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Numerical Method for Solving the Time-Optimization Problem for Linear Non-Stationary Discrete-Time Systems of General Form

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Abstract—In the paper, we construct a software-implemented numerical procedure for solving the time-optimization problem for linear discrete-time systems with arbitrary variable matrices of the system and convex sets of geometric constraints on control. We also prove the convergence of the sequence of control processes produced by the algorithm to a solution to the problem. The efficiency is demonstrated on a number of examples.

Keywords: time-optimization problem, linear discrete-time systems, sequential global improvement, Krotov method

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1. INTRODUCTION

The time-optimization problem is one of the classical problems of optimal control theory. In its most generality, it is posed as the problem of transferring a given dynamical object from a fixed initial state to some known set in phase space (target domain) in the least possible time. The corresponding dynamical system itself can operate in either continuous or discrete time, can be linear or nonlinear, and the target domain can be either a given smooth surface or a one-point set (singleton). In this paper, the time-optimization problem is understood as the problem of transferring a linear dynamical object operating in discrete time from a given initial state to the origin in the least possible number of steps. With the exception of some special cases, the main tool for studying problems of this kind has always been and remains the method of directly enumerating the moments of time of possible termination of the transient process and directly solving the resulting finite-dimensional problems of terminal stabilization. For these reasons, relatively few works are devoted directly to the time-optimization problem in the above sense. Among them, we highlight the fairly general considerations in [1–3], as well as their continuations [4–8].

In this paper, we develop a fundamentally different algorithm for solving the time-optimization problem, which is based on the method of constructing sequential global improvements of control processes proposed in the works of V.F. Krotov [9, 10]. This method is also applied to problems of terminal stabilization of a given linear discrete-time system at various fixed moments of time of the end of the transient process. However, unlike the classical approach, these problems are not supposed to be solved completely. Improvements are constructed until either an optimal process in terms of time-optimization is found, or the impossibility of reaching the origin in a given number of steps is proven. For these purposes, we construct an estimate of the optimal value of the quality functional based on the current approximation. A similar approach can be implemented using other numerical methods of conditional finite-dimensional optimization and improvement, including those developed specifically for studying discrete optimal control problems. The latter include the methods from [11, Chapter 6] and some methods from [12, Part 2 and Part 4]. A distinctive feature of Krotov method is the speed of improvement, as well as the fact that, within the framework of the considered problems of terminal stabilization of linear discrete-time systems, it is possible to establish the presence of the key property of strict improvement of non-optimal processes. General results on non-strict improvement in optimal control problems for discrete and continuous systems were obtained by V.F. Krotov. Some particular results on strict improvement, close to those discussed here, but mainly in the case of continuous systems, are presented in [13].

The paper directly continues the research begun in [14], where we propose an approach to constructing two-sided optimal time estimates for stationary linear systems with a non-singular diagonalizable matrix of the system, and a numerical method for improving the upper bound and constructing a guaranteeing process. Here, the results obtained in [14] are further developed in the following directions. First, it will be shown that the iterative procedure from [14] not only allows one to improve the upper bounds for the optimal time estimate and construct guaranteeing processes, but is also actually capable of approximately finding optimal controls in the corresponding time-optimization problem without any additional assumptions. Second, a new procedure will be proposed that allows one to solve the time-optimization problem and construct optimal processes in the absence of any known optimal time estimates. Third, all results will be generalized to the case of non-stationary linear discrete-time systems of general form. In particular, all of them are applicable to stationary systems with a singular matrix.

2. PROBLEM STATEMENT

Consider a linear non-stationary system with discrete time

$$x(k+1) = A(k)x(k) + u(k), \quad k = 0, 1, \dots, \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state of the system, $u(k) \in U(k)$ is the control, $U(k)$ are convex compact sets in $\mathbb{R}^n$ containing the origin, $A(k) \in \mathbb{R}^{n\times n}$ are arbitrary given matrices. The initial condition for the system (1) is fixed:

$$x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

It is required to calculate the minimum number of steps $N_{\min}$, in which it is possible to transfer the system (1) from a given initial state x_0 to the origin and to construct an optimal process $\{x^*(k), u^*(k-1)\}_{k=1}^{N_{\min}}$, satisfying the condition $x^*(N_{\min}) = 0$. The number $N_{\min}$ will be called the optimal time of the system (1) with the initial condition (2) and we will assume that the problem is solvable, i.e. $N_{\min} < \infty$.

In the stationary case $A(k) \equiv A \in \mathbb{R}^{n\times n}$ and $U(k) \equiv U \subset \mathbb{R}^n$, the well-known sufficient conditions for the solvability of the time-optimization problem for the system (1)–(2) are the Schur stability of the matrix A and the inclusion $0 \in \mathrm{int}U$ or the Kalman controllability of the system (1) and the inclusion $0 \in \mathrm{ri}U$.

3. FIXED-TIME PROBLEM AND OPTIMALITY CONDITIONS

In [14], in the case of stationary systems with a non-singular matrix $A(k) \equiv A$, we propose an algorithm that allows one to construct guaranteeing processes, using known estimates for the optimal time, and improve the upper estimate. This includes solving several problems with a fixed operation time of the system sequentially. Let us discuss these problems.

First, introduce the following notation. Let $N > 0$ be some fixed value of discrete time and $k = 0, \dots, N-1$ in the system (1). Let

$$\mathcal{U}_N := \{k \mapsto u(k) : \{0, \dots, N-1\} \to \mathbb{R}^n \mid u(k) \in U(k)\},$$
$$\mathcal{X}_N := \{k \mapsto x(k) : \{0, \dots, N\} \to \mathbb{R}^n\}.$$

We will identify the set $\mathcal{X}_N$ with the Euclidean space $\mathbb{R}^{n(N+1)}$, and the elements of the set $\mathcal{U}_N$ with the corresponding vectors of the Euclidean space $\mathbb{R}^{nN}$.

Consider for the system (1)–(2) the problem

$$J_N(u) = \|x(N)\|^2 \to \min_{u \in \mathcal{U}_N}, \tag{3}$$

where $x \in \mathcal{X}_N$ is the solution to (1)–(2) for $k = 0, \dots, N-1$ and fixed $u \in \mathcal{U}_N$. Here and below, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.

The problem (3) for the system (1)–(2) satisfies the conditions [11, p. 124, conditions 1–4], which guarantee [11, Theorem 5.6.2] that for the optimality of control $\hat{u} \in \mathcal{U}_N$ in this problem it is necessary and sufficient to require the fulfillment of the relations of the classical discrete maximum principle [11, Theorems 5.3.1 and 5.6.1]. Since the system (1) is linear, these relations reduce to testing N maximum conditions

$$\langle \hat{\psi}(k+1), \hat{u}(k) \rangle = \max_{u \in U(k)} \langle \hat{\psi}(k+1), u \rangle, \tag{4}$$

for $k = 0, \dots, N-1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, and the values of the dual variable $\hat{\psi}(k)$ are uniquely determined by

$$\hat{x}(k+1) = A(k)\hat{x}(k) + \hat{u}(k), \quad \hat{x}(0) = x_0, \tag{5}$$
$$\hat{\psi}(k) = A(k)^{\mathrm{T}} \hat{\psi}(k+1), \quad \hat{\psi}(N) = -2\hat{x}(N). \tag{6}$$

It is easy to see that for $N \geqslant N_{\min}$ the optimal controls in problem (3) are singular [15], since in this case $\hat{\psi}(k) \equiv 0$. For what follows, we will need an alternative form of the optimality conditions in this problem, with respect to which the optimal controls will no longer be singular for any value of N.

Theorem 1. *Let $\hat{u} \in \mathcal{U}_N$. Then for the optimality of $\hat{u}$ in the problem* (1)–(3) *it is necessary and sufficient that for $k = 0, \dots, N-1$ the inclusions*

$$\hat{u}(k) \in \operatorname{Arg} \min_{u \in U(k)} \|\mathcal{A}_N(k+1)u + \mathcal{A}_N(k)\hat{x}(k) - \hat{\xi}(k+1)\| \tag{7}$$

hold, where

$$\mathcal{A}_N(k) := A(N-1) \dots A(k), \quad k = 0, \dots, N-1, \quad \mathcal{A}_N(N) := I, \tag{8}$$

I is the identity matrix of size $n \times n$, and the values $\hat{x}(k)$ and $\hat{\xi}(k)$ are defined by the equalities (5) *and*

$$\hat{\xi}(k) = \hat{\xi}(k+1) - \mathcal{A}_N(k+1)\hat{u}(k), \quad \hat{\xi}(N) = 0. \tag{9}$$

For the convenience of the reader, the proofs of the assertions presented in the paper are included in Appendix A.

Necessary conditions in Theorem 1 up to changes of variables and renaming were obtained in [14] for the case $A(k) \equiv A$, $\det A \neq 0$, $U(k) \equiv U$. Sufficient conditions are obtained here for the first time.

The set of conditions (5), (7) and (9), unlike (4)–(6), does not degenerate in the case $N \geqslant N_{\min}$. Thus, for example, for $k = N-1$ from (7) we have

$$\hat{u}(N-1) \in \operatorname{Arg} \min_{u \in U(N-1)} \|A(N-1)\hat{x}(N-1) + u\|,$$

and this condition is always meaningful and reflects the geometric meaning of optimal control in the problem (1)–(3).

As a direct consequence of Theorem 1, we can obtain the statement that the algorithm proposed in [14] allows one to approximately find optimal processes in the time-optimization problem for stationary non-degenerate linear systems. This strengthens the theoretical results presented in the indicated work. The algorithm for solving the time-optimization problem proposed below also relies heavily on the assertion of Theorem 1.

4. IMPROVEMENT THEOREMS

Let $\hat{u} \in \mathcal{U}_N$ be some arbitrary control. Let us set the goal of constructing a new control $\tilde{u} \in \mathcal{U}_N$ that improves $\hat{u}$ in the sense of the value of the functional J_N in the problem (3).

Theorem 2. *Let $\hat{u} \in \mathcal{U}_N$ be an arbitrary control, and $\hat{x}, \hat{\xi} \in \mathcal{X}_N$ be the corresponding solutions to the equations* (5) *and* (9). *Let us define the control $\tilde{u} \in \mathcal{U}_N$ by the condition*

$$\tilde{u}(k) \in \operatorname{Arg} \min_{u \in U(k)} \|\mathcal{A}_N(k+1)u + \mathcal{A}_N(k)\tilde{x}(k) - \hat{\xi}(k+1)\| \quad \forall k \in \{0, \ldots, N-1\}, \tag{10}$$

where $\mathcal{A}_N(k)$ is determined from (8), *and $\tilde{x}(k)$ satisfies the equalities*

$$\tilde{x}(k+1) = A(k)\tilde{x}(k) + \tilde{u}(k), \quad \tilde{x}(0) = x_0. \tag{11}$$

Then, with respect to controls $\hat{u}, \tilde{u} \in \mathcal{U}_N$, there is a non-strict improvement in (3), *i.e.*

$$J_N(\tilde{u}) \leqslant J_N(\hat{u}).$$

Theorem 2 was proved up to notation in [14] under the assumptions $A(k) \equiv A$, $\det A \neq 0$, $U(k) \equiv U$, which are inessential. The proof given in Appendix 1 is entirely based on V.F. Krotov's global improvement constructions.

