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PID Controller Design for Suppressing Bounded Exogenous Disturbances

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Abstract—This paper proposes a novel approach to suppressing nonrandom bounded exogenous disturbances in linear control systems using a PID controller. The approach involves reducing the original problem to a nonconvex matrix optimization problem. A gradient method for finding the PID controller parameters is derived and justified. The recursive procedure proposed is simple to implement and yields controllers that are quite satisfactory in terms of engineering performance indices.

Keywords: linear system, exogenous disturbances, PID controller, optimization, Lyapunov equation, gradient method

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1. INTRODUCTION

PID controllers are the most common type of automatic controllers; according to various estimates, they account for over 90% of controllers used today. In 2019, the IFAC Industry Committee conducted a survey [1] of its members to identify the control technologies mostly demanded by modern industry; multiple answers were allowed. PID control was ranked first with a large gap: by 91% of respondents. Furthermore, according to medium-term forecasts, this share will even remain exceptionally high, nearly 80%. The extensive application of PID controllers in industry is due to several circumstances. In addition to their suitability for solving most practical problems and low cost, a high demand for such controllers is associated with their simplicity: one needs to correctly choose only three coefficients (gains) to tune a PID controller.

However, the procedures for their practical tuning remain much heuristic: in real plants, PID controllers are often tuned manually, based on an intuitive understanding of the industrial process and the influence of separate PID control components on the latter. Known analytical approaches to PID controller design consider "fixed" models of plants, i.e., they are not universal; moreover, they often neglect the effect of uncertainties. Accordingly, the problem of developing regular approaches to PID controller tuning still retains its meaningfulness and topicality.

Among the publications devoted to this range of problems, note [2], where PID controllers were designed based on genetic algorithms, and [3], where evolutionary algorithms were used for this purpose. Also, we mention the works [4, 5], related to the so-called active disturbance rejection control (ADRC, belonging to model-free control), and [6, 7], with the magnitude optimum method applied for these purposes. The internal model method is also widespread for tuning PID controllers [8–10]. The interested reader will find an extensive bibliography in the recent review [11] as well.

The general trend of the latest decades is the transition to numerical PID controller design methods based on solving optimization problems. The optimization approach to the problem of suppressing bounded exogenous disturbances, proposed in [12] and originating from [13], lies in this vein. Recall that the classical problem of suppressing nonrandom bounded exogenous disturbances is stated as follows. Consider a linear control system

$$\begin{aligned}\dot{x} &= Ax + Bu + Dw, \quad x(0) = x_0,\\ y &= C_1 x,\\ z &= C_2 x + B_1 u\end{aligned}$$

with the state vector $x(t) \in \mathbb{R}^n$, the measured output $y(t) \in \mathbb{R}^\ell$, the controlled output $z(t) \in \mathbb{R}^r$, the control input $u(t) \in \mathbb{R}^p$, and an exogenous disturbance $w(t) \in \mathbb{R}^m$ such that $\|w(t)\| \leqslant \bar{w}$ for all $t \geqslant 0$. The problem is to find a stabilizing feedback to reduce the "peak" of the output z, i.e., the value of $\sup_{t\geqslant 0} \max_{\|w\|\leqslant \bar{w}} \|z(t)\|$. In [12], this problem was reduced to a nonconvex matrix optimization problem, a gradient method for finding a static linear state-feedback ($u = Kx$) or output-feedback ($u = Ky$) control law was derived, and its justification was provided.

On the other hand, the optimization approach was applied to the PID controller design problem in [14]. More precisely, a regular approach to finding its parameters was proposed, involving the solution of a nonconvex matrix optimization problem. In this case, the controller's performance was evaluated by a quadratic criterion of the system output: the controller was tuned against uncertainty in the initial conditions to make the system output uniformly small. The recursive procedure proposed therein proved to be very effective and yielded quite satisfactory controllers in terms of engineering performance indices. Later on, the optimization approach was adopted for suppressing bounded exogenous disturbances using a PI controller [15]. This paper continues and develops the above line of research: we design a PID controller for suppressing bounded exogenous disturbances. Note that the approach presented below can be extended to various robust statements of the problem.

From now on, $\|\cdot\|$ is the Euclidean norm of a vector and the spectral norm of a matrix; $\|\cdot\|_F$ indicates the Frobenius norm of a matrix; $^{\mathrm{T}}$ is the transpose symbol; tr stands for the trace of a matrix; I denotes an identity matrix of appropriate dimension; $\lambda_i(A)$ are the eigenvalues of a matrix A; finally, $\sigma(A) \doteq -\max_i \mathrm{Re}\big(\lambda_i(A)\big) > 0$ means the stability degree of a Hurwitz matrix A.

2. PROBLEM STATEMENT

Consider a linear SISO control system described by

$$\begin{aligned}\dot{x} &= Ax + bu + Dw, \quad x(0) = x_0,\\ y &= c^{\mathrm{T}} x,\\ z &= Cx,\end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $D \in \mathbb{R}^{n\times m}$, $c \in \mathbb{R}^n$, $C \in \mathbb{R}^{r\times n}$, with the state vector $x(t) \in \mathbb{R}^n$, the control input $u(t) \in \mathbb{R}$, the measured output $y(t) \in \mathbb{R}$, the controlled output $z(t) \in \mathbb{R}^r$, and an exogenous disturbance $w(t) \in \mathbb{R}^m$ satisfying the constraint

$$\|w(t)\| \leqslant \bar{w} \quad \text{for all } t \geqslant 0. \tag{2}$$

By assumption, the pair (A, D) is controllable, and the pair (A, C) is observable.

We will find the control input in the form of a PID controller

$$u(t) = -k_P y(t) - k_I \int_0^t y(\tau)d\tau - k_D \dot{y}(t) \tag{3}$$

that stabilizes the closed-loop system and suppresses the effect of exogenous disturbances w, minimizing the bounding ellipsoid for the output z.

Let us conceptually recall the method of invariant ellipsoids; more details can be found in [17]. Consider a linear time-invariant dynamical system described by

$$\begin{aligned} \dot{x} &= Ax + Dw, \quad x(0) = x_0, \\ z &= Cx \end{aligned} \tag{4}$$

with the state vector $x(t) \in \mathbb{R}^n$, the output $z(t) \in \mathbb{R}^r$, and a measurable exogenous disturbance $w(t) \in \mathbb{R}^\ell$ that is bounded at each time instant: $\|w(t)\| \leqslant 1$ for all $t \geqslant 0$. Assume that system (4) is stable (i.e., the matrix A is Hurwitz), and the pair (A, D) is controllable.

An ellipsoid centered at the origin is said to be *invariant* for the dynamical system (4) if any of its trajectories evolving from a point inside the ellipsoid will remain in this ellipsoid at any time instant under all admissible exogenous disturbances of the system.

When evaluating the effect of exogenous disturbances on the system output, it is natural to consider the minimal ellipsoids containing this output (in a certain sense). Obviously, if an ellipsoid

$$\mathcal{E} = \{x \in \mathbb{R}^n: \quad x^{\mathrm{T}} P^{-1} x \leqslant 1\} \tag{5}$$

with a positive definite matrix P ($P \succ 0$) is invariant, then the output of system (4) with $x_0 \in \mathcal{E}$ belongs to the so-called *bounding* ellipsoid

$$\mathcal{E}_z = \{z \in \mathbb{R}^r: \quad z^{\mathrm{T}} (CPC^{\mathrm{T}})^{-1} z \leqslant 1\}. \tag{6}$$

In the literature, the linear function $f(P) = \operatorname{tr} CPC^{\mathrm{T}}$ (the sum of the squared semi-axes of the bounding ellipsoid) is often introduced as a minimality criterion.

An invariance criterion for ellipsoids in terms of linear matrix inequalities (LMIs) was established in the book [16]. Here, we formulate it as follows [17].

