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NONLINEAR SYSTEMS

Pseudo-Optimal Solution of a Variational Problem with a Free Right End and a Specified Time of Ending the Transient Process

V. N. Afanas'ev[*,**]

[*] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
[**] *National Research University Higher School of Economics, Moscow, Russia*
e-mail: afanval@mail.ru

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Abstract—The problem of optimal control of the final state of the system in a sense is the core of any other optimization problem. The formulation of such problems includes a description of the dynamic object itself, constraints imposed on the controls and states of the object, and a quality functional, in general, the Bolz's functional. The necessary optimality conditions in the problem of synthesizing the corresponding controls have written in the form of the canonical Euler–Lagrange system with the assignment of the corresponding boundary conditions. The synthesis of the corresponding controls faces the problem of the need to find solutions to boundary value problems, which has usually realized by numerical methods. The paper proposes an alternative to such methods for solving two-point boundary value problems based on the assumption of the validity of R. Bellman's inverse optimality principle, which consists in preserving the functional relationship between the components of a two-point boundary value problem in the entire control interval. The obtained theoretical results have confirmed by modeling the control system with synthesized control.

Keywords: optimal control, Hamiltonian of the system, canonical Euler–Lagrange system, boundary conditions of the canonical system

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1. INTRODUCTION

Control systems described by ordinary differential equations quite fully reflect many real processes and therefore are the most common objects to which mathematical methods of constructing controls are applied. The task of constructing an optimal dynamic control system with complete information in relation to a set of goals, a quality functional, a set of admissible controls, a set of states and the initial state of the object at the start of control is to find a control belonging to an admissible set of controls that minimizes a given quality functional on the solutions of the object equation [1–5]. The solution to the problem has carried out using variation methods [4–7].

The synthesis of an optimal control system has carried out using necessary and sufficient conditions for the minimum of the quality functional [2–5]. It should be note that the existence of optimal control is not necessary: the set of admissible controls may not include controls at all that transfer the object from the initial state to a given set of goals.

The necessary conditions for the existence of optimal control are described by a two-point boundary value problem and the condition for choosing the control itself in the form of a certain function depending on the behavior of the Hamiltonian on the optimal trajectory. The main problem of finding optimal control is associated with finding a solution to a two-point boundary value problem. Such boundary value problems for systems of differential equations have rarely solved analytically and require the use of numerical methods, which are divided into two types — iterative and non-iterative. For linear problems, a solution can be obtained without iterations, while iterative methods are indispensable when solving nonlinear problems [8–12]. Such methods include: Euler's method, linear interpolation method, finite difference method, shooting method [10]. The essence of this method is to reduce the boundary value problem to a multiple solution of the Cauchy problem. A certain development of the shooting method is the differential sweep method, in which auxiliary Cauchy problems have solved not for the original differential equation, but for other equations of a lower order.

In general, the problem of solving two-point problems is relevant, and methods for solving it have proposed today, some of them based on the use of neural networks [13–19].

One of the methods for solving boundary value problems is the method of successive approximations. This method, which has not yet received wide application, reduces the original problem to a certain sequence of linear-quadratic problems. The paper [15] presents a method for synthesizing optimal control with feedback for one class of nonlinear systems using a quadratic criterion. This method has based on a special method of successive approximations, the convergence of which allows one to prove the existence of optimal control and obtain a procedure for constructing it. The paper considers an analytical and numerical study of this method and its implementation in the MathCloud system.

To solve variation problems, two non-classical methods has intensively developed since the end of the 20th century. The first method has based on the Differential Transform Method (DTM), which seeks an analytical solution in the form of a certain functional series [18–20]. The second method is based on a neural network based on mathematical models of natural sciences (Physics-Informed Neural Network, PINN), where artificial intelligence in the form of a neural network is used to solve a differential equation [19, 20].

The first method, DTM, has characterized by its flexibility both in the form of the differential equation and in the boundary conditions. One of its strengths is its scalability to handle approximate solutions of various orders, and sometimes it even allows predicting the exact solution based on the form of the found coefficients of the original equation. The disadvantage of this method is the difficulty of automating the process of solving a given problem.

The second method (PINN) uses neural networks to solve the corresponding differential equation. One of the advantages of this method is its relatively simple implementation and flexibility. Once a differential equation model has trained, it can provide solutions for different grids without recalculating the problem each time.

In this paper, an alternative to numerical methods for solving two-point boundary value problems is proposed, based on the assumption of the validity of R. Bellman's inverse optimality principle [21], which consists in preserving the functional connection between the components of a two-point boundary value problem in the entire control interval.

2. TERMINAL OPTIMAL CONTROL PROBLEM

2.1. Statement of the Problem

The article considers an object that is described by an ordinary differential equation

$$\begin{gathered}\frac{d}{dt}x(t) = f(t, x(t)) + g(t)u(t), \quad x(t_0) = x_0, \\ f, g : T \times \Omega_s \to \mathbb{R}^n, \quad (t, x) \to f(t, x), \, g(t).\end{gathered} \tag{1}$$

Here T is the control interval $[t_0, t_f]$; $u \in \{u(t) \in \mathbb{R}^r \subset U, t \in [t_0, t_f]\}$ is the control to be found, the matrices $f(t,x)$, $g(t)$ are real and continuous; U is a compact set of admissible controls; Ω_x is the set of trajectories $x(\cdot) : [t_0, t_f] \to \mathbb{R}^n$, that satisfy the initial condition of $x(t_0) = x_0$ and the differential inclusion $\{dx(t)/dt\} \in \mathrm{co}\{[f(t,x(t)) + g(t)u(t)] : u \in U\}$.