Theorem 3. *Let $\hat{u} \in \mathcal{U}_N$, $\hat{x} \in \mathcal{X}_N$ satisfy the equation* (5) *and for the pair $(\tilde{x}, \tilde{u})$ the conditions* (10), (11) *hold. Then the equality $J_N(\tilde{u}) = J_N(\hat{u})$ holds if and only if the control $\hat{u}$ is optimal in* (1)–(3).

Theorem 3 states that any non-optimal control in (1)–(3) can be strictly improved with respect to the values of the functional J_N by constructing a new control according to the relations (10), (11). Since all sets $U(k)$ are assumed to be compact, these relations are always solvable (possibly not uniquely).

5. ESTIMATION OF THE OPTIMAL VALUE OF THE QUALITY FUNCTIONAL

When studying the problem (3) in the context of the original time-optimization problem for the system (1)–(2), it is important to be able to estimate from below the optimal value of the functional J_N with sufficient accuracy. For this purpose, one can use the information obtained in the process of constructing improvements. Here is one such estimate.

Theorem 4. *Let $\hat{u} \in \mathcal{U}_N$, the values $\hat{x}(k)$ and $\hat{\psi}(k)$ be determined from* (5) *and* (6), *and J_N^* be the optimal value of the functional J_N in* (3). *Then*

$$J_N^* \geqslant \|\hat{x}(N)\|^2 - \sum_{k=0}^{N-1} \max_{u \in U(k)} \langle \hat{\psi}(k+1), u - \hat{u}(k) \rangle. \tag{12}$$

From Theorem 4 we deduce the following: if for some $N > 0$ and some control $\hat{u} \in \mathcal{U}_N$ the right-hand side of inequality (12) is strictly positive, then $N_{\min} > N$ is guaranteed to hold, since $J_N^* = 0$ for all $N \geqslant N_{\min}$ due to the conditions $0 \in U(k)$. In particular, substituting the control $\hat{u} = 0 \in \mathcal{U}_N$ into (12), we obtain the following estimate of the optimal time:

$$N_{\min} \geqslant \min \left\{ N \geqslant 0 \;\middle|\; \|\mathcal{A}_N(0)x_0\|^2 + 2\sum_{k=0}^{N-1} \min_{u \in U(k)} \langle \mathcal{A}_N(k+1)^{\mathrm{T}} \mathcal{A}_N(0) x_0, u \rangle \leqslant 0 \right\}.$$

We should note that a similar to (12) inequality is obtained in [11, p. 166].

6. ITERATIVE ALGORITHM FOR SOLVING THE TIME-OPTIMIZATION PROBLEM

Now we proceed to constructing a general algorithm for solving the initial time-optimization problem for the system (1)–(2). To do this, we will sequentially improve the control processes in the problem (3) for values $N = 1, 2, \ldots$, applying the results of section 4, and use the inequality (12) for a quick estimate of the theoretical possibility of reaching the origin for the current value of N. Excluding the trivial case, we will assume that $N_{\min} > 0$.

We have the following algorithm:

0. Set the value of the permissible calculation error $\varepsilon > 0$, put $N = 1$.
1. Set/supplement the initial approximation with the equality $u^{(0)}(N-1) = 0$, put $l = 0$ and calculate the matrices $\mathcal{A}_N(k) = A(N-1) \ldots A(k)$, $k = 0, \ldots, N-1$, set the matrix $\mathcal{A}_N(N)$ to be unitary, of size $n \times n$.
2. Find the solution $\xi^{(l)}$ to the system of equations
$$\xi(k) = \xi(k+1) - \mathcal{A}_N(k+1)u^{(l)}(k), \quad \xi(N) = 0.$$
3. For each $k \in \{0, \ldots, N-1\}$, sequentially find and fix some solution $u^{(l+1)}(k)$ to the extremal problem
$$\|\mathcal{A}_N(k+1)u + \mathcal{A}_N(k)x^{(l+1)}(k) - \xi^{(l)}(k+1)\| \to \min_{u \in U(k)},$$
where the values of $x^{(l+1)}(k)$ are calculated using the formulas
$$x^{(l+1)}(k+1) = A(k)x^{(l+1)}(k) + u^{(l+1)}(k), \quad k = 0, \ldots, N-1, \quad x^{(l+1)}(0) = x_0.$$
4. Check the external stopping condition
$$\|x^{(l+1)}(N)\| < \varepsilon,$$
when executed, finish the calculations with the answer $N_\varepsilon = N$, $u_\varepsilon = u^{(l+1)}$.
5. Find the solution $\psi^{(l+1)}$ to the system of equations
$$\psi(k) = A(k)^{\mathrm{T}}\psi(k+1), \quad k = 0, \ldots, N-1, \quad \psi(N) = -2x^{(l+1)}(N).$$
6. Calculate the estimate $E_N^{(l+1)}$ of the possible approximation to origin by the formula
$$E_N^{(l+1)} = \|x^{(l+1)}(N)\|^2 - \sum_{k=0}^{N-1} \max_{u \in U(k)} \langle \psi^{(l+1)}(k+1), u - u^{(l+1)}(k) \rangle.$$

7. Check the internal stopping condition

$$E_N^{(l+1)} > 0,$$

when executing, fix the found $u^{(l+1)}$ as a new initial approximation, i.e. set $u^{(0)}(k) = u^{(l+1)}(k)$, $k = 0, \ldots, N-1$, increase N by one and go to step 1, otherwise increase l by one and go to step 2.

Denote by $\mathcal{U}^*$ the set of all time-optimal controls for the system (1)–(2). By virtue of the assumptions made, the inclusions $\mathcal{U}^* \subset \mathcal{U}_{N_{\min}} \subset \mathbb{R}^{nN_{\min}}$ hold. Recall that the distance from a point to a set in the space $\mathbb{R}^{nN_{\min}}$ is defined as

$$\operatorname{dist}(z, Z) := \inf_{\zeta \in Z} \|z - \zeta\|, \quad z \in \mathbb{R}^{nN_{\min}}, \quad Z \subset \mathbb{R}^{nN_{\min}}.$$

Theorem 5. *Let $N_{\min} < \infty$. Then, for any value of $\varepsilon > 0$, the algorithm constructed above terminates its work after a finite number of iterations with an answer N_ε, u_ε. In this case, $N_\varepsilon \leqslant N_{\min}$, and if the number $\varepsilon > 0$ is small enough, then $N_\varepsilon = N_{\min}$. Moreover, for such values of ε, $u_\varepsilon \in \mathbb{R}^{nN_{\min}}$ holds and the convergence*

$$\operatorname{dist}(u_\varepsilon, \mathcal{U}^*) \to 0, \quad \varepsilon \to 0$$

takes place.

Thus, the constructed algorithm allows us to approximately find solutions to the time-optimization problem for the system (1)–(2).

7. DISCUSSION AND COMMENTS

To implement the proposed algorithm, it is necessary to determine a method for solving two types of finite-dimensional convex optimization problems at steps 3 and 6.

The problems at step 3 are essentially problems of metric projection of given vectors onto given convex compact sets in $\mathbb{R}^n$. The latter are the sets $\mathcal{A}_N(k+1)U(k)$, where all $U(k)$ are known in advance, and the linear transformations $\mathcal{A}_N(k+1)$ are calculated from the known matrices of the system $A(k)$ and depend on the current values of k and N. Thus, for any values of k, N, the set $\mathcal{A}_N(k+1)U(k)$ can be considered known in advance. In applications, $U(k)$ are often linear transformations of some base set U (e.g., U is the unit ball and $U(k)$ is an ellipsoid; U is the unit cube and $U(k)$ is a zonotope, etc.), and in this case $\mathcal{A}_N(k+1)U(k)$ have the same structure. For many such sets, the metric projection problem has been extensively studied, and various high-speed algorithms have been developed for its solution [16, 17]. It is also important to note that the computational errors associated with one or another method of approximate solution to the problems at step 3 do not accumulate in subsequent iterations, since each time a nonlocal improvement of the previously obtained (inaccurate) control program is performed anew. Note also that the optimization problems at step 3 can be rewritten in terms of trajectories. Namely, instead of performing steps 2 and 3 for each $k \in \{0, \ldots, N-1\}$, the following optimization problems can be solved sequentially:

$$x^{(l+1)}(k+1) \in \operatorname{Arg} \min_{x \in A(k)x^{(l+1)}(k) + U(k)} \|\mathcal{A}_N(k+1)(x - x^{(l)}(k+1)) + x^{(l)}(N)\|,$$

where for $k = 0$ we have

$$x^{(l+1)}(0) = x_0.$$

In this case, the number of intermediate calculations required to construct the improved trajectory is somewhat reduced. In this case, the corresponding control values are determined by the equalities

$$u^{(l+1)}(k) = x^{(l+1)}(k+1) - A(k)x^{(l+1)}(k), \quad k = 0, \ldots, N-1.$$

More important is the exact solving of the maximization problems at step 6. Calculating the estimate value $E_N^{(l+1)}$ with the minimal possible error is important for correctly checking the condition $E_N^{(l+1)} > 0$ at step 7, which guarantees that the current value of time N is strictly less than the optimal time $N_{\min}$. The presence of computational errors for $E_N^{(l+1)}$ can lead to the issue that the optimal time is found incorrectly even in the case when the value $\varepsilon > 0$ is sufficiently small. In this regard, it is helpful to use support functions of the sets $U(k)$, which can be determined in advance, since the structure of all $U(k)$ is assumed to be known. With this approach, instead of solving problems of maximizing a linear function on a convex compact set, it will be sufficient to calculate the values of known support functions at given points. The latter allows increasing the accuracy of calculations and additionly reducing the overall running time.

The algorithm proposed in Section 6 can be implemented in any software environment in an arbitrary programming language. To demonstrate the results, an implementation was made in the freely distributed high-level language Python3. For the numerical solving of extremal problems at steps 3 and 6, we use a separate convex optimization library *cvxpy* and the methods *cvxpy.SCS* and *cvxpy.CLARABEL*. Note that in the case where the set U is a zonotope, it is advisable to use the *cvxpy.CLARABEL* method both in step 6 and in step 3. Detailed documentation and source codes of these methods can be found on the official website of the library https://www.cvxpy.org/.

The program listing is included in Appendix B. To feed the initial data from the following examples to the program input, one can use the json code provided in Appendix C.

8. EXAMPLES

Example 1. Consider a one-dimensional stationary system

$$x(k+1) = x(k) + u(k), \quad k = 0, 1, \ldots, \quad x(0) = x_0 \in (2; 3]$$

with a control constraint $|u(k)| \leqslant 1$ and solve the time-optimization problem for this system using the algorithm from Section 6.