Theorem 1. *Assume that the matrix A is Hurwitz, the pair (A, D) be controllable, and the matrix $P(\alpha) \succ 0$ satisfies the Lyapunov equation*

$$\left(A + \frac{\alpha}{2} I\right) P + P \left(A + \frac{\alpha}{2} I\right)^{\mathrm{T}} + \frac{1}{\alpha} D D^{\mathrm{T}} = 0$$

on the interval $0 < \alpha < 2\sigma(A)$.

Then the minimal bounding ellipsoid for system (4) *is obtained by minimizing the function* $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ *on the interval* $0 < \alpha < 2\sigma(A)$.

The strict convexity of the function $f(\alpha)$ on the interval $0 < \alpha < 2\sigma(A)$ was shown in [12] (under assumptions that can certainly be weakened). Also note that if α^* is the minimum point in the above problem of Theorem 1 and $x(0) = x_0$ satisfies the condition $x_0^{\mathrm{T}} P^{-1}(\alpha^*) x_0 \leqslant 1$, then the uniform estimate

$$\|z(t)\| \leqslant \sqrt{\|CP(\alpha^*)C^{\mathrm{T}}\|} \leqslant \sqrt{f(\alpha^*)}$$

obviously holds for all $t \geqslant 0$.

3. SOLUTION APPROACH

Let us introduce an auxiliary scalar variable ξ as follows:

$$\dot{\xi} = y, \quad \xi(0) = 0.$$

With the extended state vector

$$g = \begin{pmatrix} x \\ \xi \end{pmatrix} \in \mathbb{R}^{n+1},$$

system (1) can be written as

$$\begin{aligned} &\dot g = \begin{pmatrix} A & 0 \\ c^{\mathrm T} & 0 \end{pmatrix} g + \begin{pmatrix} b \\ 0 \end{pmatrix} u + \begin{pmatrix} D \\ 0 \end{pmatrix} w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ &y = \begin{pmatrix} c^{\mathrm T} & 0 \end{pmatrix} g, \\ &z = \begin{pmatrix} C & 0 \end{pmatrix} g. \end{aligned} \tag{7}$$

According to (1) and (3), we have

$$\begin{aligned} u &= -k_P y(t) - k_I \int_0^t y(\tau) d\tau - k_D \dot y(t) \\ &= -k_P c^{\mathrm T} x - k_I \xi - k_D c^{\mathrm T} \dot x = -k_P c^{\mathrm T} x - k_I \xi - k_D c^{\mathrm T}(Ax + bu + Dw) \\ &= -k_P \begin{pmatrix} c^{\mathrm T} & 0 \end{pmatrix} g - k_I \begin{pmatrix} 0 & 1 \end{pmatrix} g - k_D \begin{pmatrix} c^{\mathrm T} A & 0 \end{pmatrix} g - k_D c^{\mathrm T} b u - k_D c^{\mathrm T} D w; \end{aligned}$$

consequently,

$$(1 + k_D c^{\mathrm T} b) u = -k_P \begin{pmatrix} c^{\mathrm T} & 0 \end{pmatrix} g - k_I \begin{pmatrix} 0 & 1 \end{pmatrix} g - k_D \begin{pmatrix} c^{\mathrm T} A & 0 \end{pmatrix} g - k_D c^{\mathrm T} D w,$$

and

$$\begin{aligned} u = &- \frac{k_P}{1 + k_D c^{\mathrm T} b} \begin{pmatrix} c^{\mathrm T} & 0 \end{pmatrix} g - \frac{k_I}{1 + k_D c^{\mathrm T} b} \begin{pmatrix} 0 & 1 \end{pmatrix} g \\ &- \frac{k_D}{1 + k_D c^{\mathrm T} b} \begin{pmatrix} c^{\mathrm T} A & 0 \end{pmatrix} g - \frac{k_D}{1 + k_D c^{\mathrm T} b} c^{\mathrm T} D w. \end{aligned} \tag{8}$$

With the new variables

$$k_1 = \frac{k_P}{1 + k_D c^{\mathrm T} b}, \quad k_2 = \frac{k_I}{1 + k_D c^{\mathrm T} b}, \quad k_3 = \frac{k_D}{1 + k_D c^{\mathrm T} b},$$

the expression (8) takes the form

$$u = -\begin{pmatrix} k_1 c^{\mathrm T} + k_3 c^{\mathrm T} A & k_2 \end{pmatrix} g - k_3 c^{\mathrm T} D w, \tag{9}$$

and the original PID controller parameters are uniquely given by

$$k_P = \frac{k_1}{1 - k_3 c^{\mathrm T} b}, \quad k_I = \frac{k_2}{1 - k_3 c^{\mathrm T} b}, \quad k_D = \frac{k_3}{1 - k_3 c^{\mathrm T} b}.$$

Thus, system (7) with the feedback control law (9) is described by

$$\begin{aligned} &\dot g = \begin{pmatrix} A - k_1 b c^{\mathrm T} - k_3 b c^{\mathrm T} A & -k_2 b \\ c^{\mathrm T} & 0 \end{pmatrix} g + \begin{pmatrix} (I - k_3 b c^{\mathrm T}) D \\ 0 \end{pmatrix} w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ &z = \begin{pmatrix} C & 0 \end{pmatrix} g. \end{aligned}$$

It can be represented as

$$\begin{aligned} &\dot g = (\mathcal{A}_0 + k_1 \mathcal{A}_1 + k_2 \mathcal{A}_2 + k_3 \mathcal{A}_3) g + (\mathcal{D}_0 + k_3 \mathcal{D}_3) w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ &z = \mathcal{C} g, \end{aligned} \tag{10}$$

where

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix}, \quad \mathcal{A}_1 = \begin{pmatrix} -bc^{\mathrm{T}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_3 = \begin{pmatrix} -bc^{\mathrm{T}}A & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{D}_0 = \begin{pmatrix} D \\ 0 \end{pmatrix}, \quad \mathcal{D}_3 = \begin{pmatrix} -bc^{\mathrm{T}}D \\ 0 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} C & 0 \end{pmatrix}.$$

Remark 1. Note that the pair $(\mathcal{A}_0, \mathcal{D}_0)$ is controllable. Indeed, otherwise there would exist a vector $0 \neq v \in \mathbb{C}^{n+1}$ such that, for some $\lambda \in \mathbb{C}$,

$$v^* \begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix} = \lambda v^*, \quad v^* \begin{pmatrix} D \\ 0 \end{pmatrix} = 0.$$

By writing the vector v as $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $v_1 \in \mathbb{C}^n$, $v_2 \in \mathbb{C}$, we obtain

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^* \begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^*, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^* \begin{pmatrix} D \\ 0 \end{pmatrix} = 0,$$

or $v_1^* A = \lambda v_1^*$, $v_1^* D = 0$, which obviously contradicts the controllability of the pair (A, D).

Thus, the "nominal" system

$$\begin{aligned} \dot{g} &= \mathcal{A}_0 g + \mathcal{D}_0 w, \\ z &= \mathcal{C} g \end{aligned} \tag{11}$$

is controllable.

Similarly, the pair $(\mathcal{A}_0, \mathcal{C})$ is observable. Indeed, otherwise there would exist a vector $0 \neq v \in \mathbb{C}^{n+1}$ such that, for some $\lambda \in \mathbb{C}$,

$$\begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix} v = \lambda v, \quad \begin{pmatrix} C & 0 \end{pmatrix} v = 0.$$

Again, by representing the vector v as $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $v_1 \in \mathbb{C}^n$, $v_2 \in \mathbb{C}$, we arrive at

$$\begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \begin{pmatrix} C & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0,$$

or $Av_1 = \lambda v_1$, $Cv_1 = 0$, which contradicts the observability of the pair (A, C). Thus, the "nominal" system (11) is observable as well.

Following the method of invariant ellipsoids, let the state g of system (10) belong to the invariant ellipsoid (5) generated by a matrix $0 \prec P \in \mathbb{S}^{n+1}$. We will minimize the size of the corresponding bounding ellipsoid (6) with respect to the output $z = \mathcal{C}g$.