It is assumed that for all (t,x) the pair of $\{f(t,x(t)), g(t)\}$ is controllable. In addition, we will assume that the functions $f(t,x(t))$, $g(t)$ are smooth enough so that one and only one solution (1) of the $x(t, t_0, x_0) \in \Omega_x$ passes through any $(t_0, x_0) \in T \times X_0$.

The set of goals in this problem is defined as $S \in \mathbb{R}^n \times [t_0, t_f]$. The elements of the set of goals are pairs (t,x), consisting of a state X and a point t from the interval of definition of the system $[t_0, t_f]$.

Considering the problem of synthesis of the control law, we introduce the Boltz's functional

$$J(x(\cdot), u(\cdot)) = K(x(t_f)) + \int_{t_0}^{t_f} \{L(t, x(t), u(t))\} dt. \tag{2}$$

Let be $L(t, x(t), u(t))$ a continuous real function defined on $\mathbb{R}^n \times \mathbb{R}^r \times [t_0, t_f]$, and $K(x(t_f))$ be a real function on $\mathbb{R}^n \times t_f$.

Proposition 1. *On the properties of a function $f(t,x)$ and a functional $L(t,x,u)$* [2–4].

1. *Functions $f(t,x)$, $L(t,x,u)$ and $K(x(t_f))$ are continuous and satisfy the constraints*

$$\|f(t,x)\| \le (1 + \|x\|)R_f, \quad |L(t,x,u)| \le (1 + \|x\|)R_L, \quad K(x(t)) \le (1 + \|x\|)R_K$$

for all $(t,x,u) \in [t_0, t_f] \times \mathbb{R}^n \times U$ (here R_f, R_L, R_K are positive numbers).

2. *We will assume that the functions $f(t,x(t))$ and $L(t,x,u)$ satisfy the Lipschitz's condition for the variable x:*

$$\|f(t, x+y) - f(t,x)\| + |L(t, x+y, u) - L(t,x,u)| \le \lambda_{up}\|y\|$$

for all $(t,x) \in [t_0, t_f] \times \mathbb{R}^n$, $y \in \mathbb{R}^n$.

3. *Functions $f(t,x(t))$, $L(t,x(t),u(t))$ and their partial derivatives with respect to t, x, u, i.e.*

$$\begin{aligned} &f_i(t,x(t)), \quad \frac{\partial f_i(t,x(t))}{\partial t}, \quad \frac{\partial f_i(t,x(t))}{\partial x}, \quad i = 1, \ldots, n, \\ &L(t,x(t),u(t)), \quad \frac{\partial L(t,x(t),u(t))}{\partial t}, \quad \frac{\partial L(t,x(t),u(t))}{\partial x}, \quad \frac{\partial L(t,x(t),u(t))}{\partial u}, \end{aligned}$$

are continuous on $\mathbb{R}^n \times \mathbb{R}^r \times [t_0, t_f]$.
Note that for a number of problems the existence of continuous partial derivatives $L(t,x(t),u(t))$ with respect to u is not required.

4. *The finite cost function $K(x(t_f))$, defined on $\mathbb{R}^n \times t_f$, is a real function such that*

$$K(x(t)), \quad \frac{\partial K(x(t))}{\partial x}, \quad \frac{\partial^2 K(x(t))}{\partial x^2}$$

are continuous on $\mathbb{R}^n \times [t_0, t_f]$.

Additional assumptions about the properties of the vector $f(t,x(t))$ will be made below (Section 4 of the article).

Let the element $\xi = (x(t), u(t), t_0, t_f)$, for which all the specified conditions and constraints of the problem are met, be an admissible controlled process. We will consider the admissible elements $\xi = (x(t), u(t), t_0, t_f)$ in the stated problem to be functions of the class $x(\cdot) \in C^1([t_0, t_f], \mathbb{R}^n)$, $u(\cdot) \in C([t_0, t_f], \mathbb{R}^r)$. The type of restrictions imposed on control are specified below.

The control problem consists of constructing an optimal strategy, i.e. finding an admissible controlled process $\xi^0 = (x^0(t), u^0(t), t_f)$, that minimizes a functional of the form (2) on an object (1), where the control objective is specified in the form $S \in \mathbb{R}^n \times [t_0, t_f]$.

2.2. Necessary Optimality Conditions

Let us write the Hamiltonian of the system $H(t, x(t), u(t), \lambda(t))$:

$$H(t, x(t), u(t), \lambda(t)) = L(t, x(t), u(t)) + \lambda^{\mathrm{T}}(t)[f(t, x) + g(t)u(t)]. \tag{3}$$

Here $\lambda(t)$ is the Lagrange function.