Since $n = 1$, all extremal problems at step 3 have unique solution, and $\mathcal{A}_N(k) = 1$ for all N and k. We assume that $\varepsilon > 0$ is so small that the complete stop condition at step 4 can be replaced by the condition $|x^{(l+1)}(N)| = 0$.

Let $N = 1$ and $u^{(0)}(0) = 0$. At step 2 we have

$$\xi^{(0)}(0) = \xi^{(0)}(1) = 0.$$

At step 3 we find

$$u^{(1)}(0) = \arg\min_{u \in [-1;1]} |u + x_0| = -1, \quad x^{(1)}(1) = x_0 - 1 \in (1; 2].$$

Checking at step 4 shows that $|x^{(1)}(1)| > 0$, so at step 5 we have

$$\psi^{(1)}(0) = \psi^{(1)}(1) = -2x^{(1)}(1) = 2 - 2x_0 < 0.$$

From here, in step 6 we calculate

$$E_1^{(1)} = |x^{(1)}(1)|^2 - \max_{u \in [-1;1]} \psi^{(1)}(1)(u - u^{(1)}(0)) = |x^{(1)}(1)|^2 = (x_0 - 1)^2 > 0.$$

In step 7 we fix the found $u^{(1)}(0) = -1$ and move on to the case $N = 2$.

For $N = 2$ we have $u^{(0)}(0) = -1$, $u^{(0)}(1) = 0$. Then we sequentially obtain

$$\xi^{(0)}(1) = \xi^{(0)}(2) = 0, \quad \xi^{(0)}(0) = 1;$$

$$u^{(1)}(0) = \arg \min_{u\in[-1;1]} |u + x_0| = -1, \quad x^{(1)}(1) = x_0 - 1 \in (1; 2];$$

$$u^{(1)}(1) = \arg \min_{u\in[-1;1]} \left|u + x^{(1)}(1)\right| = -1, \quad x^{(1)}(2) = x_0 - 2 \in (0; 1].$$

Since $|x^{(1)}(2)| > 0$, we find

$$\psi^{(1)}(0) = \psi^{(1)}(1) = \psi^{(1)}(2) = -2x^{(1)}(2) = 4 - 2x_0 < 0;$$

$$E_2^{(1)} = |x^{(1)}(2)|^2 - \max_{u\in[-1;1]} \psi^{(1)}(1)(u - u^{(1)}(0)) - \max_{u\in[-1;1]} \psi^{(1)}(2)(u - u^{(1)}(1))$$
$$= |x^{(1)}(2)|^2 = (x_0 - 2)^2 > 0.$$

We fix the found $u^{(1)}(0) = u^{(1)}(1) = -1$ and finally move on to the case $N = 3$.

For $N = 3$ we have $u^{(0)}(0) = u^{(0)}(1) = -1$, $u^{(0)}(2) = 0$. Here we get

$$\xi^{(0)}(2) = \xi^{(0)}(3) = 0, \quad \xi^{(0)}(0) = \xi^{(0)}(1) = 1;$$

$$u^{(1)}(0) = \arg \min_{u\in[-1;1]} |u + x_0 - 1| = -1, \quad x^{(1)}(1) = x_0 - 1 \in (1; 2];$$

$$u^{(1)}(1) = \arg \min_{u\in[-1;1]} \left|u + x^{(1)}(1)\right| = -1, \quad x^{(1)}(2) = x_0 - 2 \in (0; 1].$$

$$u^{(1)}(2) = \arg \min_{u\in[-1;1]} \left|u + x^{(1)}(2)\right| = -x^{(1)}(2) = 2 - x_0, \quad x^{(1)}(2) = 0.$$

The check at step 4 shows that the algorithm is finished and a solution to the time-optimization problem has been found: $N_{\min} = 3$, $u^*(0) = u^*(1) = -1$, $u^*(2) = 2 - x_0$, $x^*(1) = x_0 - 1$, $x^*(2) = x_0 - 2$, $x^*(3) = 0$.

Example 2. Let in (1)–(2) $n = 2$, the system is stationary and it is known that

$$A(k) \equiv \begin{pmatrix} \frac{4}{5}(\cos(1) + \sin(1)) & -\frac{8}{5}\sin(1) \\ \frac{4}{5}\sin(1) & \frac{4}{5}(\cos(1) - \sin(1)) \end{pmatrix};$$

$$U(k) \equiv \left\{v \in \mathbb{R}^2 \mid (2v_1 + v_2)v_1 + (v_1 + 3v_2)v_2 \leqslant 1\right\};$$

$$x_0 = (-37.79, \ -26.1); \quad \varepsilon = 0.0001.$$

Applying the proposed algorithm, we obtain the following results: $N_{\min} = 10$, the time-optimal process has the form shown in Figs. 1 and 2. Table 1 provides detailed information on the conver-

Table 1. Convergence of the algorithm in Example 2

N	$l+1$	$\|x(N)\|$	$\sqrt{E}$
1	1	19.72	19.72
2	1	16.24	16.24
3	1	20.4	20.4
4	1	8.91	8.9
5	1	4.58	4.54
6	1	7.07	7.07
7	1	2.52	2.51
8	3	0.55	0.25
9	1	0.22	0.19
10	1	10^{-5}	0

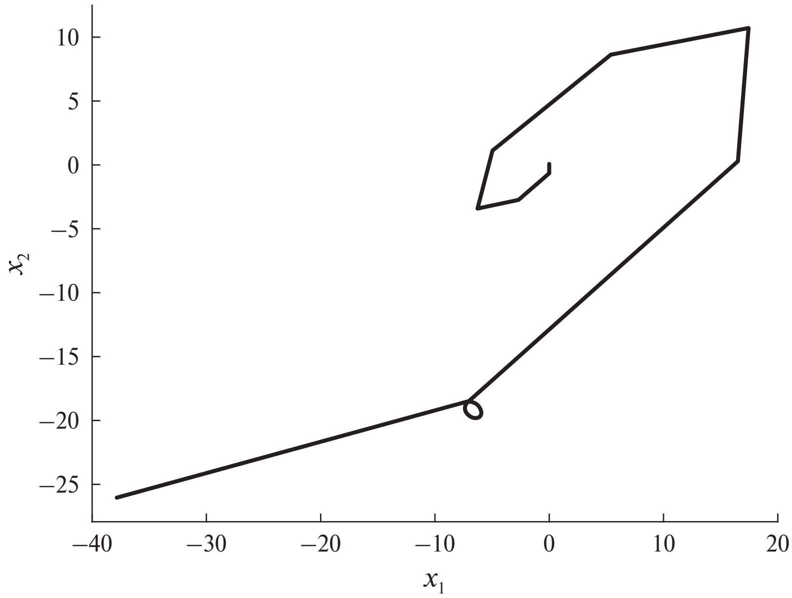

Fig. 1. Optimal trajectory in Example 2.

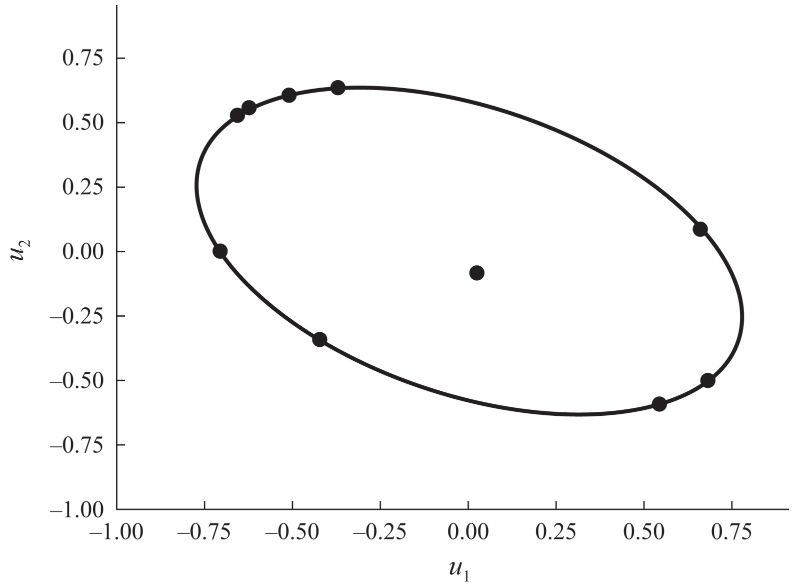

Fig. 2. Optimal control in Example 2.

gence of the algorithm. In this and subsequent tables, N and l correspond to the notations in the paper, $\|x(N)\|$ should be read as $\|x^{(l+1)}(N)\|$, and $\sqrt{E}$ as $\sqrt{\max\{E_N^{(l+1)}, 0\}}$.

Example 3. Let in the system (1)–(2) $n = 2$ and it is known that

$$A(k) \equiv \begin{pmatrix} \cos\left(\dfrac{\pi}{4}\right) & -\sin\left(\dfrac{\pi}{4}\right) \\ \sin\left(\dfrac{\pi}{4}\right) & \cos\left(\dfrac{\pi}{4}\right) \end{pmatrix};$$

$$x_0 = (9.33,\ 0.2); \quad \varepsilon = 0.0001.$$

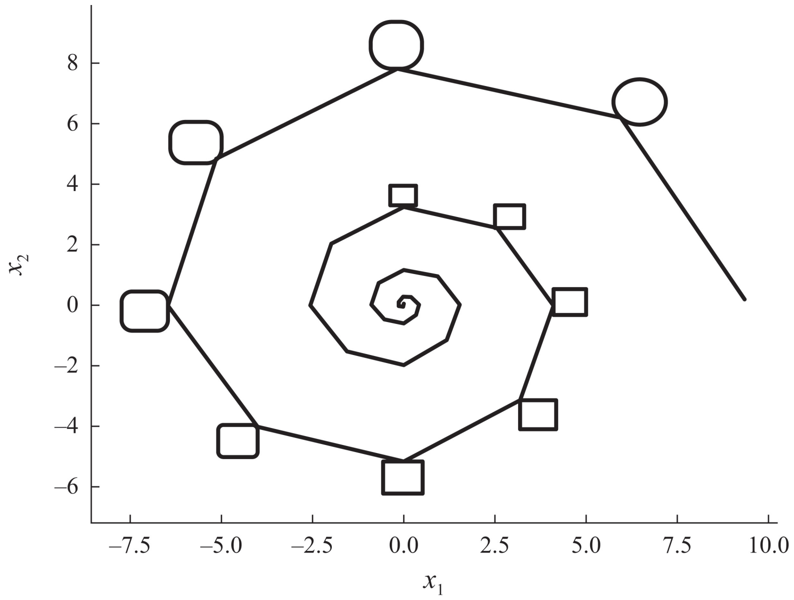

Fig. 3. Optimal trajectory in Example 3.

