residuals.

To visualize the conditions introduced above and indicate the convergence of the process, we separate 3 intervals containing 10 time instants each. During the first 10 time instants, the agents intentionally take small steps to ensure convergence. During the next 10 time instants, larger steps are selected to show the divergence of the process. During the final 10 time instants, the process is again made convergent due to small steps taken by some agents.

Consider the process dynamics in more detail. At each of the first 10 time instants, the weighted means $\tilde{\gamma}^t$ do not exceed $2/(1+n) = 0.4$ (there are two negative values), and the aggregate estimate $\bar{\tilde{\gamma}}_+^\tau$ (their *arithmetic mean* over t time instants, see column 12) lies in the range $(0, 0.4)$. Therefore, we observe a trend for decreasing the absolute value of the total residuals: from $\varepsilon^0 = (-60; -110; 90; -110)$ and $\left|\sum_{j=1}^4 \varepsilon_j^0\right| = 190$ to $\varepsilon^{10} = (4.7; -7.0; 19.1; -17.1)$ and $\left|\sum_{j=1}^4 \varepsilon_j^{10}\right| = 0.3$.

During the next 10 time instants (from the 11th to the 20th), $\tilde{\gamma}^t$ and the aggregate dynamics estimate $\bar{\tilde{\gamma}}_-^\tau$ (column 13) exceed $4/(1+n) = 0.8$, which determines a trend for increasing the absolute value of the total residuals to $\varepsilon^{20} = (-39.9; -6.7; -26.6; -39.9)$ and $\left|\sum_{j=1}^4 \varepsilon_j^{20}\right| = 113.1$.

From the 21st to the 30th time instants, the *arithmetic weighted means* $\tilde{\gamma}^t$ and their *arithmetic mean* $\bar{\tilde{\gamma}}_-^\tau$ lie in the range $(0.4, 0.8)$. In other words, the convergence conditions of Proposition 3 and Corollary 1 are satisfied for $t_0 = 21$, and we observe a trend for decreasing the total residuals: $\varepsilon^{30} = (-1.4; -1.9; -3.0; -5.7)$ and $\left|\sum_{j=1}^4 \varepsilon_j^{30}\right| = 12.0$.

Also, we have $\bar{\tilde{\gamma}}^{30} = \frac{1}{30}\sum_{t=1}^{30} \tilde{\gamma}^t = 0.76$. According to the table data, the dynamics evolve towards the static Nash equilibrium.

Figures 1 and 2 demonstrate the "synchronization" of the dynamics of individual and total costs. Obviously, if the residuals of each agent converge, the total residuals of all agents will converge as well. The converse also follows from Proposition 2: if the total residuals vanish, the residuals of each agents will vanish too.

4. CONCLUSIONS

This paper has considered the problem of identifying convergence conditions of collective behavior models to equilibrium. The peculiarity and novelty of the approach presented above consist in the following. Traditionally, such conditions are formulated as admissible ranges for the values of the parameters γ for each agent at each time instant: if each agent chooses his parameter within such a range at each time instant, the dynamics surely converge. In this study, the convergence conditions have been formulated for a set of time instants (i.e., over a time interval) as an admissible range for the *arithmetic mean of the parameters*, which are the *arithmetic weighted means* of the set $\{\gamma_j^t\}_{j\in N}$ for each time instant t included in the time interval. If the mean over the time interval belongs to the range, the dynamics of each agent will evolve towards the equilibrium. In addition, if the set of time instants is unbounded, the dynamics of each agent will surely converge to the equilibrium.

APPENDIX

Proof of Proposition 1. First, we note the existence of a unique static Nash equilibrium in model (9) and $\varepsilon_i^* = 0 \quad \forall i \in N$.

The validity of Proposition 1 follows from the fact that one process is obtained from the other via a transformation based on the residuals as the change of variables. Therefore, the convergence of the process (5)–(6) directly implies the convergence of the process (8).

Now we employ mathematical induction to show that the convergence of the process (5)–(6) follows from the convergence of the process (8).