The necessary optimality conditions have the form [1, 2, 7]

$$\frac{d}{dt}x(t) = \left\{\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial \lambda}\right\}^{\mathrm{T}}, \quad x(t_0) = x_0, \tag{4}$$

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x}\right\}^{\mathrm{T}}, \quad \lambda(t_f) = \left\{\frac{\partial K(x(t_f))}{\partial x}\right\}^{\mathrm{T}}. \tag{5}$$

It is known [1–4] that the optimal control $u(t)$, the synthesis of which is performed using the Hamiltonian (3), leads to the need to solve the two-point boundary value problem (4), (5). However, it should be noted that the Hamiltonian does not contain any information about the functional relationship of the processes $x(t)$ and $\lambda(t)$.

In the problem under consideration, taking into account the assumptions made above, the necessary condition must be met

$$H^0(t, x(t), u(t), \lambda(t)) = \min_{u(t)} H(t, x(t), u(t), \lambda(t)), \quad t \in [t_0, t_f], \tag{6}$$

where $\lambda(t)$ is the solution of system (5). In the case where the set of admissible controls U coincides with the entire space $\mathbb{R}^n$, i.e. $U = \mathbb{R}^n$, then condition (6) can be satisfied only at a stationary point, i.e.

$$\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial u} = \frac{\partial L(t, x(t), u(t))}{\partial u} + g^{\top}(t)\lambda(t) = 0, \quad t \in [t_0, t_f]. \tag{7}$$

However, if the set U is closed and $U \neq \mathbb{R}^n$, then relation (7) is not satisfied in the general case and Pontryagin's principle [2–4] should be applied to synthesize optimal control.

Let us assume that there exists an optimal control satisfying the necessary conditions (6) or (7), which we write in the form

$$u(t) = -\varphi(x(t))\lambda(t), \tag{8}$$

where is a matrix function $\varphi(x(t)) \in \mathbb{R}^{r \times n}$ such that

$$g(t)\varphi(x(t)) \leq \Phi \in \mathbb{R}^{n \times n}. \tag{9}$$

Here Φ is a parametrically specified matrix that determines the set of possible values of the parameters of the matrix $\varphi(x(t))$ given the known matrix $g(t)$; which means that the restrictions imposed on the control $u(t) \subset U$, are specified in the form of condition (9). The issue of the connection between the established restrictions on the parameters of the regulator will be considered in the fourth section of the article when specifying the type of functions $L(t, x(t), u(t))$ and $K(x(t_f))$ the quality criterion (2).

Thus, the synthesis of the optimal admissible process $\xi = \{x(t), u(t), t_0, t_f\}$ is replaced by the search for a matrix $\varphi(x(t))$, in which control $u(t) = -\varphi(x(t))\lambda(t) \subset U$ ensures the fulfillment of condition (9) and minimizes the functional (2).

Let us rewrite conditions (4), (5) taking into account (8):

$$\frac{d}{dt}x(t) = f(t, x(t)) - g(t)\varphi(x(t))\lambda(t), \quad x(t_0) = x_0, \tag{10}$$

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial f(t, x(t))}{\partial x}\right\}^{\mathrm{T}} \lambda(t) - \left\{\frac{\partial L(t, x(t), u(t))}{\partial x}\right\}^{\mathrm{T}}, \quad \lambda(t_f) = \left\{\frac{\partial K(x(t_f))}{\partial x}\right\}^{\mathrm{T}}. \tag{11}$$

Thus, the successful solution of the optimal control synthesis problem in the form $u(t) = -\varphi(x(t))\lambda(t)$ depends on the possibility of successfully solving the two-point boundary value problem (10), (11).

3. PSEUDO-OPTIMAL SOLUTION OF THE SYNTHESIS PROBLEM

It should be noted that the above stages of constructing optimal control are based on the use of the properties of the Hamiltonian (3), however, the Hamiltonian does not contain information regarding the boundary conditions of the control system. Only the condition $K(x(t_f))$ in the functional (2), that is $\lambda(t_f) = \{\partial K(x(t_f))/\partial x\}^{\mathrm{T}}$ in the boundary condition of equation (11), can connect the variables $x(t)$ and $\lambda(t)$ at the moment t_f. In addition, it should be noted that since system (1) is described by a nonlinear differential equation, then constraint (9), imposed on the optimal control, in the general case, may not contain control actions that achieve the control objective for given initial conditions $x_0 \in X_0$ and, moreover, provide uniform. asymptotic stability to the system [1–4].

To justify the proposed method for solving the problem of constructing pseudo-optimal control, we use the inverse Bellman optimality principle [4, 21], which assumes that optimal control has the property that for any initial state and the initial control used, the value of the criterion on the final interval is affected by the control on this interval and the value of the phase vector at the end of the interval. For the problem considered in this paper, we will clarify this definition.

Definition 1. *Inverse principle of optimality.* For the optimality of the admissible process $\xi^0 = (x^0(t), u^0(t), t_0, t_f)$ in problem (1), (2) it is necessary that for any of the subintervals $[t_0, \tau] \subset [t_0, t_f]$, $\tau \leq t_f$ the admissible process starting at time t_0, be optimal with respect to the control on this interval and the value of the phase vector at the end of the interval $x(\tau)$, which determines the value of the function of the auxiliary variable $\lambda(\tau)$, i.e. $\xi^0 = (x^0(t), u^0(t), t_0, \tau)$.