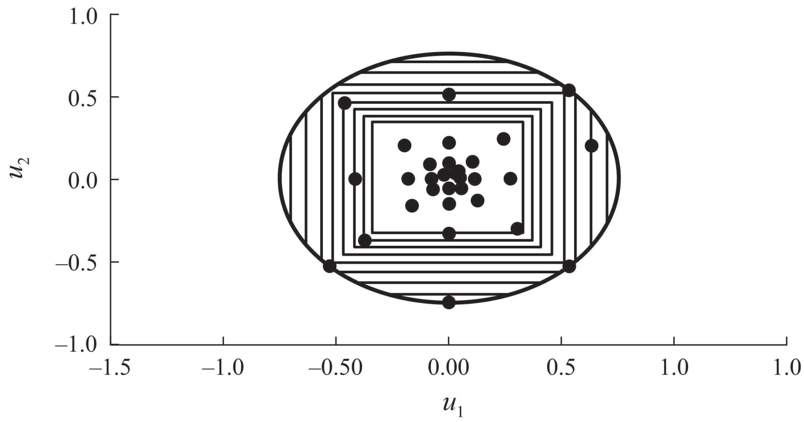

Fig. 4. Optimal control in Example 3.

Let us assume that the sets $U(k)$ change according to the following rule:

$$U(k) = \left\{ v \in \mathbb{R}^2 \;\middle|\; \max\{|v_1|, |v_2|\} \leqslant (0.9)^k \frac{\sqrt{3}}{2},\ \|v\| \leqslant \frac{3}{4} \right\}.$$

Applying the proposed algorithm, we obtain the following results, presented in Table 2 and Figs. 3 and 4.

Table 2. Convergence of the algorithm in Example 3

N	$l+1$	$\|x(N)\|$	$\sqrt{E}$
1	1	8.58	8.58
2	1	7.83	7.83
3	1	7.08	7.08
4	1	6.45	6.45
5	1	5.7	5.7
...	...	...	...
28	1	0.13	0.13
29	1	0.07	0.07
30	1	0.03	0.03
31	1	10^{-6}	0

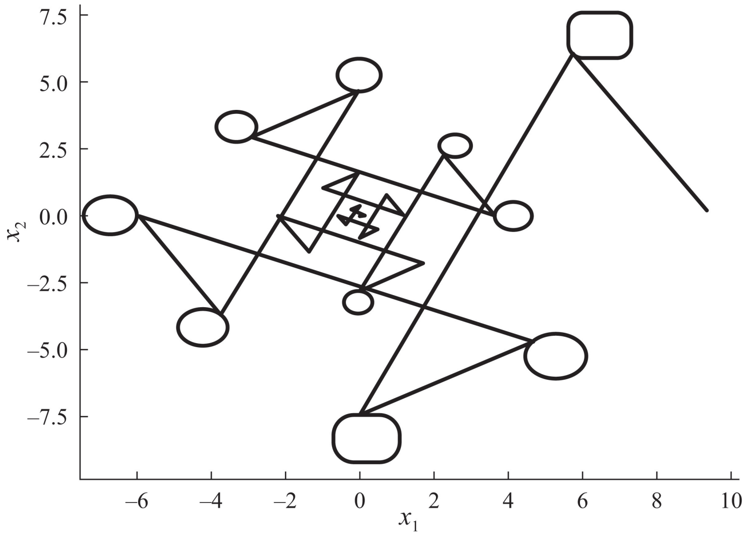

Fig. 5. Optimal trajectory in Example 4.

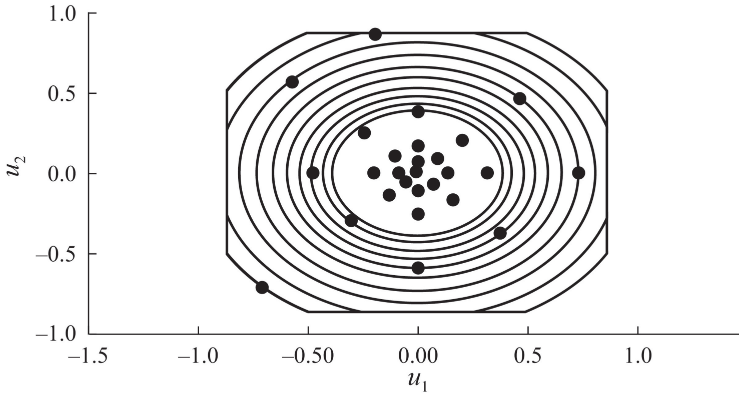

Fig. 6. Optimal control in Example 4.

Example 4. In the conditions of the previous example, suppose that both matrices $A(k)$ and sets $U(k)$ change according to the following rules:

$$A(k) = (-1)^k \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix};$$

$$U(k) = \left\{ v \in \mathbb{R}^2 \;\middle|\; \max\{|v_1|, |v_2|\} \leqslant \frac{\sqrt{3}}{2},\ \|v\| \leqslant (0.9)^k \right\}.$$

We take the values of x_0 and ε from Example 3. Applying the proposed algorithm, we obtain the following results, presented in Table 3 and Figs. 5 and 6.

Table 3. Convergence of the algorithm in Example 4

N	$l+1$	$\|x(N)\|$	$\sqrt{E}$
1	1	8.33	8.33
2	1	7.46	7.46
3	1	6.65	6.65
4	1	5.92	5.92
5	1	5.27	5.27
...	...	...	...
24	1	0.16	0.16
25	1	0.08	0.08
26	1	0.1	0.1
27	1	10^{-5}	0

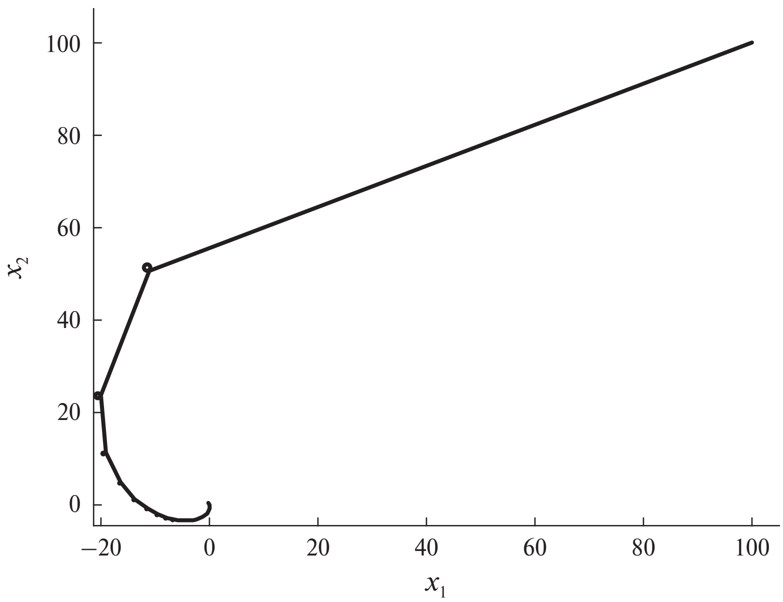

Fig. 7. Optimal trajectory in Example 5.

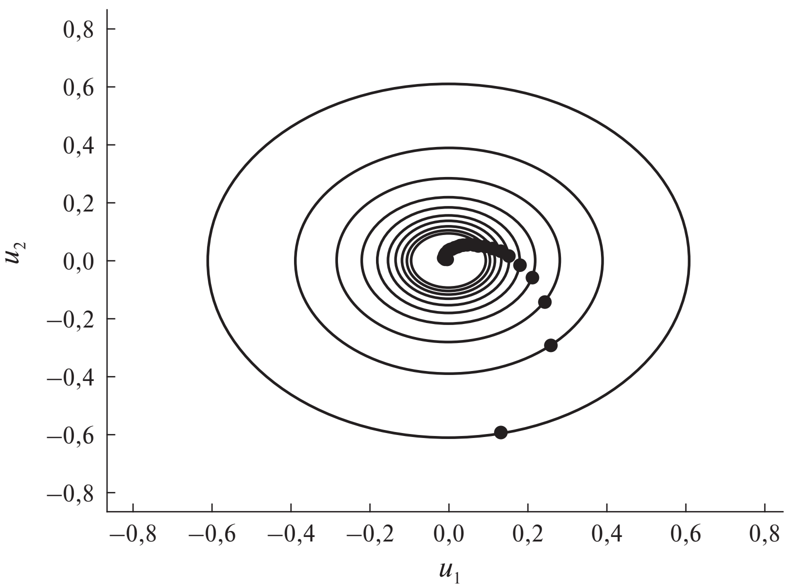

Fig. 8. Optimal control in Example 5.

Example 5. Consider the following example of a non-stationary system (1)–(2). Let $n = 2$ and it is known that

$$A(k) = \begin{pmatrix} e^{-1/(k+1)}\cos(1/(k+1)) & -e^{-1/(k+1)}\sin(1/(k+1)) \\ e^{-1/(k+1)}\sin(1/(k+1)) & e^{-1/(k+1)}\cos(1/(k+1)) \end{pmatrix};$$

$$U(k) = \left\{ v \in \mathbb{R}^2 \;\middle|\; 2v_1^2 + 2v_2^2 \leqslant 1 + e^{-2/(k+1)} - 2e^{-1/(k+1)}\cos(1/(k+1)) \right\};$$

$$x_0 = (100,\ 100); \quad \varepsilon = 0.0001.$$

Here the matrices $A(k)$ and the sets $U(k)$ correspond to a system of the form (1)–(2) obtained by discretizing a continuous-time system of the form

$$\dot{z}(t) = A_c z(t) + w(t), \quad z(0) = x_0,$$

where $t \in [0; +\infty)$,

$$A_c = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix},$$

the control function $w(t)$ satisfies the geometric constraint $\|w(t)\| \leqslant 1$, $t \geqslant 0$, and is piecewise constant on time intervals $[t_k; t_{k+1})$, $k = 0, 1, \dots$, where $t_0 = 0$ and the discretization step $\Delta_k := t_{k+1} - t_k$ is not constant, but changes according to the rule $\Delta_k = 1/(k+1)$.

Applying the proposed algorithm, we obtain the following results, presented in Table 4 and Figs. 7 and 8.

Table 4. Convergence of the algorithm in Example 5

N	$l+1$	$\|x(N)\|$	$\sqrt{E}$
1	1	51.42	51.42
2	1	30.78	30.78
3	1	21.79	21.79
4	1	16.75	16.75
5	1	13.53	13.53
...	...	...	...
77	1	0.03	0.03
78	1	0.02	0.02
79	1	0.01	0.01
80	1	10^{-5}	0

9. CONCLUSION

The algorithm developed in this paper is a method for sequential global improvement of the control program in the time-optimization problem for a general linear discrete-time system. In the absence of additional assumptions about the problem and the properties of its solution, we establish the convergence to an optimal process. It should be emphasized that despite the presence of some enumeration procedures in the structure of the algorithm, the information obtained at earlier stages is not lost when moving to the next steps, but is used to construct a new approximation.