In view of condition (2), according to Theorem 1, we appropriately scale the matrix D to arrive the problem of minimizing $\operatorname{tr} \mathcal{C}P\mathcal{C}^{\mathrm{T}}$ subject to the constraint

$$\left(\mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2 + k_3\mathcal{A}_3 + \frac{\alpha}{2}I\right) P + P\left(\mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2 + k_3\mathcal{A}_3 + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{\bar{w}^2}{\alpha}(\mathcal{D}_0 + k_3\mathcal{D}_3)(\mathcal{D}_0 + k_3\mathcal{D}_3)^{\mathrm{T}} = 0.$$

By introducing the designations

$$A_k \doteq \mathcal{A}_0 + \{\mathcal{A}, k\} \doteq \mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2 + k_3\mathcal{A}_3,$$
$$D_k \doteq \begin{pmatrix} (I - k_3 bc^{\mathrm{T}})D \\ 0 \end{pmatrix}$$

for convenience, we write this constraint as

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right) P + P\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{\bar{w}^2}{\alpha} D_k D_k^{\mathrm{T}} = 0. \tag{12}$$

Here, optimization is performed with respect to the matrix variable $0 \prec P \in \mathbb{S}^{n+1}$, the vector variable $k \in \mathbb{R}^3$, and the scalar parameter $\alpha > 0$.

Finally, as the performance criterion, we take the function

$$f(k, \alpha) = \operatorname{tr} \mathcal{C} P \mathcal{C}^{\mathrm{T}} + \rho \|k\|^2, \quad \rho > 0, \tag{13}$$

which includes a control penalty. (The coefficient $\rho > 0$ adjusts its significance.)

Thus, the original problem (the design of a PID controller to suppress exogenous disturbances) has been reduced to the nonconvex matrix optimization problem (13)–(12). Note that for given k and α, the matrix P is found from the Lyapunov equation (12); thus, the independent variables are k and α.

Nonconvexity in this optimization problem is due to the presence of the bilinear terms $A_i k_i P$, $i = 1, \dots, 3$, in the Lyapunov equation (12), which defines the constraint in the parameter space. Essentially, we are dealing with a bilinear matrix equation, and its solution (as in the case of bilinear matrix inequalities) is an NP-hard problem [18–20]. In some particular cases, bilinear matrix equations (or inequalities) can be linearized by special variable changes, e.g., the problem of stabilizing a linear time-invariant system of the form $\dot{x} = Ax + Bu$ by a proportional (P) controller $u = Kx$. However, even in the problem of stabilizing this system by the output $y = Cx$ (i.e., with a controller $u = KCx$), such an approach becomes fundamentally impossible, and other solution methods are required; one alternative is to impose stronger conditions on the system, e.g., in the spirit of the passification conditions (see the feedback Kalman–Yakubovich–Popov lemma [19]). Despite the fundamental difference between stabilization and suppression of exogenous disturbances, similar challenges arise in such problems, which demonstrates the complexity of solving the problem stated above. In the next section, after studying some properties of the objective function, we will apply direct optimization in the parameter space to solve this problem.

4. SOME PROPERTIES OF THE FUNCTION $f(k, \alpha)$

The objective function $f(k, \alpha)$ can be minimized with respect to α, e.g., using Newton's method, as proposed in [12]. Consider the problem

$$\min f(\alpha), \quad f(\alpha) = \operatorname{tr} P C^{\mathrm{T}} C,$$

subject to the constraint

$$\left(A + \frac{\alpha}{2}I\right) P + P\left(A + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha} D D^{\mathrm{T}} = 0$$

with respect to the matrix variable $P \in \mathbb{S}^n$ and the scalar parameter $0 < \alpha < 2\sigma(A)$; the matrix A is assumed stable (Hurwitz).

Let us choose an initial approximation $0 < \alpha_0 < 2\sigma(A)$ and apply the iterative process

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)},$$

where

$$f'(\alpha) = \operatorname{tr} Y \left(P - \frac{1}{\alpha^2} DD^{\mathrm{T}}\right), \quad f''(\alpha) = 2 \operatorname{tr} Y \left(X + \frac{1}{\alpha^3} DD^{\mathrm{T}}\right),$$

and the matrices Y and X are the solutions of the Lyapunov equations

$$\left(A + \frac{\alpha}{2} I\right)^{\mathrm{T}} Y + Y \left(A + \frac{\alpha}{2} I\right) + C^{\mathrm{T}} C = 0$$

and

$$\left(A + \frac{\alpha}{2} I\right) X + X \left(A + \frac{\alpha}{2} I\right)^{\mathrm{T}} + P - \frac{1}{\alpha^2} DD^{\mathrm{T}} = 0,$$

respectively. The method converges globally (faster than the geometric progression with a ratio of 1/2), with quadratic convergence in the neighborhood of the solution.

On the other hand, due to the convexity of the function $f(\alpha)$ (see [12]), it can be effectively minimized on the interval $(0, 2\sigma(A))$ by simpler techniques, e.g., the golden section method.

Next, we introduce the function

$$f(k) \doteq \min_{\alpha} f(k, \alpha).$$

Obviously, $f(k)$ is well-defined and nonnegative on the set $\mathcal{S}$ of all stabilizing controllers (their gains k). Moreover, the set $\mathcal{S}$ can be nonconvex and disconnected, and its boundaries can be nonsmooth.

Assumption. Let

$$k^{(0)} = \begin{pmatrix} k_1^{(0)} \\ k_2^{(0)} \\ k_3^{(0)} \end{pmatrix}$$

be the gains of a known stabilizing PID controller, i.e., the matrix $A_{k^{(0)}} = \mathcal{A}_0 + \{\mathcal{A}, k^{(0)}\}$ is Hurwitz.

We proceed to some properties of the gradient of the objective function.

Lemma 1. *The function $f(k, \alpha)$ is well-defined on the set of stabilizing gains k for $0 < \alpha < 2\sigma(A_k)$. It is differentiable on this admissible set, and the gradient is given by*

$$f'_{\alpha}(k, \alpha) = \operatorname{tr} Y \left(P - \frac{\bar{w}^2}{\alpha^2} D_k D_k^{\mathrm{T}}\right),$$

$$\frac{1}{2} \frac{\partial f(k, \alpha)}{\partial k_i} = \operatorname{tr} P Y \mathcal{A}_i + \rho k_i - \delta_{i3} \frac{\bar{w}^2}{\alpha} \operatorname{tr} Y D_k \begin{pmatrix} bc^{\mathrm{T}} D \\ 0 \end{pmatrix}^{\mathrm{T}}, \quad i = 1, \ldots, 3, \tag{14}$$

where δ_{ij} denotes the Kronecker delta, and the matrices P and Y are the solutions of equation (12) *and the Lyapunov equation*

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2} I\right)^{\mathrm{T}} Y + Y \left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2} I\right) + \mathcal{C}^{\mathrm{T}} \mathcal{C} = 0, \tag{15}$$

respectively.

The function $f(k,\alpha)$ *achieves minimum at an inner point of the admissible set that is determined by the conditions*

$$f'_\alpha(k,\alpha)=0, \quad f'_{k_i}(k,\alpha)=0, \quad i=1,\dots,3.$$

In addition, $f(k,\alpha)$ *as a function of* α *is strictly convex on* $0<\alpha<2\sigma(A_k)$ *and achieves minimum at an inner point of this interval.*

The Jacobian of the function $f(k)$ has the following properties.

Lemma 2. *The function* $f(k)$ *is twice differentiable, and*

$$\frac{1}{2}(f''(k)v,v)=2\operatorname{tr}PY'\{\mathcal{A},v\}+\rho(v,v)-\frac{\bar{w}^2}{\alpha}v_3\operatorname{tr}\left[2Y'D_k-v_3Y\begin{pmatrix}bc^{\mathrm{T}}D\\0\end{pmatrix}\right]\begin{pmatrix}bc^{\mathrm{T}}D\\0\end{pmatrix}^{\mathrm{T}}, \tag{16}$$

where $v\in\mathbb{R}^3$ *and the matrices* P, Y, *and* Y' *are the solutions of equation* (12), *equation* (15), *and the Lyapunov equation*

$$\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)^{\mathrm{T}}Y'+Y'\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)+\{\mathcal{A},v\}^{\mathrm{T}}Y+Y\{\mathcal{A},v\}=0, \tag{17}$$

respectively.