Let ε^0 be the initial data vector of the process (8) under which it converges. We define by $q^0 = q^* - \varepsilon^0$ the initial data vector of the process (5)–(6) and calculate ε^1 using formula (8). In view of $h_i = Q^* + q_i^*$ and (6), we establish that $q^1 = q^* - \varepsilon^1$:

$$\begin{aligned} q_i^* - \varepsilon_i^1 &= q_i^* - \varepsilon_i^0 - \gamma_i^1\left(-\frac{\sum_{j\neq i}\varepsilon_j^0}{2} - \varepsilon_i^0\right) = q_i^0 - \gamma_i^1\left(-\frac{\sum_{j\neq i}(q_j^* - q_j^0)}{2} - (q_i^* - q_i^0)\right) \\ &= q_i^0 + \gamma_i^1\left(\frac{2q_i^* + \sum_{j\neq i}(q_j^* - q_j^0)}{2} - q_i^0\right) = q_i^0 + \gamma_i^1\left(\frac{Q^* + q_i^* - \sum_{j\neq i} q_j^0}{2} - q_i^0\right) \\ &= q_i^0 + \gamma_i^1\left(\frac{h_i - \sum_{j\neq i} q_j^0}{2} - q_i^0\right) = q_i^0 + \gamma_i^1(x_i(q_{-i}^0) - q_i^0) = q_i^1. \end{aligned}$$

By analogy, it can be shown that $q^t = q^* - \varepsilon^t$ implies $q^{t+1} = q^* - \varepsilon^{t+1}$ for ε^{t+1} calculated using formula (8). Therefore, if $\varepsilon^t \to 0$, then $q^t \to q^*$.

The proof of Proposition 1 is complete.

Proof of Proposition 2. Suppose that $\sum_{j\in N}\varepsilon_j^t \to 0$. Then $\forall\delta > 0\ \exists t'$ such that $\left|\sum_{j\in N}\varepsilon_j^t\right| < \delta$ for $t > t'$.

Letting $t = t' + 1$ and using (8) yield

$$\left|\varepsilon_i^{t+1}\right| \leqslant (1-\gamma_i^{t+1}/2)\left|\varepsilon_i^t\right| + \gamma_i^{t+1}/2\left|\sum_{j\in N}\varepsilon_j^t\right| < (1-\gamma_i^{t+1}/2)\left|\varepsilon_i^t\right| + \delta\gamma_i^{t+1}/2 < (1-\gamma_i^{t+1}/2)\left|\varepsilon_i^t\right| + \delta.$$

Based on (8) and the previous inequality, we have

$$\left|\varepsilon_i^{t+2}\right| \leqslant (1-\gamma_i^{t+2}/2)\left|\varepsilon_i^{t+1}\right| + \gamma_i^{t+2}/2\left|\sum_{j\in N}\varepsilon_j^{t+1}\right| < (1-\gamma_i^{t+2}/2)\left|\varepsilon_i^{t+1}\right| + \delta\gamma_i^{t+2}/2$$
$$< (1-\gamma_i^{t+2}/2)(1-\gamma_i^{t+1}/2)\left|\varepsilon_i^{t}\right| + \left[(1-\gamma_i^{t+2}/2) + \gamma_i^{t+2}/2\right]\delta$$
$$< (1-\gamma_i^{t+2}/2)(1-\gamma_i^{t+1}/2)\left|\varepsilon_i^{t}\right| + \delta.$$

Following the method of mathematical induction, we assume that at some time instant $(t+m)$,

$$\left|\varepsilon_i^{t+m}\right| < \left|\varepsilon_i^{t}\right| \times \prod_{l=1}^{m}(1-\gamma_i^{t+l}/2) + \delta.$$

Based on (8) and the last inequality,

$$\left|\varepsilon_i^{t+m+1}\right| \leqslant (1-\gamma_i^{t+m+1}/2)\left|\varepsilon_i^{t+m}\right| + \gamma_i^{t+m+1}/2\left|\sum_{j\in N}\varepsilon_j^{t+m}\right| < (1-\gamma_i^{t+m+1}/2)\left|\varepsilon_i^{t+m}\right| + \delta\gamma_i^{t+m+1}/2$$
$$< (1-\gamma_i^{t+m+1}/2)\left[\left|\varepsilon_i^{t}\right| \times \prod_{l=1}^{m}(1-\gamma_i^{t+l}/2) + \delta\right] + \delta\gamma_i^{t+m+1} = \left|\varepsilon_i^{t}\right| \times \prod_{l=1}^{m+1}(1-\gamma_i^{t+l}/2) + \delta.$$

In other words, the same inequality holds at time instant $(t+m+1)$.

In the inequalities derived for each time instant, the first term vanishes as $m\to\infty$ under $\gamma\neq 0$ and an arbitrarily small number δ. Therefore, the conclusion follows, and the proof of Proposition 2 is complete.

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