Examples show that there are problems in which this approach is more effective in comparison with the classical enumeration-optimization approach. However, at the moment it has not been proven that the algorithm will work as effectively in the general case. Moreover, computational practice shows that the convergence rate directly depends on both the method of software implementation and the initial data of the problem. At the same time, various additional information that may arise when solving specific applied problems (for example, any estimates of the optimal time and/or a known approximation to the optimal process) are naturally built into the structure of the algorithm and can be used to increase the rate of convergence to the solution. Theoretical study of the convergence properties of the developed iterative procedure seems to be a relevant direction for further research.

We also note that the results of this paper can be easily generalized to a more meaningful practical case of non-stationary linear systems of the form $x(k+1) = A(k)x(k) + B(k)u(k)$, where the vector $u(k)$ has dimension m, and, generally speaking, $m \neq n$, and the matrices $B(k) \in \mathbb{R}^{n\times m}$ are considered to be known.

APPENDIX A

Lemma 1. *Let $\hat{u} \in \mathcal{U}_N$ be an arbitrary control and $\hat{x} \in \mathcal{X}_N$ be found from* (5)*. Let also $\hat{\psi}, \hat{\xi} \in \mathcal{X}_N$ be defined by* (6) *and* (9)*, respectively. Then for each $k \in \{0, \ldots, N\}$ the equality*

$$\hat{\psi}(k) = 2\mathcal{A}_N(k)^{\mathrm{T}} \left(\hat{\xi}(k) - \mathcal{A}_N(k)\hat{x}(k)\right) \tag{A.1}$$

holds.

Proof. Indeed, for $k = N$ the equality (A.1) follows from the initial conditions on the right-hand sides of the formulas (6) and (9), taking into account $\mathcal{A}_N(N) = I$. Suppose that (A.1) holds for some $k \in \{1, \ldots, N\}$. Then

$$\hat{\psi}(k-1) \overset{(6)}{=} A(k-1)^{\mathrm{T}}\hat{\psi}(k) \overset{(A.1)}{=} 2\mathcal{A}_N(k-1)^{\mathrm{T}} \left(\hat{\xi}(k) - \mathcal{A}_N(k)\hat{x}(k)\right) \overset{(5)}{=}$$
$$2\mathcal{A}_N(k-1)^{\mathrm{T}} \left(\hat{\xi}(k) - \mathcal{A}_N(k-1)\hat{x}(k-1) - \mathcal{A}_N(k)\hat{u}(k-1)\right) \overset{(9)}{=}$$
$$2\mathcal{A}_N(k-1)^{\mathrm{T}} \left(\hat{\xi}(k-1) - \mathcal{A}_N(k-1)\hat{x}(k-1)\right).$$

Consequently, (A.1) holds for all $k \in \{0, \ldots, N\}$. Lemma 1 is proved.

Proof of Theorem 1. Since the relations (4)–(6) are a necessary and sufficient condition for optimality in the problem (1)–(3), it suffices to establish their equivalence to the relations (5), (7) and (9). The relation (5) is present in both sets. Moreover, for a fixed $\hat{u} \in \mathcal{U}_N$, it uniquely determines the trajectory $\hat{x}$, and the relations (6) and (9) uniquely determine the values of the dual variables $\hat{\psi}(k)$ and $\hat{\xi}(k)$, which are related by the formula (A.1) by virtue of Lemma 1. Thus, it remains to check the equivalence of the conditions (4) and (7).

The condition (7) is equivalent to the condition

$$\hat{u}(k) \in \operatorname{Arg} \min_{u \in U(k)} \left(\|\mathcal{A}_N(k+1)u\|^2 + 2\langle \mathcal{A}_N(k+1)u, \mathcal{A}_N(k)\hat{x}(k) - \hat{\xi}(k+1)\rangle\right).$$

Given (5) this is also equivalent to

$$\hat{u}(k) \in \operatorname{Arg} \min_{u \in U(k)} \left(\|\mathcal{A}_N(k+1)u\|^2 + 2\langle \mathcal{A}_N(k+1)u, \mathcal{A}_N(k+1)(\hat{x}(k+1) - \hat{u}(k)) - \hat{\xi}(k+1)\rangle\right),$$

and by virtue of (A.1) it is equivalent to

$$\hat{u}(k) \in \operatorname{Arg} \min_{u \in U(k)} \left(\|\mathcal{A}_N(k+1)u\|^2 - \langle u, \hat{\psi}(k+1) + 2\mathcal{A}_N(k+1)^{\mathrm{T}}\mathcal{A}_N(k+1)\hat{u}(k)\rangle\right).$$

Therefore, the condition (7) is equivalent to the condition

$$\langle \hat{\psi}(k+1), \hat{u}(k)\rangle + \|\mathcal{A}_N(k+1)\hat{u}(k)\|^2$$
$$\geqslant \langle \hat{\psi}(k+1), u\rangle + 2\langle \mathcal{A}_N(k+1)^{\mathrm{T}}\mathcal{A}_N(k+1)\hat{u}(k), u\rangle - \|\mathcal{A}_N(k+1)u\|^2 \quad \forall u \in U(k)$$

or, what is the same,

$$\langle \hat{\psi}(k+1), \hat{u}(k) - u\rangle + \|\mathcal{A}_N(k+1)(\hat{u}(k) - u)\|^2 \geqslant 0 \quad \forall u \in U(k).$$

Clearly, the latter holds if (4) holds. Suppose that (4) does not hold. Then there exists $u' \in U(k)$ such that

$$\langle \hat{\psi}(k+1), \hat{u}(k) - u'\rangle < 0.$$

But then for $v_\varepsilon = (1-\varepsilon)\hat{u}(k) + \varepsilon u'$ with $\varepsilon > 0$ we have

$$\langle \hat{\psi}(k+1), \hat{u}(k) - v_\varepsilon \rangle = \varepsilon \langle \hat{\psi}(k+1), \hat{u}(k) - u' \rangle < 0.$$

Moreover, for values $\varepsilon \in (0;1]$ we have $v_\varepsilon \in U(k)$ since the set $U(k)$ is convex. In addition, $v_\varepsilon \to \hat{u}(k)$ for $\varepsilon \to 0$. Since

$$\begin{aligned} &\langle \hat{\psi}(k+1), \hat{u}(k) - v_\varepsilon \rangle + \|\mathcal{A}_N(k+1)(\hat{u}(k) - v_\varepsilon)\|^2 \\ &= \varepsilon \langle \hat{\psi}(k+1), \hat{u}(k) - u' \rangle + \varepsilon^2 \|\mathcal{A}_N(k+1)(\hat{u}(k) - u')\|^2 \\ &\leqslant \varepsilon \left(\langle \hat{\psi}(k+1), \hat{u}(k) - u' \rangle + \varepsilon \|\mathcal{A}_N(k+1)\|^2 \|\hat{u}(k) - u'\|^2 \right), \end{aligned}$$

then it follows that for a sufficiently small value of $\varepsilon > 0$ it will be true that

$$\langle \hat{\psi}(k+1), \hat{u}(k) - v_\varepsilon \rangle + \|\mathcal{A}_N(k+1)(\hat{u}(k) - v_\varepsilon)\|^2 < 0,$$

where $v_\varepsilon \in U(k)$, so the condition (7) is also not satisfied in this case. We finally establish that the conditions (4) and (7) are equivalent. Theorem 1 is proved.

To prove Theorems 2, 3 and 5 we need the following constructions (see also [9, 10, 14]). Fix $\hat{u} \in \mathcal{U}_N$, determine the values of $\hat{\xi}(k)$ from the equation (9) and set for arbitrary $k \in \{0, \ldots, N-1\}$ and $x, u \in \mathbb{R}^n$

$$\begin{aligned} \hat{\varphi}(k,x) &= 2\langle \mathcal{A}_N(k)^{\mathrm{T}} \hat{\xi}(k), x \rangle - \|\mathcal{A}_N(k)x\|^2, \\ \hat{R}(k,x,u) &= \hat{\varphi}(k+1, A(k)x + u) - \hat{\varphi}(k,x). \end{aligned}$$

Lemma 2. *Let $\hat{u} \in \mathcal{U}_N$ be an arbitrary control, and $\hat{x}, \hat{\xi} \in \mathcal{X}_N$ be the corresponding solutions to the equations* (5) *and* (9). *Then the condition* (10) *is satisfied if and only if*

$$\hat{R}(k, \tilde{x}(k), \tilde{u}(k)) = \max_{u \in U(k)} \hat{R}(k, \tilde{x}(k), u) \quad \forall k \in \{0, \ldots, N-1\}, \tag{A.2}$$

where $\tilde{x}$ satisfies (11). *In particular, the control $\hat{u}$ is optimal exactly when*

$$\hat{R}(k, \hat{x}(k), \hat{u}(k)) = \max_{u \in U(k)} \hat{R}(k, \hat{x}(k), u) \quad \forall k \in \{0, \ldots, N-1\}. \tag{A.3}$$

Proof. By definition we have

$$\begin{aligned} &\hat{R}(k, \tilde{x}(k), u) = \hat{\varphi}(k+1, A(k)\tilde{x}(k) + u) - \hat{\varphi}(k, \tilde{x}(k)) \\ &= 2\langle \mathcal{A}_N(k+1)^{\mathrm{T}} \hat{\xi}(k+1), A(k)\tilde{x}(k) + u \rangle - \|\mathcal{A}_N(k+1)(A(k)\tilde{x}(k) + u)\|^2 - \hat{\varphi}(k, \tilde{x}(k)) \\ &= -\|\mathcal{A}_N(k)\tilde{x}(k) + \mathcal{A}_N(k+1)u\|^2 + 2\langle \hat{\xi}(k+1), \mathcal{A}_N(k)\tilde{x}(k) + \mathcal{A}_N(k+1)u \rangle - \hat{\varphi}(k, \tilde{x}(k)) \\ &= -\|\mathcal{A}_N(k)\tilde{x}(k) + \mathcal{A}_N(k+1)u - \hat{\xi}(k+1)\|^2 + \|\hat{\xi}(k+1)\|^2 - \hat{\varphi}(k, \tilde{x}(k)). \end{aligned}$$

Since the second and third terms obtained do not depend on u, we arrive at the first of the assertions to be proved. The second assertion follows directly from the first and Theorem 1. Lemma 2 is proved.