The gradient of the function $f(k)$ is not Lipschitz on the set $\mathcal{S}$ of all stabilizing controllers, but it possesses this property on a subset $\mathcal{S}_0\subset\mathcal{S}$. The corresponding result will be presented below.

For obtaining simple quantitative estimates in Lemmas 3 and 4 below, we incorporate the regularizing terms ε and δ into the optimization problem (13), (12) as follows:

$$\min f(k,\alpha), \quad f(k,\alpha)=\operatorname{tr}P(\mathcal{C}^{\mathrm{T}}\mathcal{C}+\varepsilon I)+\rho\|k\|^2, \quad 0<\varepsilon\ll 1,$$

subject to the constraint

$$\left(A_k+\frac{\alpha}{2}I\right)P+P\left(A_k+\frac{\alpha}{2}I\right)^{\mathrm{T}}+\frac{\bar{w}^2}{\alpha}(D_kD_k^{\mathrm{T}}+\delta I)=0, \quad 0<\delta\ll 1. \tag{18}$$

The requirement for their introduction can be significantly weakened, but the current aim is to obtain the simplest and most illustrative results.

Lemma 3. *The function* $f(k)$ *is coercive on the set* $\mathcal{S}$ *of all stabilizing controllers (i.e., tends to infinity on its boundary) and, moreover,*

$$f(k)\geqslant\frac{\bar{w}^2}{4\sigma(A_k)}\frac{\varepsilon}{\|A_k\|+\sigma(A_k)}\|D_k\|_F^2, \tag{19}$$

$$f(k)\geqslant\rho\|k\|^2.$$

Let us introduce the level set

$$\mathcal{S}_0=\{k\in\mathcal{S}\colon\quad f(k)\leqslant f(k^{(0)})\}.$$

Obviously, Corollary 1 is immediate from Lemma 3.

Corollary 1. *For any controller* $k^{(0)}\in\mathcal{S}$, *the set* $\mathcal{S}_0$ *is bounded.*

On the other hand, the function $f(k)$ has a minimum point on the set $\mathcal{S}_0$ (as a continuous function on a compact set), but the set $\mathcal{S}_0$ shares no points with the boundary of $\mathcal{S}$ due to (19). According to the above considerations, $f(k)$ is differentiable on $\mathcal{S}_0$. Consequently, Corollary 2 is true.

Corollary 2. *There exists a minimum point* k_* *on the set* S, *and* $f'(k_*)=0$.

Lemma 4. *On the set $\mathcal{S}_0$, the gradient of the function $f(k)$ is Lipschitz with the constant*

$$L = 2\sqrt{3(n+1)}\frac{f(k_0)}{\varepsilon}\frac{2(n+1)^{3/2}f(k_0)}{\bar{w}^2\varepsilon\delta}$$
$$\times\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\|+\max_i\|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2+3\max_i\|\mathcal{A}_i\|^2\right]$$
$$\times\left(\|\mathcal{A}_0\|+\max_i\|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)\max_i\|\mathcal{A}_i\|+2\rho$$
$$+2\bar{w}^2\left(2\frac{(n+1)f(k_0)}{\bar{w}^2\varepsilon\delta}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\|+\max_i\|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2+3\max_i\|\mathcal{A}_i\|^2\right]\right.$$
$$\left.\times\left(1+\sqrt{\frac{f(k_0)}{\rho}}\|bc^{\mathrm{T}}\|\right)\|D\|_F+\frac{f(k_0)}{\bar{w}^2\delta}\|bc^{\mathrm{T}}D\|_F\right)\|bc^{\mathrm{T}}D\|_F.$$

These properties of the objective function and its derivatives allow constructing an optimization procedure and justifying its convergence.

5. OPTIMIZATION ALGORITHM

We propose an iterative approach to solve the problem posed. This approach is based on the application of the gradient method with respect to variable k and convex minimization with respect to α. The corresponding algorithm includes several steps as follows.

(1) Choose some values of the parameters $\varepsilon > 0$, $\gamma > 0$, and $0 < \tau < 1$ and an initial stabilizing approximation $k^{(0)}$. Compute
$$\alpha_0 = \sigma(\mathcal{A}_0 + \{\mathcal{A}, k^{(0)}\}).$$

(2) On the jth iteration, the values of $k^{(j)}$ and α_j are given. Compute $A_{k^{(j)}} = \mathcal{A}_0 + \{\mathcal{A}, k^{(j)}\}$ and solve equations (12) and (15) to find the matrices P and Y. Compute the gradient
$$H_j = \nabla_k f(k^{(j)}, \alpha_j)$$
from the relations (14). If $\|H_j\| \leqslant \varepsilon$, then take $k^{(j)}$ as an approximate solution and terminate the algorithm.

(3) Perform the gradient method step:
$$k^{(j+1)} = k^{(j)} - \gamma_j H_j.$$
Adjust the step length $\gamma_j > 0$ by fractionating until the following conditions are satisfied:
(a) The matrix $\mathcal{A}_0 + \{\mathcal{A}, k^{(j+1)}\} + \frac{\alpha_j}{2}I$ is Hurwitz.
(b) $f(k^{(j+1)}) \leqslant f(k^{(j)}) - \tau\gamma_j\|H_j\|^2$.

(4) Minimize $f(k^{(j+1)}, \alpha)$ with respect to α (see the beginning of Section 4) and find α_{j+1}. Revert to Step 2.

This method converges in the following sense.

Theorem 2. *In Algorithm 1, only a finite number of fractions are realized for γ_j on each iteration, the function $f(k^{(j)})$ is monotonically decreasing, and its gradient vanishes with an exponential rate (like a geometric progression):*
$$\lim_{j\to\infty}\|H_j\| = 0.$$

Indeed, Algorithm 1 is well-defined at the initial point since $k^{(0)}$ is a stabilizing controller by the above assumption. For sufficiently small γ_j, the function $f(k)$ monotonically decreases (moves in the direction of its antigradient); with this step adjustment, the values of $k^{(j)}$ remain in the domain $\mathcal{S}_0$, where Lemma 4 ensures the Lipschitz property of the gradient. Thus, the gradient method for unconstrained minimization is convergent [21]. In particular, condition (b) at Step 3 of Algorithm 1 will be satisfied after a finite number of fractions, and the gradient method will demonstrate gradient convergence with a linear rate.

Naturally, it is difficult to expect convergence to a global minimum: the definitional domain of $f(k)$ may even be disconnected.

Let us finally emphasize the following aspect.

Remark 2. With the Euclidean norm in the objective function (13) being replaced by the weighted

$$\|x\|_\rho \doteq \sqrt{\sum_i \rho_i x_i^2}, \quad \rho_i > 0,$$

one can tune the PID controller parameters more flexibly via assigning different weights.

6. EXAMPLES

Consider an illustrative example from the paper [22]. The transfer function has the form

$$G(s) = \frac{1}{(1+s)(1+\alpha s)(1+\alpha^2 s)(1+\alpha^3 s)}, \quad \alpha = 0.5.$$

MATLAB's procedure `tf2ss` gave the following matrices of system (4) in the state space:

$$A = \begin{pmatrix} -15 & -70 & -120 & -64 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 64 \end{pmatrix}.$$

Let us choose the matrix

$$D = \begin{pmatrix} 10 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the controlled output matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We assign $\rho = 0.1$ and the stabilizing PID controller

$$k_0 = \begin{pmatrix} 0.6389 \\ 0.9853 \\ 0.3243 \end{pmatrix}$$

as an initial one.