Based on this definition, we make the following proposition:

Proposition 2. *Noting that the condition connecting the variables $x(t)$ and $\lambda(t)$ at the moment t_f, is defined as $\lambda(t_f) = \{\partial K(x(t_f))/\partial x\}^{\mathrm{T}}$, we will assume, based on the inverse principle of optimality, that the relation $\tilde{\lambda}(\tau) = \{\partial K(\tilde{x}(\tau))/\partial \tilde{x}\}^{\mathrm{T}}$ is valid for all $\tau \in [t_0, t_f]$.*

The fulfillment of Proposition 2 means that for all $t \in [t_0, t_f]$ the variable $\tilde{\lambda}(t)$ is a function of the state $\tilde{x}(t)$.

Definition 2. The control $\tilde{u}(t) = -\varphi(\tilde{x}(t))\tilde{\lambda}(t)$, whose synthesis is based on the adoption of Proposition 2 is called **pseudo-optimal control**.

Let us rewrite equations (10), (11) with admissible control $\tilde{u}(t) = -\varphi(\tilde{x}(t))\tilde{\lambda}(t)$ in the form

$$\frac{d}{dt}\tilde{x}(t) = f(t, \tilde{x}(t)) - g(t)\varphi(\tilde{x}(t))\tilde{\lambda}(t), \quad \tilde{x}(t_0) = x_0, \tag{12}$$

$$\frac{d}{dt}\tilde{\lambda}(t) = -\left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} \tilde{\lambda}(t) - \left\{\frac{\partial L(t, \tilde{x}(t), u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}, \quad \tilde{\lambda}(t_f) = \lambda(t_f) = \left\{\frac{\partial K(x(t_f))}{\partial x}\right\}^{\mathrm{T}}. \tag{13}$$

When Proposition 2 is fulfilled, the first full derivatives of the main and auxiliary equations are related by the relation

$$\frac{d}{dt}\tilde{\lambda}(t)=\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}\frac{d}{dt}\tilde{x}(t),\quad t\in[t_0,t_f]. \tag{14}$$

Substituting $dx(t)/dt$ and $d\lambda(t)/dt$, defined in (12) and (13), we get

$$-\left\{\frac{\partial f(t,\tilde{x}(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}\tilde{\lambda}(t)-\left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}$$
$$=\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}f(t,\tilde{x}(t))-\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}g(t)\varphi(\tilde{x}(t))\tilde{\lambda}(t).$$

From where, bringing similar terms and taking into account (9), we obtain

$$\begin{gathered}\tilde{\lambda}(t)=\left[-\left\{\frac{\partial f(t,\tilde{x}(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}+\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}\Phi\right]^{-1}\\ \times\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}f(t,\tilde{x}(t))+\left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}\right].\end{gathered} \tag{15}$$

It is obvious that the expression $\{\partial^2 K(\tilde{x}(t))/\partial\tilde{x}^2\}\,\Phi-\{\partial f(t,\tilde{x}(t))/\partial\tilde{x}\}^{\mathrm{T}}$ must not be negative or equal to zero. This condition can be ensured by choosing the appropriate elements of the terminal penalty function of the functional (2). Let us establish for definiteness that the matrix

$$\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}\Phi-\left\{\frac{\partial f(t,\tilde{x}(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}\right]^{-1}$$

is positive definite, which will be taken into account when analyzing the stability of a system with synthesized control.

Pseudo-optimal control of the form (8) will be determined by the following expression

$$\begin{gathered}\tilde{u}(t)=-\varphi(\tilde{x}(t))\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}\Phi-\left\{\frac{\partial f(t,\tilde{x}(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}\right]^{-1}\\ \times\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial\tilde{x}^2}\right\}f(t,\tilde{x}(t))+\left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial\tilde{x}}\right\}^{\mathrm{T}}\right].\end{gathered} \tag{16}$$

To establish the uniform asymptotic requirements that system (12) with control (16), must meet, we introduce the Lyapunov function

$$V_L(t)=\tilde{x}^{\mathrm{T}}(t)\tilde{x}(t). \tag{17}$$

The total derivative of the Lyapunov function (17) has the form

$$\frac{d}{dt}V_L(t)=f^{\mathrm{T}}(t,\tilde{x}(t))\tilde{x}(t)-\tilde{\lambda}^{\mathrm{T}}(t)\Phi^{\mathrm{T}}\tilde{x}(t)+\tilde{x}^{\mathrm{T}}(t)f(t,\tilde{x}(t))-\tilde{x}^{\mathrm{T}}(t)\Phi\tilde{\lambda}(t)\le 0, \tag{18}$$

where $\tilde{\lambda}(t)$ is determined by equation (15). Thus, system (12) is uniformly asymptotically stable if the condition

$$f^{\mathrm{T}}(t,\tilde{x}(t))\tilde{x}(t)+\tilde{x}^{\mathrm{T}}(t)f(t,\tilde{x}(t))\le\tilde{\lambda}^{\mathrm{T}}(t)\Phi^{\mathrm{T}}\tilde{x}(t)+\tilde{x}^{\mathrm{T}}(t)\Phi\tilde{\lambda}(t) \tag{19}$$

is satisfied.