Proof of Theorem 2. By virtue of the introduced notation, we have

$$\begin{aligned} J_N(\tilde{u}) &= -\hat{\varphi}(N, \tilde{x}(N)) = -\hat{\varphi}(0, x_0) + \hat{\varphi}(0, x_0) - \hat{\varphi}(N, \tilde{x}(N)) \\ &= -\hat{\varphi}(0, x_0) - \sum_{k=0}^{N-1} \Big(\hat{\varphi}(k+1, \tilde{x}(k+1)) - \hat{\varphi}(k, \tilde{x}(k)) \Big) \\ &= -\hat{\varphi}(0, x_0) - \sum_{k=0}^{N-1} \hat{R}(k, \tilde{x}(k), \tilde{u}(k)) \end{aligned}$$

and, similarly,

$$J_N(\hat{u}) = -\hat{\varphi}(0, x_0) - \sum_{k=0}^{N-1} \hat{R}(k, \hat{x}(k), \hat{u}(k)).$$

Using that according to (9)

$$\begin{gathered}\hat{R}(k, x, \hat{u}(k)) = \hat{\varphi}(k+1, A(k)x + \hat{u}(k)) - \hat{\varphi}(k, x)\\ = 2\langle \mathcal{A}_N(k+1)^{\mathrm{T}}\hat{\xi}(k+1), A(k)x + \hat{u}(k)\rangle - \|\mathcal{A}_N(k+1)(A(k)x + \hat{u}(k))\|^2\\ -2\langle \mathcal{A}_N(k)^{\mathrm{T}}\hat{\xi}(k), x\rangle + \|\mathcal{A}_N(k)x\|^2 \equiv 2\langle \mathcal{A}_N(k+1)^{\mathrm{T}}\hat{\xi}(k), \hat{u}(k)\rangle + \|\mathcal{A}_N(k+1)\hat{u}(k)\|^2\end{gathered}$$

does not depend on x, then by (10) and Lemma 2 for each $k \in \{0, \ldots, N-1\}$ we have

$$\hat{R}(k, \tilde{x}(k), \tilde{u}(k)) \geqslant \hat{R}(k, \tilde{x}(k), \hat{u}(k)) = \hat{R}(k, \hat{x}(k), \hat{u}(k)). \tag{A.4}$$

Therefore, $J_N(\tilde{u}) \leqslant J_N(\hat{u})$. Theorem 2 is proved.

Let us perform one more general construction, which will be used below in proving Theorems 3 and 5.

Consider the auxiliary system

$$y(k+1) = y(k) + v(k), \quad k = 0, \ldots, N-1, \quad y(0) = y_0 := \mathcal{A}_N(0)x_0 \tag{A.5}$$

with geometric constraints $v(k) \in V(k) := \mathcal{A}_N(k+1)U(k)$. It is clear that all sets $V(k)$ are convex, compact, and contain the origin. Let $\hat{u} \in \mathcal{U}_N$ be given. In the system (A.5), we set $v(k) = \hat{v}(k) = \mathcal{A}_N(k+1)\hat{u}(k)$. Then, since $\hat{u}(k) \in U(k)$, we have $\hat{v}(k) \in V(k)$ and the solution $\hat{y}$ to the system (A.5) satisfies the relation

$$\hat{y}(k) = \mathcal{A}_N(k)\hat{x}(k), \quad k = 0, \ldots, N.$$

We write constructions for the system (A.5) similar to those given above. Namely, let us set for $k \in \{0, \ldots, N-1\}$ and $y, v \in \mathbb{R}^n$

$$\begin{gathered}\hat{\phi}(k, y) = 2\langle \hat{\xi}(k), y\rangle - \|y\|^2,\\ \hat{\mathcal{R}}(k, y, v) = \hat{\phi}(k+1, y+v) - \hat{\phi}(k, y),\end{gathered}$$

where the values of $\hat{\xi}(k)$ are found from (9). Then the following relation holds:

$$\hat{\mathcal{R}}(k, \mathcal{A}_N(k)x, \mathcal{A}_N(k+1)u) = \hat{R}(k, x, u) \quad \forall x, u \in \mathbb{R}^n, \quad k = 0, \ldots, N-1. \tag{A.6}$$

Indeed,

$$\begin{gathered}\hat{\mathcal{R}}(k, \mathcal{A}_N(k)x, \mathcal{A}_N(k+1)u) = \hat{\phi}(k+1, \mathcal{A}_N(k)x + \mathcal{A}_N(k+1)u) - \hat{\phi}(k, \mathcal{A}_N(k)x)\\ = 2\langle \hat{\xi}(k+1), \mathcal{A}_N(k)x + \mathcal{A}_N(k+1)u\rangle - \|(\mathcal{A}_N(k)x + \mathcal{A}_N(k+1)u)\|^2\\ -2\langle \hat{\xi}(k), \mathcal{A}_N(k)x\rangle + \|\mathcal{A}_N(k)x\|^2 = 2\langle \mathcal{A}_N(k+1)^{\mathrm{T}}\hat{\xi}(k+1), A(k)x + u\rangle\\ -\|\mathcal{A}_N(k+1)(A(k)x + u)\|^2 - 2\langle \mathcal{A}_N(k)^{\mathrm{T}}\hat{\xi}(k), x\rangle + \|\mathcal{A}_N(k)x\|^2\\ = \hat{\varphi}(k+1, A(k)x + u) - \hat{\varphi}(k, x) = \hat{R}(k, x, u).\end{gathered}$$

Lemma 3. *Let $\hat{u} \in \mathcal{U}_N$ and the values of $\hat{\xi}(k)$ be found from (9). Then there exists unique pair $(\tilde{y}, \tilde{v})$ satisfying the conditions*

$$\hat{\mathcal{R}}(k, \tilde{y}(k), \tilde{v}(k)) = \max_{v \in V(k)} \hat{\mathcal{R}}(k, \tilde{y}(k), v), \quad k = 0, \ldots, N-1, \tag{A.7}$$

$$\tilde{y}(k+1) = \tilde{y}(k) + \tilde{v}(k), \quad k = 0, \ldots, N-1, \quad \tilde{y}(0) = y_0. \tag{A.8}$$

In this case, the condition (A.7) *is equivalent to the condition*

$$\tilde{v}(k) \in \operatorname{Arg}\min_{v \in V(k)} \|v + \tilde{y}(k) - \hat{\xi}(k+1)\| \quad \forall k \in \{0, \ldots, N-1\} \tag{A.9}$$

and for any pair $(\tilde{x}, \tilde{u})$ *satisfying the relations* (10)–(11),

$$\tilde{y}(k) = \mathcal{A}_N(k)\tilde{x}(k), \quad k = 0, \ldots, N, \quad \tilde{v}(k) = \mathcal{A}_N(k+1)\tilde{u}(k), \quad k = 0, \ldots, N-1. \tag{A.10}$$

Proof. Let us consider the condition (A.7). By definition we have

$$\begin{aligned}
\hat{\mathcal{R}}(k, \tilde{y}(k), v) &= \hat{\phi}(k+1, \tilde{y}(k) + v) - \hat{\phi}(k, \tilde{y}(k)) \\
&= 2\langle \hat{\xi}(k+1), \tilde{y}(k) + v \rangle - \|(\tilde{y}(k) + v)\|^2 - \hat{\phi}(k, \tilde{y}(k)) \\
&= -\|\tilde{y}(k) + v\|^2 + 2\langle \hat{\xi}(k+1), \tilde{y}(k) + v \rangle - \hat{\phi}(k, \tilde{y}(k)) \\
&= -\|\tilde{y}(k) + v - \hat{\xi}(k+1)\|^2 + \|\hat{\xi}(k+1)\|^2 - \hat{\phi}(k, \tilde{y}(k)).
\end{aligned}$$

As in the proof of Lemma 2, we find that (A.7) is equivalent to (A.9). In the case $k = 0$, the condition (A.9) means that the value $\tilde{v}(0)$ is found by solving the extremal problem

$$\|v + \mathcal{A}_N(0)x_0 - \hat{\xi}(1)\| \to \min_{v \in V(0)}.$$

It is clear that for a nonempty convex compact set $V(0)$ the solution to this problem exists and is unique. Moreover, by (A.8) the value of $\tilde{y}(1)$ is uniquely determined from here. At the same time, any $\tilde{u}(0)$ satisfying (10) is one of the solutions to the problem

$$\|\mathcal{A}_N(1)u + \mathcal{A}_N(0)x_0 - \hat{\xi}(1)\| \to \min_{u \in U(0)},$$

whence $\tilde{v}(0) = \mathcal{A}_N(1)\tilde{u}(0)$, since the sets $U(0)$ and $V(0)$ are related by the equality $V(0) = \mathcal{A}_N(1)U(0)$. For $\tilde{x}(1)$ from (11) we obtain $\tilde{y}(1) = \mathcal{A}_N(1)\tilde{x}(1)$, since $y_0 = \mathcal{A}_N(0)x_0$.

Let $\tilde{v}(k-1)$ and $\tilde{y}(k)$ be known for some $k \in \{1, \ldots, N-1\}$. Then $\tilde{v}(k)$ is defined as the unique solution to the problem

$$\|v + \tilde{y}(k) - \hat{\xi}(k+1)\| \to \min_{v \in V(k)},$$

where $V(k)$ is nonempty, convex, and compact. For known $\tilde{v}(k)$ and $\tilde{y}(k)$, the value $\tilde{y}(k+1)$ is uniquely determined from (A.8). Thus, the pair $(\tilde{y}, \tilde{v})$ satisfying conditions (A.7) and (A.8) is completely and uniquely determined.

Carrying out similar comparisons of the values of $\tilde{v}(k)$ and $\tilde{u}(k)$, $\tilde{y}(k+1)$ and $\tilde{x}(k+1)$ for $k = 1, \ldots, N-1$ and taking into account $V(k) = \mathcal{A}_N(k+1)U(k)$, we establish the validity of (A.10). Lemma 3 is proved.