The iterative process of the optimization algorithm terminated with the PID controller

$$k_* = \begin{pmatrix} 0.6670 \\ 0.5466 \\ 0.1081 \end{pmatrix}$$

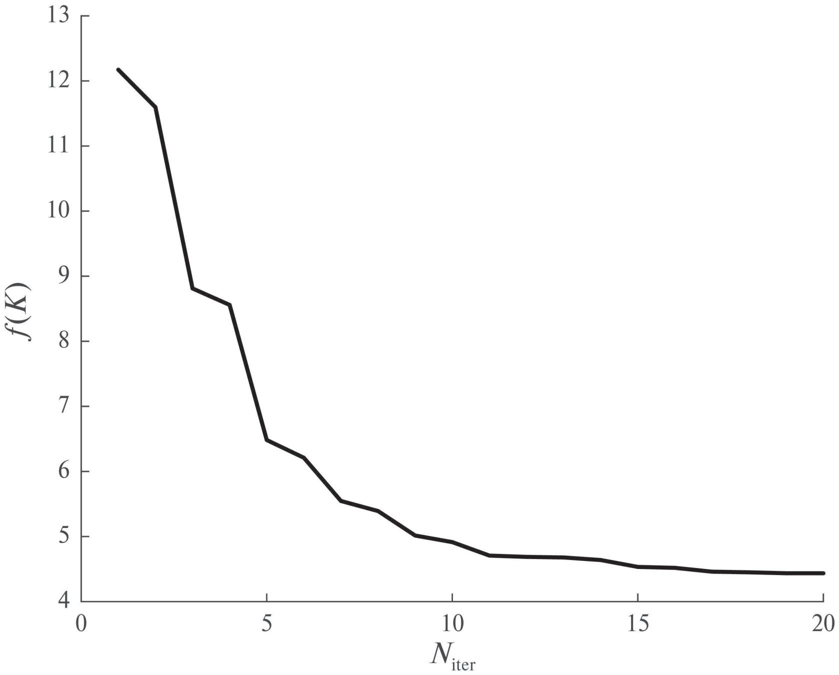

Fig. 1. Optimization procedure.

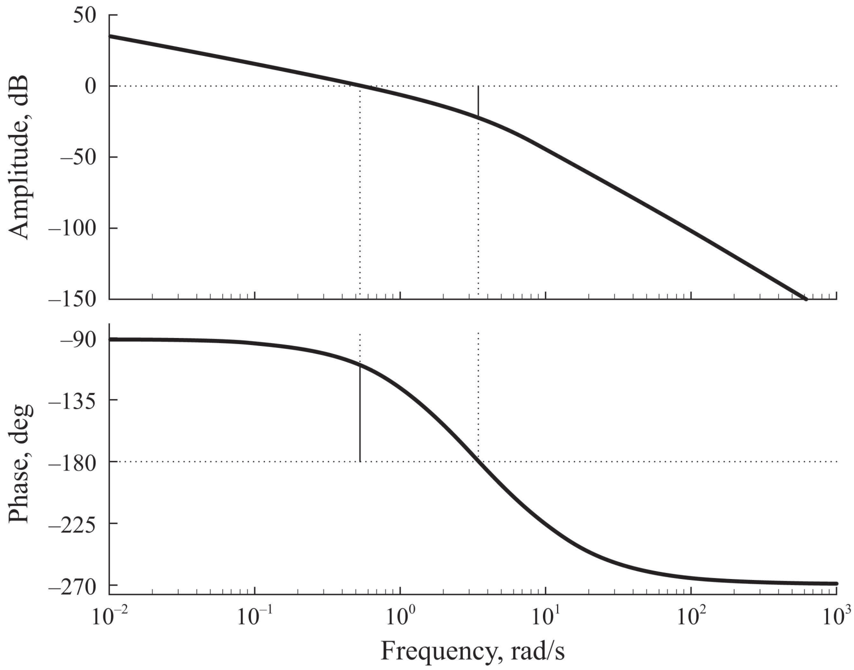

Fig. 2. Bode plots of the closed-loop system.

and the bounding ellipse matrix

$$P_* = \begin{pmatrix} 4.0336 & -0.1999 \\ -0.1999 & 0.3259 \end{pmatrix}, \quad \operatorname{tr} P_* = 4.3595.$$

The dynamics of the criterion $f(k)$ are shown in Fig. 1.

The closed-loop system with the PID controller k_* is stable by the Nyquist criterion; its minimal gain and phase margins are 23.3 dB and 70.3°, respectively (Fig. 2).

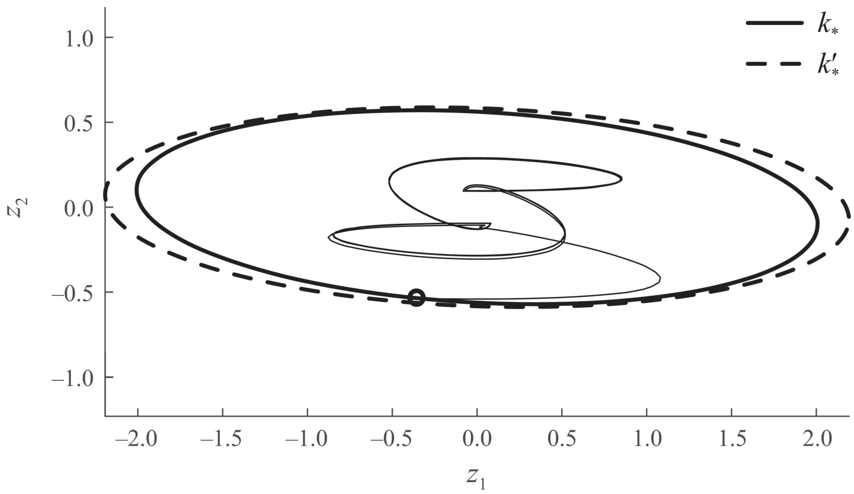

Fig. 3. Bounding ellipses.

By setting $\rho = 10$, we arrived at the PID controller

$$k'_* = \begin{pmatrix} 0.1232 \\ 0.2410 \\ -0.0710 \end{pmatrix}$$

and the bounding ellipse matrix

$$P'_* = \begin{pmatrix} 4.8090 & -0.1560 \\ -0.1560 & 0.3453 \end{pmatrix}, \quad \operatorname{tr} P'_* = 5.15437.$$

Therefore, the norm of the vector of PID controller gains was reduced three times, at the "price" of a less than 20% increase in the size of the bounding ellipse.

Figure 3 shows the resulting bounding ellipses and the trajectory of the closed-loop system with the PID controller k_* under some admissible exogenous disturbance.

All computations were carried out in Matlab using `cvx` [23].

In the future, we intend to conduct extensive numerical simulations and discuss all computational aspects thoroughly. In this paper, it is important to demonstrate the principal effectiveness of the novel approach in disturbance suppression.

7. CONCLUSIONS

This paper has proposed a novel and easily implementable approach to the design problem of PID controllers suppressing nonrandom bounded exogenous disturbances in linear control systems. The approach is based on reducing the original problem to a nonconvex matrix optimization problem, which is then solved by the gradient method. The corresponding algorithm has been constructed and justified.

Although only SISO systems have been considered, the approach is fully transferable to the multidimensional case; here, the constructs will become somewhat more cumbersome, while the conceptual side will change little.

An important direction for further research is to extend the above results to systems with uncertainties.

FUNDING

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APPENDIX

Lemma A.1 [12]. *Let X and Y be the solutions of the dual Lyapunov equations with a Hurwitz matrix A:*

$$A^{\mathrm{T}}X + XA + W = 0 \quad \text{and} \quad AY + YA^{\mathrm{T}} + V = 0.$$

Then $\operatorname{tr}(XV) = \operatorname{tr}(YW)$.