It can be noted that the fulfillment of condition (19) depends on the matrix Φ, which limits the control capabilities.

Let us formulate the theorem.

Theorem 1. *A pseudo-optimal solution to the control problem of a nonlinear dynamic object* (1) *with functional* (2) *exists if and only if*

$$\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} > 0, \quad \forall (t,\tilde{x}) \in [t_0,t_f]\times\Omega_x. \tag{20}$$

In this case, the trajectory $\tilde{x}^0(t)$ of system (1), *originating from $\tilde{x}(t_0)=x(t_0)$ and corresponding to pseudo-optimal control $\tilde{u}^0(t)$, is a solution to the equation*

$$\begin{gathered}\frac{d}{dt}\tilde{x}(t) = f(t,\tilde{x}(t)) - g(t)\varphi(\tilde{x}(t))\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1} \\ \times\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\} f(t,\tilde{x}(t)) + \left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right].\end{gathered} \tag{21}$$

Satisfaction of condition (20) *taking into account the restrictions* (9) *imposed on the parameters of the control action allows us to determine the region of initial conditions of the system* (21), *under which pseudo-optimal control* (16) *will provide the nonlinear system with uniform asymptotic stability:*

$$\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}_{t=t_0}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}_{t=t_0} \succ 0. \tag{22}$$

4. PROBLEM WITH A QUADRATIC QUALITY FUNCTIONAL

The problem of dynamic system control by a quadratic criterion is a classical problem of modern control theory. For linear systems, this problem has been completely solved and its solution has been fully and thoroughly described in many books (see, for example, [1–3]). For nonlinear systems, one of its first considerations has given in book [4]. It describes a certain scheme of successive approximations, which in some cases provides optimal control in the form of a law with feedback. A feature of the method is that the system operator changes from iteration to iteration. This complicates the analytical study of the problem and greatly complicates the computational procedure. Therefore, this method has not found wide application. In works [6–8], various aspects of the problem has considered (from constructing optimal control to proving the existence of its solution). In work [17], a modified scheme of successive approximations for the problem of optimal control of a nonlinear system by a quadratic quality functional has described. This scheme provides a solution to the problem in many important cases. However, the question of the convergence of the method on an arbitrary finite horizon remains open and is a topic for further research.

This article considers the problem of controlling an object of the form (1)

$$\begin{aligned}&\frac{d}{dt}x(t) = f(t,x(t)) + g(t)u(t), \quad x(t_0) = x_s,\\ &y(t) = Cx(t),\\ &f,g : T\times\Omega_x \to \mathbb{R}^n, \quad (t,x)\to f(t,x(t)),\, g(t),\end{aligned} \tag{23}$$

with the functional

$$J(x(\cdot),u(\cdot)) = \frac{1}{2}x^{\mathrm{T}}(t_f)Fx(t_f) + \frac{1}{2}\int\limits_{t_0}^{t_f}\left\{x^{\mathrm{T}}(t)C^{\mathrm{T}}QCx(t) + u^{\mathrm{T}}(t)Ru(t)\right\}dt. \tag{24}$$

The matrices F, Q, R are positive definite.

The control problem consists of constructing an optimal strategy, i.e. finding an admissible controlled process $\xi^0 = (x^0(t), u^0(t), t_0, t_f)$, that minimizes a functional of the form (24) on the object (23), where the control objective is specified in the form $S \in \mathbb{R}^n \times [t_0, t_f]$.

Let us assume that there exists an optimal control satisfying the necessary conditions (6) or (7), which we write in the form

$$u(t) = -\varphi(x(t))\lambda(t), \tag{25}$$

where the matrix function $\varphi(x(t)) \in \mathbb{R}^{r\times n}$, taking into account (10), is such that

$$g(t)R^{-1}g^{\mathrm{T}}(t) = g(t)\varphi(x(t)) \le \Phi \in \mathbb{R}^{n\times n}. \tag{26}$$

Here Φ is a parametrically specified matrix that elementwise determines the set of possible values of the parameters of the matrix $\varphi(x(t))$ with a known matrix $g(t)$, which means that the restrictions imposed on the control of $u(t) \subset U$, are specified in the form of condition (26).

Proposition 3. *By assigning the penalty matrices F, Q, R of the quality functional* (24) *it is possible, when determining the optimal control, to ensure that condition* (26) *is satisfied.*

The two-point boundary value problem in this subproblem, taking into account (26) has the form

$$\frac{d}{dt}x(t) = f(t, x(t)) - \Phi\lambda(t), \quad x(t_0) = x_0, \tag{27}$$

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial f(t, x(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\lambda(t) - C^{\mathrm{T}}QCx(t), \quad \lambda(t_f) = Fx(t_f). \tag{28}$$

Thus, the successful solution of the optimal control synthesis problem in the form $u(t) = -\varphi(x(t))\lambda(t)$ depends on the possibility of successfully solving the two-point boundary value problem (27), (28).