Proof of Theorem 3. Suppose that the control $\hat{u}$ is not optimal in problem (1)–(3). Then by Lemma 2 there exists $r \in \{0, \ldots, N-1\}$ such that

$$\hat{R}(r, \hat{x}(r), \hat{u}(r)) < \max_{u \in U(r)} \hat{R}(r, \hat{x}(r), u).$$

Take the smallest such r. Then by (A.6) for any $k \in \{0, \ldots, r-1\}$ we have

$$\hat{\mathcal{R}}(k, \hat{y}(k), \hat{v}(k)) = \hat{R}(k, \hat{x}(k), \hat{u}(k)) = \max_{u \in U(k)} \hat{R}(k, \hat{x}(k), u) = \max_{v \in V(k)} \hat{\mathcal{R}}(k, \hat{y}(k), v),$$

where $\hat{y}(k) = \mathcal{A}_N(k)\hat{x}(k)$, $\hat{v}(k) = \mathcal{A}_N(k+1)\hat{u}(k)$. Since by Lemma 3 there exists unique pair $(\tilde{y}, \tilde{v})$ satisfying conditions (A.7) and (A.8), we have

$$\tilde{v}(k) = \hat{v}(k), \quad k = 0, \ldots, r-1,$$

and, consequently, $\tilde{y}(r) = \hat{y}(r)$. Hence, by (A.6) and (A.10), for any pair $(\tilde{x}, \tilde{u})$ satisfying (10)–(11), we have

$$\hat{R}(r, \hat{x}(r), \hat{u}(r)) < \max_{u \in U(r)} \hat{R}(r, \hat{x}(r), u) = \max_{v \in V(r)} \hat{\mathcal{R}}(r, \hat{y}(r), v)$$
$$= \max_{v \in V(r)} \hat{\mathcal{R}}(r, \tilde{y}(r), v) = \max_{u \in U(r)} \hat{R}(r, \tilde{x}(r), u) = \hat{R}(r, \tilde{x}(r), \tilde{u}(r)).$$

Returning now to the proof of Theorem 2, we find that for $k = r$ the inequality in (A.4) is strict and therefore

$$J_N(\tilde{u}) < J_N(\hat{u}).$$

If the control $\hat{u}$ is optimal in the problem (1)–(3), then according to Lemma 2 and (A.6) for all $k \in \{0, \ldots, N-1\}$ we have

$$\hat{\mathcal{R}}(k, \hat{y}(k), \hat{v}(k)) = \hat{R}(k, \hat{x}(k), \hat{u}(k)) = \max_{u \in U(k)} \hat{R}(k, \hat{x}(k), u) = \max_{v \in V(k)} \hat{\mathcal{R}}(k, \hat{y}(k), v),$$

and hence the pair $(\tilde{y}, \tilde{v}) = (\hat{y}, \hat{v})$ satisfies the conditions (A.7)–(A.8). Therefore, by Lemma 3 and (A.6), for an arbitrary pair $(\tilde{x}, \tilde{u})$ satisfying (10)–(11), we have

$$\hat{R}(k, \tilde{x}(k), \tilde{u}(k)) = \hat{\mathcal{R}}(k, \hat{y}(k), \hat{v}(k)) = \hat{R}(k, \hat{x}(k), \hat{u}(k)) \quad \forall k \in \{0, \ldots, N-1\}.$$

But then, returning to the proof of Theorem 2, we find that all inequalities in (A.4) for values $k = 0, \ldots, N-1$ are satisfied as equalities and, therefore, $J_N(\tilde{u}) = J_N(\hat{u})$. Theorem 3 is proved.

Proof of Theorem 4. Let $\hat{u} \in \mathcal{U}_N$ be an arbitrary control, and let $\hat{x}(k)$ and $\hat{\psi}(k)$ be defined by (5) and (6). Consider a function $\hat{L} : \mathbb{R}^n \times \mathcal{U}_N \to \mathbb{R}$ of the form

$$\hat{L}(x, u) = \|x\|^2 - 2\langle \hat{x}(N), x \rangle - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \langle \hat{\psi}(k+1), u(k) \rangle.$$

We emphasize that in this notation x is a vector from $\mathbb{R}^n$, and u is an element of the set $\mathcal{U}_N$. We show that for the vector $x(N)$ found from (1)–(2) for fixed $u \in \mathcal{U}_N$, it holds

$$\hat{L}(x(N), u) = \|x(N)\|^2 = J_N(u).$$

Indeed, due to (1) and (6) we obtain

$$\hat{L}(x(N), u) = \|x(N)\|^2 - 2\langle \hat{x}(N), x(N) \rangle - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \langle \hat{\psi}(k+1), u(k) \rangle$$
$$= \|x(N)\|^2 + \langle \hat{\psi}(N), x(N) \rangle - \langle \hat{\psi}(0), x_0 \rangle$$
$$- \sum_{k=0}^{N-1} \Big(\langle \hat{\psi}(k+1), A(k)x(k) + u(k) \rangle - \langle \hat{\psi}(k+1), A(k)x(k) \rangle \Big)$$
$$= \|x(N)\|^2 + \langle \hat{\psi}(N), x(N) \rangle - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \Big(\langle \hat{\psi}(k+1), x(k+1) \rangle - \langle \hat{\psi}(k), x(k) \rangle \Big) = \|x(N)\|^2.$$

In particular, for any optimal process (x^*, u^*) in problem (3) we have

$$\hat{L}(x^*(N), u^*) = J_N^* := J_N(u^*).$$

On the other hand, at the point of its global minimum, the function $\hat{L}$ takes the value

$$\hat{L}^* = \min_{\substack{x \in \mathbb{R}^n \\ u \in \mathcal{U}_N}} \hat{L}(x, u) = -\|\hat{x}(N)\|^2 - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \max_{u \in U(k)} \langle \hat{\psi}(k+1), u \rangle.$$

Consequently, due to (5) the inequality holds

$$J_N^* = \hat{L}(x^*(N), u^*) \geqslant \hat{L}^* = -\|\hat{x}(N)\|^2 - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \max_{u \in U(k)} \langle \hat{\psi}(k+1), u \rangle$$

$$= \|\hat{x}(N)\|^2 - 2\|\hat{x}(N)\|^2 - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \max_{u \in U(k)} \Big(\langle \hat{\psi}(k+1), A(k)\hat{x}(k) + u \rangle - \langle \hat{\psi}(k+1), A(k)\hat{x}(k) \rangle \Big)$$

$$= \|\hat{x}(N)\|^2 + \langle \hat{\psi}(N), \hat{x}(N) \rangle - \langle \hat{\psi}(0), x_0 \rangle - \sum_{k=0}^{N-1} \max_{u \in U(k)} \Big(\langle \hat{\psi}(k+1), A(k)\hat{x}(k) + u \rangle - \langle \hat{\psi}(k), \hat{x}(k) \rangle \Big)$$

$$= \|\hat{x}(N)\|^2 - \sum_{k=0}^{N-1} \max_{u \in U(k)} \Big(\langle \hat{\psi}(k+1), A(k)\hat{x}(k) + u \rangle - \langle \hat{\psi}(k+1), \hat{x}(k+1) \rangle \Big)$$

$$= \|\hat{x}(N)\|^2 - \sum_{k=0}^{N-1} \max_{u \in U(k)} \langle \hat{\psi}(k+1), u - \hat{u}(k) \rangle.$$

Theorem 4 is proved.

Let us make one more construction necessary for proving Theorem 5. Let $N > 0$ be given. For an arbitrary $\hat{u} \in \mathcal{U}_N$, we define two numbers $f_I(\hat{u})$ and $f_E(\hat{u})$ as follows. Let

$$f_I(\hat{u}) = \|\tilde{y}(N)\|^2,$$

where $\tilde{y}$ is determined by the conditions (A.9) and (A.8), in which $\hat{\xi}$ is found from (9). In addition, we set

$$f_E(\hat{u}) = \|\hat{x}(N)\|^2 - \sum_{k=0}^{N-1} \max_{u \in U(k)} \langle \hat{\psi}(k+1), u - \hat{u}(k) \rangle,$$

where $\hat{x}$ and $\hat{\psi}$ are the solutions to the equations (5) and (6).

Lemma 4. *The functions $f_I, f_E : \mathcal{U}_N \to \mathbb{R}$ are well defined and continuous. Moreover,*

$$f_I(\hat{u}) = J_N(\tilde{u})$$

for any $\tilde{u}$ satisfying the conditions (10)–(11)*, and each of the two equalities*

$$f_I(\hat{u}) = J_N(\hat{u}) = f_E(\hat{u})$$

holds exactly when $\hat{u}$ is an optimal control in the problem (1)–(3)*.*

Proof. The well-definedness of the function f_I follows from Lemma 3. The function f_E is well defined due to the compactness of all sets $U(k)$.

The solution $\hat{\xi}$ to the equation (9) depends continuously on the parameters $\hat{u}(k)$, and the solution $\tilde{y}$ to the equation (A.8) depends continuously on the parameters $\tilde{v}(k)$. In addition, for each $k \in \{0, \dots, N-1\}$ the value $\tilde{v}(k)$ is determined by the condition (A.9) as the metric projection of the point $\hat{\xi}(k+1) - \tilde{y}(k)$ onto a nonempty convex compact set $V(k)$ in the space $\mathbb{R}^n$. As is known, the operator of metric projection onto a convex and closed set in a finite-dimensional Euclidean space is well defined and continuous. Therefore, the function f_I is continuous. Let $\tilde{u} \in \mathcal{U}_N$ and $\tilde{x} \in \mathcal{X}_N$ satisfy the conditions (10)–(11). Then, by Lemma 3, we have

$$f_I(\hat{u}) = \|\tilde{y}(N)\|^2 = \|\mathcal{A}_N(N)\tilde{x}(N)\|^2 = \|\tilde{x}(N)\|^2 = J_N(\tilde{u}).$$

In particular, according to Theorem 3, the equality $f_I(\hat{u}) = J_N(\hat{u})$ is satisfied if and only if $\hat{u}$ is an optimal control in the problem (1)–(3).

The solution to the equation (5) depends continuously on the parameters $\hat{u}(k)$, the solution to the equation (6) depends continuously on the parameters $\hat{x}(k)$, and the quantity $\max_{u\in U(k)}\langle\hat{\psi}(k+1),u\rangle$ is the value of the support function of the nonempty compact set $U(k)$ at the point $\hat{\psi}(k+1)$, which is also continuous. Therefore, the function f_E is continuous. Moreover, if $\hat{u}$ is an optimal control in the problem (1)–(3), then the maximum condition (4) is satisfied, whence

$$f_E(\hat{u}) = \|\hat{x}(N)\|^2 = J_N(\hat{u}).$$

Finally, by Theorem 4 for an arbitrary control $\hat{u}\in\mathcal{U}_N$ the two-sided estimate

$$f_E(\hat{u}) \leqslant J_N^* \leqslant J_N(\hat{u})$$

holds, therefore, in the case of equality of the left and right parts, the control $\hat{u}$ is optimal in the problem (1)–(3). Lemma 4 is proved.

Proof of Theorem 5. Since by the condition of the problem $0\in U(k)$ for all k, then for any fixed $N>0$ the control $u^{(0)}$ constructed according to the rule

$$u^{(0)}(k)\in U(k), \quad k=0,\dots,N-2, \quad u^{(0)}(N-1)=0,$$

satisfies the geometric constraints, and step 1 of the algorithm is thus well defined. Steps 3 and 6 are well defined due to the compactness of all sets $U(k)$.

Let $N>0$ be arbitrary. Let J_N^* denote the optimal value of the function J_N in problem (1)–(3), and let $\mathcal{U}_N^*\subset\mathcal{U}_N$ denote the set of all optimal controls in it. Let us consider the sequence $\{u^{(l)}\}\subset\mathcal{U}_N$ constructed by the algorithm in steps 2–5. According to Theorem 2, the sequence of non-negative numbers $J_N(u^{(l)})$ is monotonically non-increasing. Therefore, it has some limit J^0. Since the set $\mathcal{U}_N$ is compact, there exists a subsequence $\{u^{(l_m)}\}$ and an element $u^*\in\mathcal{U}_N$ such that $u^{(l_m)}\to u^*$ as $m\to\infty$. Let us show that $u^*\in\mathcal{U}_N^*$. Consider the function f_I constructed above. By Lemma 4, it satisfies the equality

$$f_I(u^{(l_m)}) = J_N(u^{(l_m+1)}),$$

where the function f_I is continuous. Passing to the limit with respect to $m\to\infty$, we obtain

$$f_I(u^*) = J^0.$$

On the other hand, the function J_N is also continuous due to the continuity of the squared norm function and the continuous dependence of the solution to the equations (1)–(2) on the parameters $u(k)$. Therefore, we have

$$J_N(u^*) = \lim_{m\to\infty} J_N(u^{(l_m)}) = J^0.$$

According to Lemma 4, the obtained equalities mean that u^* is an optimal control in the problem (1)–(3). In particular, $J^0 = J_N(u^*) = J_N^*$.