Lemma A.2 [24]. 1. *Real matrices A and B of compatible dimensions satisfy the relations*

$$\begin{gathered}\|AB\|_F \leqslant \|A\|_F\|B\|,\\ |\operatorname{tr} AB| \leqslant \|A\|_F\|B\|_F,\\ \|A\| \leqslant \|A\|_F,\\ AB + B^{\mathrm{T}}A^{\mathrm{T}} \leqslant \varepsilon AA^{\mathrm{T}} + \frac{1}{\varepsilon}B^{\mathrm{T}}B \quad \text{for any } \varepsilon > 0.\end{gathered}$$

2. *Positive semidefinite matrices A and B satisfy the relations*

$$0 \leqslant \lambda_{\min}(A)\lambda_{\max}(B) \leqslant \lambda_{\min}(A)\operatorname{tr} B \leqslant \operatorname{tr} AB \leqslant \lambda_{\max}(A)\operatorname{tr} B \leqslant \operatorname{tr} A \operatorname{tr} B.$$

Proof of Lemma 1. Differentiating equation (12) with respect to α gives

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)\frac{\partial P}{\partial \alpha} + \frac{\partial P}{\partial \alpha}\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + P - \frac{\bar{w}^2}{\alpha^2}D_kD_k^{\mathrm{T}} = 0. \tag{A.1}$$

With Lemma A.1 applied to the dual Lyapunov equations (A.1) and (15), we obtain

$$f'_\alpha(k,\alpha) = \operatorname{tr}\mathcal{C}\frac{\partial P}{\partial \alpha}\mathcal{C}^{\mathrm{T}} = \operatorname{tr}\frac{\partial P}{\partial \alpha}\mathcal{C}^{\mathrm{T}}\mathcal{C} = \operatorname{tr} Y\left(P - \frac{\bar{w}^2}{\alpha^2}D_kD_k^{\mathrm{T}}\right).$$

To differentiate with respect to k, we add the increment Δk and denote the corresponding increment of P by ΔP:

$$\begin{gathered}\left(\mathcal{A}_0 + \{\mathcal{A}, k + \Delta k\} + \frac{\alpha}{2}I\right)(P + \Delta P)\\ +(P + \Delta P)\left(\mathcal{A}_0 + \{\mathcal{A}, k + \Delta k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{\bar{w}^2}{\alpha}D_{k+\Delta k}D_{k+\Delta k}^{\mathrm{T}} = 0,\end{gathered}$$

where

$$D_{k+\Delta k} = \begin{pmatrix}(I - (k_3 + \Delta k_3)bc^{\mathrm{T}})D\\ 0\end{pmatrix} = D_k - \Delta k_3\begin{pmatrix}bc^{\mathrm{T}}D\\ 0\end{pmatrix}.$$

Let us apply linearization and subtract this and the previous equations to get

$$\begin{gathered}\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)\Delta P + \Delta P\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}}\\ + \{\mathcal{A}, \Delta k\}P + P\{\mathcal{A}, \Delta k\}^{\mathrm{T}} - \Delta k_3\frac{\bar{w}^2}{\alpha}\left[D_k\begin{pmatrix}bc^{\mathrm{T}}D\\ 0\end{pmatrix}^{\mathrm{T}} + \begin{pmatrix}bc^{\mathrm{T}}D\\ 0\end{pmatrix}D_k^{\mathrm{T}}\right] = 0.\end{gathered} \tag{A.2}$$

The increment of $f(k)$ is calculated by linearizing the corresponding terms:

$$\Delta f(k) = \operatorname{tr} \mathcal{C}(P + \Delta P)\mathcal{C}^{\mathrm{T}} + \rho\|k + \Delta k\|^2 - (\operatorname{tr} \mathcal{C}P\mathcal{C}^{\mathrm{T}} + \rho\|k\|^2) = \operatorname{tr} \Delta P\mathcal{C}^{\mathrm{T}}\mathcal{C} + 2\rho k^{\mathrm{T}}\Delta k.$$

Due to Lemma A.1, for the dual Lyapunov equations (A.2) and (15), we have

$$\begin{aligned}\Delta f(k) &= 2\operatorname{tr} Y\left[\{\mathcal{A}, \Delta k\}P - \Delta k_3\frac{\bar{w}^2}{\alpha}D_k\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}\right] + 2\rho k^{\mathrm{T}}\Delta k \\ &= 2\operatorname{tr} PY\{\mathcal{A}, \Delta k\} - 2\frac{\bar{w}^2}{\alpha}\operatorname{tr} YD_k\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}\Delta k_3 + 2\rho k^{\mathrm{T}}\Delta k.\end{aligned}$$

Thus,

$$df(k) = 2\operatorname{tr} PY\sum_{i=1}^{3}\mathcal{A}_i dk_i - 2\frac{\bar{w}^2}{\alpha}\operatorname{tr} YD_k\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}dk_3 + 2\rho\sum_{i=1}^{3}k_i dk_i.$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3\end{pmatrix} \in \mathbb{R}^3$. The value of $(f''(k)v, v)$ is obtained by differentiating $f'(k)$ in the direction v. For this purpose, linearizing the corresponding terms and using the convenient designation

$$[\operatorname{tr} PY\mathcal{A}] \doteq \begin{pmatrix} \operatorname{tr} PY\mathcal{A}_1 \\ \operatorname{tr} PY\mathcal{A}_2 \\ \operatorname{tr} PY\mathcal{A}_3\end{pmatrix},$$

we calculate the increment of $f'(k)$ in the direction v :

$$\begin{aligned}\frac{1}{2}\Delta f'(k)v &= [\operatorname{tr}(P + \Delta P)(Y + \Delta Y)\mathcal{A}] + \rho(k + \delta v) - \frac{\bar{w}^2}{\alpha}\operatorname{tr}(Y + \Delta Y)D_{k+\delta v}\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}e_3 \\ &- \left([\operatorname{tr} PY\mathcal{A}] + \rho k - \frac{\bar{w}^2}{\alpha}\operatorname{tr} YD_k\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}e_3\right) = [\operatorname{tr}(P + \delta P'(k)v)(Y + \delta Y'(k)v)\mathcal{A}] + \rho(k + \delta v) \\ &- \frac{\bar{w}^2}{\alpha}\operatorname{tr}(Y + \delta Y'(k)v)D_{k+\delta v}\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}e_3 - \left([\operatorname{tr} PY\mathcal{A}] + \rho k - \frac{\bar{w}^2}{\alpha}\operatorname{tr} YD_k\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}e_3\right) \\ &= \delta[\operatorname{tr}(PY'(k)v + P'(k)vY)\mathcal{A}] + \delta\rho v - \frac{\bar{w}^2}{\alpha}\operatorname{tr}\left[(Y + \delta Y')\left(D_k - \delta v_3\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right) - YD_k\right]\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}e_3 \\ &= \delta[\operatorname{tr}(PY'(k)v + P'(k)vY)\mathcal{A}] + \delta\rho v - \delta\frac{\bar{w}^2}{\alpha}\operatorname{tr}\left[Y'D_k - v_3 Y\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right]\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}e_3,\end{aligned}$$

where $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1\end{pmatrix}$ and

$$\begin{aligned}\Delta P &= P(k + \delta v) - P(k) = \delta P'(k)v, \\ \Delta Y &= Y(k + \delta v) - Y(k) = \delta Y'(k)v.\end{aligned}$$

Thus, with $P' = P'(k)v$ and $Y' = Y'(k)v$, we have

$$\frac{1}{2}(f''(k)v, v) = \mathrm{tr}\,(PY' + P'Y)\{\mathcal{A}, v\} + \rho(v, v) - \frac{\bar{w}^2}{\alpha}\,\mathrm{tr}\left[Y'D_k - v_3 Y \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}\right] \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}^{\mathrm{T}} v_3.$$

Furthermore, $P = P(k)$ is the solution of equation (12). We write it in increments in the direction v :

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta v\} + \frac{\alpha}{2}I\right)(P + \delta P') + (P + \delta P')\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta v\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{\bar{w}^2}{\alpha} D_{k+\delta v} D^{\mathrm{T}}_{k+\delta v} = 0,$$

or

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta v\} + \frac{\alpha}{2}I\right)(P + \delta P') + (P + \delta P')\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta v\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{\bar{w}^2}{\alpha}\left[D_k - \delta v_3 \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}\right]\left[D_k - \delta v_3 \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}\right]^{\mathrm{T}} = 0.$$