When Proposition 2 made above is fulfilled in the problem under consideration with a quadratic quality functional, the functions $\tilde{\lambda}(t)$ and $\tilde{x}(t)$ are linearly related by the relation $\tilde{\lambda}(t) = F\tilde{x}(t)$. Thus, the full derivatives of the main and auxiliary equations have the form

$$\frac{d}{dt}\tilde{\lambda}(t) = F\frac{d}{dt}\tilde{x}(t), \quad t \in [t_0, t_f]. \tag{29}$$

Substituting $d\tilde{x}(t)/dt$ and $d\tilde{\lambda}(t)/dt$, defined in (28), we have

$$-\left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\tilde{\lambda}(t) + F\Phi\tilde{\lambda}(t) = Ff(t, \tilde{x}(t)) + C^{\mathrm{T}}QC\tilde{x}(t).$$

Whence, by reducing similar terms, we obtain

$$\tilde{\lambda}(t) = \left[F\Phi - \left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1}\left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}}QC\tilde{x}(t)\right]. \tag{30}$$

Here we assume that

$$F\Phi - \left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} > 0, \quad \forall (t, \tilde{x}) \in [t_0, t_f] \times \Omega_x. \tag{31}$$

As can be seen, the fulfillment of condition (31) under the given constraints (26) depends on the purpose of the matrix F in the functional (24).

Control (27) taking into account (30) takes the form

$$\tilde{u}(t) = -\varphi(x(t)) \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}} QC\tilde{x}(t) \right]. \tag{32}$$

Let us write the original system (23) with control (32) (taking into account (25) and (26)):

$$\begin{aligned} &\frac{d}{dt}\tilde{x}(t) = f(t, \tilde{x}(t)) - \Phi \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}} QC\tilde{x}(t) \right], \\ &\tilde{x}(t_0) = x_0. \end{aligned} \tag{33}$$

To check the stability of the system (33), let's first introduce some notations: let

$$S(\tilde{x}(t)) = \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} > 0. \tag{34}$$

Let us rewrite (33) taking into account the notation made:

$$\frac{d}{dt}\tilde{x}(t) = [I - \Phi S(\tilde{x}(t))F] f(t, \tilde{x}(t)) - \Phi S(\tilde{x}(t)) C^{\mathrm{T}} QC\tilde{x}(t), \quad \tilde{x}(t_0) = x_0. \tag{35}$$

Let us introduce the Lyapunov function [22] in the form

$$V_L\tilde{x}(t) = \tilde{x}^{\mathrm{T}}(t)\tilde{x}(t). \tag{36}$$

The total derivative of the Lyapunov function is determined by the expression

$$\begin{aligned} \frac{d}{dt}V_L\tilde{x}(t) &= f^{\mathrm{T}}(t, \tilde{x}(t))[I - \Phi S(\tilde{x}(t))F]^{\mathrm{T}}\tilde{x}(t) - C^{\mathrm{T}} QC\tilde{x}(t) S^{\mathrm{T}}(\tilde{x}(t))\Phi^{\mathrm{T}}\tilde{x}(t) \\ &+ \tilde{x}^{\mathrm{T}}(t)[I - \Phi S(\tilde{x}(t))F] f(t, \tilde{x}(t)) - \tilde{x}^{\mathrm{T}}(t)\Phi S(\tilde{x}(t)) C^{\mathrm{T}} QC\tilde{x}(t) \le 0. \end{aligned} \tag{37}$$

By assigning the matrices Q and F in the functional (24) accordingly, it is possible to ensure that inequality (37) is satisfied, under which the dynamic system (33) has the property of uniform asymptotic stability.

Generalizing the result obtained above, we formulate Theorem 2.

Theorem 2. *A pseudo-optimal solution to the control problem of a nonlinear dynamic object* (23) *with functional* (24) *exists if and only if*

$$F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} > 0, \quad \forall (t, \tilde{x}) \in [t_0, t_f] \times \Omega_x. \tag{38}$$

In this case, the trajectory $\tilde{x}^0(t)$ *of system* (23), *originating from* $\tilde{x}(t_0) = x(t_0)$ *and corresponding to pseudo-optimal control* $\tilde{u}^0(t)$, *is a solution to the equation*

$$\frac{d}{dt}\tilde{x}(t) = f(t, \tilde{x}(t)) - \Phi \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}} QC\tilde{x}(t) \right]. \tag{39}$$

Satisfaction of condition (31) *allows us to determine the region of initial conditions of system* (33), *under which pseudo-optimal control* (32) *will provide the nonlinear system with uniform asymptotic stability:*

$$\left\{ F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right\}_{t=t_0} \succ 0. \tag{40}$$

The result presented above will be extended to a certain class of nonlinear systems represented using the SDC-parameterization method (State Dependent Coefficient, [23, 24]). To do this, we will make several assumptions.

Proposition 4. *The vector function* $f = f(t, \tilde{x}(t))$ *is continuous and differentiable by* $x \in \Omega_x$, *i.e.* $f(\cdot) \in C^1(\Omega_x)$.

Proposition 5. *Without loss of generality, we assume that the condition* $\tilde{x} = 0 \subset \Omega_x$ *is an equilibrium point of the system such that* $f(t, 0) = 0$.