Let us now consider the function f_E constructed above, as well as the function $\operatorname{dist}(\cdot,\mathcal{U}_N^*)$. Both of these functions are uniquely defined on $\mathcal{U}_N$. By Lemma 4, the function f_E is continuous and $f_E(u^*)=J_N^*$ for any $u^*\in\mathcal{U}_N^*$. The function $\operatorname{dist}(\cdot,\mathcal{U}_N^*)$ is continuous by definition and $\operatorname{dist}(u^*,\mathcal{U}_N^*)=0$ for any $u^*\in\mathcal{U}_N^*$. Let us show that the sequences of numbers $f_E(u^{(l)})$ and $\operatorname{dist}(u^{(l)},\mathcal{U}_N^*)$ have limits J_N^* and 0, respectively. Assume this is not true and, for definiteness, the sequence $\{f_E(u^{(l)})\}$ does not tend to J_N^*. Then for some $\delta>0$ there exists a subsequence $\{u^{(l_m)}\}$ such that $|f_E(u^{(l_m)})-J_N^*|>\delta$ for all m. Passing once again to a subsequence, in view of the compactness of the set $\mathcal{U}_N\subset\mathbb{R}^{nN}$, we can assume that there exists an element $u^*\in\mathcal{U}_N$

such that $u^{(l_m)} \to u^*$ for $m \to \infty$. By what was proved above, the inclusion $u^* \in \mathcal{U}_N^*$ holds, i.e. $f_E(u^*) = J_N^*$ by Lemma 4. But at the same time, the function f_E is continuous, and therefore $|f_E(u^*) - J_N^*| \geqslant \delta > 0$. The resulting contradiction shows that for the sequence of numbers $E_N^{(l)} = f_E(u^{(l)})$ constructed in steps 2–5 of the algorithm, the following convergence holds:

$$E_N^{(l)} \to J_N^*, \quad l \to \infty.$$

Similar reasoning leads to the fact that

$$\operatorname{dist}(u^{(l)}, \mathcal{U}_N^*) \to 0, \quad l \to \infty. \tag{A.11}$$

Since $N > 0$ was chosen arbitrarily, the latter is also true for $N = N_{\min}$.

Thus, it is shown that for any $N > 0$ the sequence of numbers $J_N(u^{(l)})$ monotonically converges to J_N^* from above and, moreover, the sequence of numbers $E_N^{(l)}$ converges to J_N^*. Moreover, by Theorem 4 we have $E_N^{(l)} \leqslant J_N^*$ for all l. Thus, it is proved that

$$J_N(u^{(l)}) \downarrow J_N^* \quad \text{and} \quad E_N^{(l)} \uparrow J_N^* \quad \text{for} \quad l \to \infty. \tag{A.12}$$

Let $\varepsilon > 0$ be given. For any $N < N_{\min}$, regardless of the value of ε, we have $J_N^* > 0$, and by virtue of (A.12) there exists an $l' = l'(N)$ such that $E_N^{(l')} > 0$. Therefore, for any such N, after a finite number of internal iterations, the stopping condition at step 7 is satisfied (if the condition at step 4 was not satisfied before that). If $N = N_{\min}$, then $J_N^* = 0$ and again by virtue of (A.12) there exists an l^* for which $J_N(u^{(l^*)}) < \varepsilon^2$, which means that after a finite number of iterations, the complete stopping condition checked at step 4 is satisfied. Consequently, the algorithm is guaranteed to complete its work in a finite total number of iterations.

Let the algorithm finish its work with an answer N_ε, u_ε. From the previous reasoning it follows that $N_\varepsilon \leqslant N_{\min}$. But if the number ε is chosen so small that

$$0 < \varepsilon < \varepsilon^* := \min_{0 < N < N_{\min}} J_N^*,$$

then the strict inequality $N_\varepsilon < N_{\min}$ is impossible, and therefore $N_\varepsilon = N_{\min}$.

By virtue of (A.11), the sequence $u^{(l)}$ constructed in steps 2–5 for $N = N_{\min}$ converges to the set $\mathcal{U}_{N_{\min}}^* = \mathcal{U}^*$ in the sense of the distance between a point and a set. Moreover, this sequence does not change depending on the choice of the value $\varepsilon \in (0; \varepsilon^*)$, since the internal stopping condition at step 7 does not depend on ε. In this case, the value of the number $l^* = l^*(\varepsilon)$, for which $u_\varepsilon = u^{(l^*)}$ holds by algorithm result, does not decrease with decreasing ε. Assuming that there exists l_0 such that $l^*(\varepsilon) = l_0$ for all sufficiently small ε, we obtain $J_{N_{\min}}(u^{(l_0)}) < \varepsilon^2$ for all small $\varepsilon > 0$, i.e. $J_{N_{\min}}(u^{(l_0)}) = 0$ and $u^{(l_0)} \in \mathcal{U}^*$. Otherwise, $l^*(\varepsilon) \to \infty$ as ε decreases. In each of these cases, $\operatorname{dist}(u_\varepsilon, \mathcal{U}^*) \to 0$ as $\varepsilon \to 0$. Theorem 5 is completely proved.

APPENDIX B

Source code of the executable program.

```
import numpy as np
import cvxpy as cp
from tkinter import filedialog
import pathlib
import json
```

```
MAX_ITER_NUM = 200
MAX_NORM = 1e3


ex_file_path = filedialog.askopenfilename(
  initialdir = pathlib.Path(__file__).parent.resolve())

with open(ex_file_path,'r') as file:
  ex_data = json.load(file)
n = ex_data['n']
A = ex_data['A']
x0 = np.array(ex_data['x0'])
epsilon = ex_data['epsilon']
U = ex_data['U']


Id = np.identity(n)
Zn = np.array([0] * n)

u = list()
A_ = list()
x = list()
xi = list()
psi = list()
v = cp.Variable(n)


N = 1
x.append(x0)
nrm = cp.norm(x0).value
E = 0
iter = 1

while nrm >= epsilon and nrm < MAX_NORM and iter < MAX_ITER_NUM:
  u.append(Zn)
  iter = 1


  A_.clear()
  A_.append(Id)
  for k in range(N):
    A_.insert(0,A_[0]@np.array(eval(A.replace('k',str(N-k-1)))))

  while iter < MAX_ITER_NUM:
    xi.clear()
    xi.append(Zn)
    for k in range(N):
      xi.insert(0,xi[0]-A_[N-k]@u[N-k-1])
```

```
        x.clear()
        x.append(x0)
        for k in range(N):
            prob = cp.Problem(cp.Minimize(
                cp.norm(A_[k+1]@v+A_[k]@x[k]-xi[k+1])),
                [eval(constr.replace('k',str(k))) for constr in U])
            prob.solve(solver=cp.SCS)
            u[k] = v.value
            x.append(np.array(eval(A.replace('k',str(k))))@x[k]+u[k])

        nrm = cp.norm(x[N]).value
        if nrm < epsilon: break

            psi.clear()
        psi.append(-2*x[N])
        for k in range(N):
            psi.insert(0,np.array(eval(
                A.replace('k',str(N-k-1))))).T@psi[0])

        E = nrm**2
        for k in range(N):
            prob = cp.Problem(cp.Maximize(
                psi[k+1].T@(v-u[k])),
                [eval(constr.replace('k',str(k))) for constr in U])
            E -= prob.solve(solver=cp.CLARABEL)

        print(N,iter,x[N],nrm,np.sqrt(np.max([E,0])))

        if E > 0:
            N += 1
            break
        iter += 1

if nrm >= MAX_NORM:
    print("No convergence! Try another example.")
elif iter >= MAX_ITER_NUM:
    print("Too many inner iterations! Break.")
else:
    print(N,x[N],nrm)
    print("Nmin = ", N)

    ex_data['Nmin'] = N
    ex_data['u_opt'] = [[u[k][j] for j in range(n)]
        for k in range(N)]
    ex_data['x_opt'] = [[x[k+1][j] for j in range(n)]
        for k in range(N)]

    with open(ex_file_path,'w') as file:
        json.dump(ex_data,file,indent=4,sort_keys=True)
```

APPENDIX C

Initial data for Example 1.


```
{
  "n": 1,
  "A": "[[1]]",
  "U": ["cp.norm(v) <= 1"],
  "x0": [2.5],
  "epsilon": 0.0001
}
```


Initial data for Example 2.


```
{
  "n": 2,
  "A": "[[4/5*(np.cos(1)+np.sin(1)),-8/5*np.sin(1)],
         [4/5*np.sin(1),4/5*(np.cos(1)-np.sin(1))]]",
  "U": [
    "cp.quad_form(v,np.array([[2, 1],[1, 3]])) <= 1"
  ],
  "x0": [-37.79,-26.1],
  "epsilon": 0.0001
}
```


Initial data for Example 3.


```
{
  "n": 2,
  "A": "[[np.cos(np.pi/4),-np.sin(np.pi/4)],
         [np.sin(np.pi/4),np.cos(np.pi/4)]]",
  "U": [
    "cp.norm(v, \"inf\") <= 0.9**k*cp.sqrt(3)/2",
    "cp.norm(v) <= 3/4"
  ],
  "x0": [9.33,0.2],
  "epsilon": 0.0001
}
```


Initial data for Example 4.


```
{
  "n": 2,
  "A": "[[(-1)**k*np.cos(np.pi/4),-(-1)**k*np.sin(np.pi/4)],
         [(-1)**k*np.sin(np.pi/4),(-1)**k*np.cos(np.pi/4)]]",
  "U": [
    "cp.norm(v, \"inf\") <= cp.sqrt(3)/2",
    "cp.norm(v) <= (0.9)**k"
  ],
  "x0": [9.33,0.2],
  "epsilon": 0.0001
}
```

Initial data for Example 5.


```
{
  "n": 2,
  "A": "[[np.exp(-1/(k+1))*np.cos(1/(k+1)),
          -np.exp(-1/(k+1))*np.sin(1/(k+1))],
         [np.exp(-1/(k+1))*np.sin(1/(k+1)),
          np.exp(-1/(k+1))*np.cos(1/(k+1))]]",
  "U": [
    "cp.norm(v) <= np.sqrt((1+np.exp(-2*1/(k+1))
          -2*np.exp(-1/(k+1))*np.cos(1/(k+1)))/2)"
  ],
  "x0": [100,100],
  "epsilon": 0.0001
}
```


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