After linearization,

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)(P + \delta P') + (P + \delta P')\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \delta\left(\{\mathcal{A}, v\}P + P\{\mathcal{A}, v\}^{\mathrm{T}}\right) + \frac{\bar{w}^2}{\alpha}\left[D_k D_k^{\mathrm{T}} - \delta v_3 D_k \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}^{\mathrm{T}} - \delta v_3 \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix} D_k^{\mathrm{T}}\right] = 0.$$

Subtracting equation (12) termwise from this expression gives

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)P' + P'\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \{\mathcal{A}, v\}P + P\{\mathcal{A}, v\}^{\mathrm{T}} - v_3\frac{\bar{w}^2}{\alpha}\left[D_k \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}^{\mathrm{T}} + \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix} D_k^{\mathrm{T}}\right] = 0. \tag{A.3}$$

Similarly, $Y = Y(k)$ is the solution of the Lyapunov equation (15). We write it in increments in the direction v :

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta v\} + \frac{\alpha}{2}I\right)^{\mathrm{T}}(Y + \delta Y') + (Y + \delta Y')\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta v\} + \frac{\alpha}{2}I\right) + \mathcal{C}^{\mathrm{T}}\mathcal{C} = 0,$$

or

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}}(Y + \delta Y') + (Y + \delta Y')\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right) + \delta(\{\mathcal{A}, v\}^{\mathrm{T}}Y + Y\{\mathcal{A}, v\}) + \mathcal{C}^{\mathrm{T}}\mathcal{C} = 0.$$

Subtracting equation (15) termwise from this expression yields the relation (17).

From (A.3) and (17) it follows that

$$\mathrm{tr}\, P'Y\{\mathcal{A}, v\} = \mathrm{tr}\, Y'\left[\{\mathcal{A}, v\}P - v_3\frac{\bar{w}^2}{\alpha} D_k \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}^{\mathrm{T}}\right] = \mathrm{tr}\, PY'\{\mathcal{A}, v\} - v_3\frac{\bar{w}^2}{\alpha}\,\mathrm{tr}\, Y'D_k \begin{pmatrix} bc^{\mathrm{T}}D \\ 0 \end{pmatrix}^{\mathrm{T}},$$

so

$$\frac{1}{2}(f''(k)v,v) = \operatorname{tr}(PY' + P'Y)\{\mathcal{A},v\} + \rho(v,v) - \frac{\bar{w}^2}{\alpha}v_3 \operatorname{tr}\left[Y'D_k - v_3 Y\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right]\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}$$

$$= 2\operatorname{tr}PY'\{\mathcal{A},v\} + \rho(v,v) - v_3\frac{\bar{w}^2}{\alpha}\operatorname{tr}Y'D_k\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}} - \frac{\bar{w}^2}{\alpha}v_3 \operatorname{tr}\left[Y'D_k - v_3 Y\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right]\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}$$

$$= 2\operatorname{tr}PY'\{\mathcal{A},v\} + \rho(v,v) - \frac{\bar{w}^2}{\alpha}v_3 \operatorname{tr}\left[2Y'D_k - v_3 Y\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right]\begin{pmatrix} bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}.$$

The proof of Lemma 2 is complete.

Proof of Lemma 3. Consider a sequence of stabilizing controllers $\{k^{(j)}\} \in \mathcal{S}$ such that $k^{(j)} \to k \in \partial\mathcal{S}$, i.e., $\sigma(A_k) = 0$. In other words, for any $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$|\sigma(A_{k^{(j)}}) - \sigma(A_k)| = \sigma(A_{k^{(j)}}) < \epsilon$$

for all $j \geqslant N(\epsilon)$.

Let P_j be the solution of the Lyapunov equation (12) associated with the controller $k^{(j)}$:

$$\left(A_{k^{(j)}} + \frac{\alpha_j}{2}I\right)P_j + P_j\left(A_{k^{(j)}} + \frac{\alpha_j}{2}I\right)^{\mathrm{T}} + \frac{\bar{w}^2}{\alpha_j}(D_{k^{(j)}}D_{k^{(j)}}^{\mathrm{T}} + \delta I) = 0.$$

Also, let Y_j be the solution of the dual Lyapunov equation

$$\left(A_{k^{(j)}} + \frac{\alpha_j}{2}I\right)^{\mathrm{T}}Y_j + Y_j\left(A_{k^{(j)}} + \frac{\alpha_j}{2}I\right) + \mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I = 0.$$

Then

$$f(k^{(j)}) = \operatorname{tr}P_j(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) + \rho\|k^{(j)}\|^2$$
$$\geqslant \operatorname{tr}P_j(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) = \operatorname{tr}Y_j\frac{\bar{w}^2}{\alpha_j}(D_{k^{(j)}}D_{k^{(j)}}^{\mathrm{T}} + \delta I)$$
$$\geqslant \frac{\bar{w}^2}{\alpha_j}\lambda_{\min}(Y_j)\operatorname{tr}(D_{k^{(j)}}D_{k^{(j)}}^{\mathrm{T}} + \delta I) \geqslant \frac{\bar{w}^2}{\alpha_j}\frac{\lambda_{\min}(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I)}{2\|A_{k^{(j)}} + \frac{\alpha_j}{2}I\|}\|D_{k^{(j)}}\|_F^2$$
$$\geqslant \frac{\bar{w}^2}{4\sigma(A_{k^{(j)}})}\frac{\varepsilon}{\|A_{k^{(j)}}\| + \sigma(A_{k_{(j)}})}\|D_{k^{(j)}}\|_F^2 \geqslant \frac{\bar{w}^2}{4\epsilon}\frac{\varepsilon}{\|A_{k^{(j)}}\| + \epsilon}\|D_{k^{(j)}}\|_F^2 \xrightarrow[\epsilon\to 0]{} +\infty$$

since

$$0 < \alpha_j < 2\sigma(A_{k_{(j)}})$$

and

$$\|A_{k^{(j)}} + \frac{\alpha_j}{2}I\| \leqslant \|A_{k^{(j)}}\| + \frac{\alpha_j}{2} \leqslant \|A_{k^{(j)}}\| + \sigma(A_{k_{(j)}}).$$

On the other hand,

$$f(k^{(j)}) = \operatorname{tr}P_j(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) + \rho\|k^{(j)}\|^2 \geqslant \rho\|k^{(j)}\|^2 \xrightarrow[\|k^{(j)}\|\to+\infty]{} +\infty.$$

The proof of Lemma 3 is complete.

Proof of Lemma 4. We establish several estimates useful for further considerations. First of all,

$$\|\{\mathcal{A},v\}\| = \|\sum_i \mathcal{A}_i v_i\| \leqslant \sum_i \|\mathcal{A}_i\||v_i| \leqslant \max_i \|\mathcal{A}_i\|\|v\|_1 \leqslant \sqrt{3}\max_i\|\mathcal{A}_i\|\|v\| \qquad \text{(A.4)}$$

since $\|a\|_1 \leqslant \sqrt{n}\|a\|$ for $a \in \mathbb{R}^n$.