Proposition 6. *We assume* [23] *that*

$$\frac{|f(t, \tilde{x}(t))|}{|\tilde{x}|} \to 0 \quad \textit{for} \quad |x| \to 0. \tag{41}$$

Taking into account the assumptions made regarding the property $f(t, \tilde{x}(t))$, we move from the description of the original system (23) to its SDC representation [23]. Writing $f(t, \tilde{x}(t))$ in the form

$$f(t, \tilde{x}(t)) = [A(t) + A(\tilde{x}(t))]x(t) = A(t, \tilde{x})\tilde{x}(t), \tag{42}$$

we have

$$\begin{aligned}
&\frac{d}{dt}\tilde{x}(t) = A(t, \tilde{x})\tilde{x}(t) + g(t)u(t), \quad \tilde{x}(t_0) = x_0,\\
&y(t) = C\tilde{x}(t),\\
&A(t, \tilde{x})\tilde{x}(t), g(t) : T \times \Omega_x \to \mathbb{R}^n, \quad (t, x) \to f(t, x),\, g(t).
\end{aligned} \tag{43}$$

Proposition 7. *Let us assume that the pair* $\{A(t, \tilde{x}), g(t)\}$ *is controlled,* $\{A(t, \tilde{x}), C\}$ *is observable.*

Let us write the equation of the object (39) with control takes the form

$$\begin{aligned}
&\frac{d}{dt}\tilde{x}(t) = A(t, \tilde{x})\tilde{x}(t) - \Phi\left[F\Phi - \left\{\frac{\partial A(t, \tilde{x})\tilde{x}(t)}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1} \times \left[FA(t, \tilde{x}) + C^{\mathrm{T}}QC\right]\tilde{x}(t),\\
&y(t) = C\tilde{x}(t), \quad x(t_0) = x_0.
\end{aligned} \tag{44}$$

Obviously, a solution to this equation exists if and only if a condition similar to condition (31) is satisfied

$$F\Phi - \left\{\frac{\partial A(t, \tilde{x})\tilde{x}(t)}{\partial \tilde{x}}\right\}^{\mathrm{T}} > 0, \quad \forall (t, \tilde{x}) \in [t_0, t_f] \times \Omega_x. \tag{45}$$

To check the stability of system (44) we first introduce some notations: let

$$S(\tilde{x}(t)) = \left[F\Phi - \left\{\frac{\partial A(t, \tilde{x})\tilde{x}(t)}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1} > 0, \tag{46}$$

i.e. the controlled system (44), taking into account the notations made, takes the form (35).

Taking into account (45) and taking into account (37), we write the total derivative of the Lyapunov function

$$\begin{aligned}
\frac{d}{dt}V_L(t, \tilde{x}) &= \tilde{x}^{\mathrm{T}}(t)A^{\mathrm{T}}(t, \tilde{x})\left[I - \Phi S(\tilde{x}(t))F\right]^{\mathrm{T}}\tilde{x}(t) - \tilde{x}^{\mathrm{T}}(t)C^{\mathrm{T}}QC\tilde{x}(t)S^{\mathrm{T}}(\tilde{x}(t))\Phi^{\mathrm{T}}\tilde{x}(t)\\
&+ \tilde{x}^{\mathrm{T}}(t)\left[I - \Phi S(\tilde{x}(t))F\right]A(t, \tilde{x})\tilde{x}(t) - \tilde{x}^{\mathrm{T}}(t)\Phi S(\tilde{x}(t))C^{\mathrm{T}}QC\tilde{x}(t) \le 0.
\end{aligned} \tag{47}$$

Thus, the asymptotic stability of the controlled system (44) must ensure that the condition is met

$$\begin{aligned}
&A^{\mathrm{T}}(t, \tilde{x})\left[I - \Phi S(\tilde{x}(t))F\right]^{\mathrm{T}} + \left[I - \Phi S(\tilde{x}(t))F\right]A(t, \tilde{x})\\
&\le C^{\mathrm{T}}QCS^{\mathrm{T}}(\tilde{x}(t))\Phi^{\mathrm{T}} + \Phi S(\tilde{x}(t))C^{\mathrm{T}}QC.
\end{aligned} \tag{48}$$

The fulfillment of conditions (26) and (48) for a given matrix Φ can be ensured by appropriately assigning the penalty matrices F, Q, R of the quality functional (24).

5. EXAMPLE

To illustrate the obtained theoretical results, let us consider an example [25] of the synthesis of pseudo-optimal control for a system of the form (23)

$$\begin{aligned}\frac{d}{dt}x_1(t) &= x_2(t) + \left[x_1^5(t) - x_1^3(t) - x_1(t) + x_1(t)x_2^4(t)\right] + x_1(t)u_1(t),\\ \frac{d}{dt}x_2(t) &= -x_1(t) - x_2(t) + \left[x_2^5(t) - x_2^3(t) - x_2(t) + x_2(t)x_1^4(t)\right] + x_2(t)u_2(t).\end{aligned} \tag{49}$$

Here, in accordance with Section 4 of the article,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} f_1(x(t)) \\ f_2(x(t)) \end{pmatrix} = \begin{pmatrix} x_1^5(t) - x_1^3(t) - x_1(t) + x_1(t)x_2^4(t) \\ x_2^5(t) - x_2^3(t) - x_2(t) + x_2(t)x_1^4(t) \end{pmatrix},$$

$$g(x(t))u(t) = \begin{pmatrix} g_1(x(t))u_1(t) \\ g_2(x(t))u_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t)u_1(t) \\ x_2(t)u_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

A quadratic quality functional of the form (24) has given with parameters

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Control interval $[t_0, t_f] = [0, 2]$.

Constraints imposed on controls for a given matrix F and

$$\Phi = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \tag{50}$$

are defined as follows:

$$g(x(t))R^{-1}g^{\mathrm{T}}(x(t)) = \begin{pmatrix} x_1^2(t) & 0 \\ 0 & x_2^2(t) \end{pmatrix} \le \Phi. \tag{51}$$

The synthesized pseudo-optimal control, according to (32), has the form

$$\tilde{u}(t) = -\varphi(x(t))\left[F\Phi - \left\{\frac{\partial f(t,x(t))}{\partial x}\right\}^{\mathrm{T}}\right]^{-1}\left[Ff(t,x(t)) + C^{\mathrm{T}}QCx(t)\right], \tag{52}$$

where the matrix $\varphi(x(t))$ satisfies condition (50), and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The constraints imposed on the controls for a given matrix F will be such that condition (40), is satisfied, in this example:

$$\begin{pmatrix} -5x_1^4(t) + 4x_1^2(t) + 1 - x_2^4(t) & -4x_2(t)x_1^3(t) + 1 \\ -4x_1(t)x_2^3(t) - 1 & -5x_2^4(t) + 4x_2^2 + 2 - x_1^4(t) \end{pmatrix} \succ 0, \forall \tilde{x}(t) \in \Omega. \tag{53}$$

System (49) with control (52) has the form

$$\begin{aligned}\frac{d}{dt}x(t) &= [Ax(t) + f(t,x(t))]\\ &\quad - \Phi\left[F\Phi - \left\{\frac{\partial f(t,x(t))}{\partial x}\right\}^{\mathrm{T}}\right]^{-1} F\left[Ax(t) + f(t,x(t)) + Qx(t)\right],\\ x(t_0) &= x_0.\end{aligned} \tag{54}$$

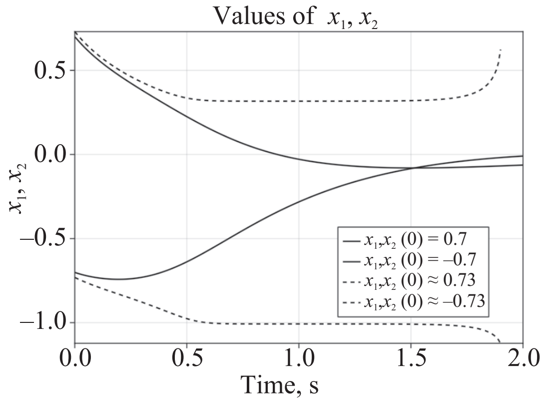

Fig. 1. Graphs of transient processes.

According to the condition (53), which defines the control capabilities of (52), the system is stable, as shown in the graphs (Figure), with initial conditions $|x_1(t_0)| \leq 0.72$; $|x_2(t_0)| \leq 0.72$ The system becomes unstable under the initial conditions $|x_1(t_0)| = 0.73$; $|x_2(t_0)| = 0.73$.

This is consistent with the conclusions of Theorem 2 ((40)) and the condition (53).

6. CONCLUSION

The Euler-Lagrange canonical system with the assignment of the corresponding boundary conditions is the basis of the necessary optimality conditions in the problem of synthesizing optimal controls for a dynamic object, while the synthesis of these controls has carried out based on the analysis of the Hamiltonian behavior on the optimal trajectory. However, the Hamiltonian does not contain any information on the relationship between the processes included in the canonical system. The success of synthesizing optimal control as a whole depends on the possibility of solving the canonical system with the specified boundary conditions. It should be noted that the functional relationship between the processes of a two-point boundary value problem exists only in the partial derivative with respect to the first term of the Bolza quality functional. In this paper, an alternative to numerical methods for solving two-point boundary value problems is proposed, based on the assumption of the validity of R. Bellman's inverse optimality principle, which consists in preserving the functional relationship between the components of a two-point boundary value problem in the entire control interval.

In this paper, an alternative to numerical methods for solving two-point boundary value problems is proposed, based on the assumption of the validity of R. Bellman's inverse optimality principle, which consists in preserving the functional connection between the components of a two-point boundary value problem in the entire control interval.

Because of the study of the control synthesis problem for a nonlinear dynamic system described by an ordinary differential equation and the Bolza functional, an analytical expression for pseudo-optimal control has obtained, and a theorem on the necessary conditions for the optimality of this control has formulated. For a problem with a quadratic quality functional, SDC representation of a nonlinear dynamic object and the corresponding pseudo-optimal control, a theorem on the asymptotic stability of the control system has proved. The conditions that must be met by the penalty matrices of the quality functional are obtained, under which the required quality of transient processes of the controlled nonlinear system is ensured when the established constraints on the control are met. The theoretical results obtained have confirmed by modeling the control system with synthesized control.

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