In view of (A.4), α can be estimated as follows:

$$\alpha < 2\sigma\big(\mathcal{A}_0 + \{\mathcal{A}, k\}\big) \leqslant 2\|\mathcal{A}_0 + \{\mathcal{A}, k\}\| \leqslant 2\big(\|\mathcal{A}_0\| + \|\{\mathcal{A}, k\}\|\big) \tag{A.5}$$
$$\leqslant 2\big(\|\mathcal{A}_0\| + \sqrt{3}\max_i \|\mathcal{A}_i\|\|k\|\big) \leqslant 2\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)$$

since

$$\|k\| \leqslant \sqrt{\frac{f(k)}{\rho}} \leqslant \sqrt{\frac{f(k_0)}{\rho}}.$$

Finally,

$$\|D_k\|_F = \left\|\begin{pmatrix}(I - k_3bc^{\mathrm{T}})D \\ 0\end{pmatrix}\right\|_F = \|(I - k_3bc^{\mathrm{T}})D\|_F \leqslant \|I - k_3bc^{\mathrm{T}}\|\|D\|_F \tag{A.6}$$
$$\leqslant (1 + |k_3|\|bc^{\mathrm{T}}\|)\|D\|_F \leqslant \left(1 + \sqrt{\frac{f(k_0)}{\rho}}\|bc^{\mathrm{T}}\|\right)\|D\|_F.$$

Applying Lemma A.2 to (16), we have

$$\frac{1}{2}\|f''(k)\| = \frac{1}{2}\sup_{\|v\|=1}(f''(k)v, v) \leqslant 2\sup_{\|v\|=1}|\operatorname{tr} PY'\{\mathcal{A}, v\}| + \rho\sup_{\|v\|=1}|(v, v)| \tag{A.7}$$
$$+\frac{\bar{w}^2}{\alpha}\sup_{\|v\|=1}\left|v_3 \operatorname{tr}\left[2Y'D_k - v_3Y\begin{pmatrix}bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right]\begin{pmatrix}bc^{\mathrm{T}}D \\ 0\end{pmatrix}^{\mathrm{T}}\right|$$
$$\leqslant \|P\|_F\|Y'\|_F\sup_{\|v\|=1}\|\{\mathcal{A}, v\}\| + \rho + \bar{w}^2\sup_{\|v\|=1}|v_3|\sup_{\|v\|=1}\left\|\frac{2}{\alpha}Y'D_k - \frac{v_3}{\alpha}Y\begin{pmatrix}bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right\|_F\left\|\begin{pmatrix}bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right\|_F$$
$$\leqslant \sqrt{3}\|P\|_F\|Y'\|_F\max_i\|\mathcal{A}_i\| + \rho + \bar{w}^2\left(2\frac{1}{\alpha}\|Y'D_k\|_F + \sup_{\|v\|=1}|v_3|\frac{1}{\alpha}\left\|Y\begin{pmatrix}bc^{\mathrm{T}}D \\ 0\end{pmatrix}\right\|_F\right)\|bc^{\mathrm{T}}D\|_F$$
$$\leqslant \sqrt{3}\underbrace{\|P\|_F}_{(A.8)}\underbrace{\|Y'\|_F}_{(A.12)}\max_i\|\mathcal{A}_i\| + \rho + \bar{w}^2\Big(2\underbrace{\frac{1}{\alpha}\|Y'\|}_{(A.11)}\underbrace{\|D_k\|_F}_{(A.6)} + \underbrace{\frac{1}{\alpha}\|Y\|}_{(A.9)}\|bc^{\mathrm{T}}D\|_F\Big)\|bc^{\mathrm{T}}D\|_F.$$

The upper bound for $\|P\|$ is derived as follows:

$$\varepsilon\|P\| \leqslant \lambda_{\min}(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I)\|P\| \leqslant \operatorname{tr} P(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) = f(k) - \rho\|k\|^2 \leqslant f(k) \leqslant f(k_0);$$

consequently,

$$\|P\| \leqslant \frac{f(k_0)}{\varepsilon}$$

and

$$\|P\|_F \leqslant \sqrt{n+1}\frac{f(k_0)}{\varepsilon}. \tag{A.8}$$

Next, by Lemma A.1, equations (18) and (15) imply

$$\operatorname{tr} Y(D_kD_k^{\mathrm{T}} + \delta I) = \operatorname{tr} P(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I).$$

As a result,

$$\frac{\bar{w}^2\delta}{\alpha}\|Y\| \leqslant \frac{\bar{w}^2}{\alpha}\lambda_{\min}(D_kD_k^{\mathrm{T}} + \delta I)\operatorname{tr} Y \leqslant \operatorname{tr} Y\frac{\bar{w}^2}{\alpha}(D_kD_k^{\mathrm{T}} + \delta I)$$
$$= \operatorname{tr} P(\mathcal{C}^{\mathrm{T}}\mathcal{C} + \varepsilon I) = f(k) - \rho\|k\|^2 \leqslant f(k) \leqslant f(k_0),$$

which gives

$$\frac{1}{\alpha}\|Y\| \leqslant \frac{f(k_0)}{\bar{w}^2\delta} \tag{A.9}$$

and, due to (A.5),

$$\|Y\| \leqslant \frac{\alpha}{\bar{w}^2\delta} f(k_0) \leqslant \frac{2f(k_0)}{\bar{w}^2\delta}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right). \tag{A.10}$$

Moreover, using Lemma A.2 together with the upper bounds (A.4) and (A.10), for $\|v\| = 1$ we have

$$\begin{aligned}\lambda_{\max}(\{\mathcal{A}, v\}^{\mathrm{T}} Y + Y\{\mathcal{A}, v\}) &= \|\{\mathcal{A}, v\}^{\mathrm{T}} Y + Y\{\mathcal{A}, v\}\| \leqslant \|Y^2 + \{\mathcal{A}, v\}^{\mathrm{T}}\{\mathcal{A}, v\}\| \\ &\leqslant \|Y\|^2 + \|\{\mathcal{A}, v\}\|^2 \leqslant \frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2.\end{aligned}$$

By Lemma A.1, equations (18) and (17) lead to

$$\operatorname{tr} Y' \frac{\bar{w}^2}{\alpha}(D_k D_k^{\mathrm{T}} + \delta I) = \operatorname{tr} P\big(\{\mathcal{A}, v\}^{\mathrm{T}} Y + Y\{\mathcal{A}, v\}\big).$$

Therefore,

$$\begin{aligned}\frac{\bar{w}^2\delta}{\alpha}\|Y'\| &\leqslant \frac{\bar{w}^2}{\alpha}\lambda_{\min}(D_k D_k^{\mathrm{T}} + \delta I)\operatorname{tr} Y' \leqslant \operatorname{tr} Y' \frac{\bar{w}^2}{\alpha}(D_k D_k^{\mathrm{T}} + \delta I) \\ &= \operatorname{tr} P\big(\{\mathcal{A}, v\}^{\mathrm{T}} Y + Y\{\mathcal{A}, v\}\big) \leqslant \lambda_{\max}\big(\{\mathcal{A}, v\}^{\mathrm{T}} Y + Y\{\mathcal{A}, v\}\big)\operatorname{tr} P \\ &\leqslant \left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2\right](n+1)\|P\| \\ &\leqslant \frac{(n+1)f(k_0)}{\varepsilon}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2\right],\end{aligned}$$

so

$$\frac{1}{\alpha}\|Y'\| \leqslant \frac{(n+1)f(k_0)}{\bar{w}^2\varepsilon\delta}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2\right]. \tag{A.11}$$

Accordingly, in view of (A.5), we obtain

$$\begin{aligned}\|Y'\| &\leqslant \alpha\frac{(n+1)f(k_0)}{\bar{w}^2\varepsilon\delta}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2\right] \\ &\leqslant \frac{2(n+1)f(k_0)}{\bar{w}^2\varepsilon\delta}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2\right] \\ &\qquad \times\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)\end{aligned}$$

and

$$\begin{aligned}&\|Y'\|_F \leqslant \sqrt{n+1}\|Y'\| \\ &\leqslant \frac{2(n+1)^{3/2}f(k_0)}{\bar{w}^2\varepsilon\delta}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i \|\mathcal{A}_i\|^2\right] \\ &\qquad \times\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right).\end{aligned} \tag{A.12}$$

Returning to (A.7), based on the upper bounds above, we finally arrive at the relation

$$\frac{1}{2}\|f''(k)\| \leqslant \sqrt{3(n+1)}\frac{f(k_0)}{\varepsilon}\frac{2(n+1)^{3/2}f(k_0)}{\bar{w}^2\varepsilon\delta}$$

$$\times\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i\|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i\|\mathcal{A}_i\|^2\right]$$

$$\times\left(\|\mathcal{A}_0\| + \max_i\|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)\max_i\|\mathcal{A}_i\| + \rho$$

$$+\bar{w}^2\left(2\frac{(n+1)f(k_0)}{\bar{w}^2\varepsilon\delta}\left[\frac{4f^2(k_0)}{\bar{w}^4\delta^2}\left(\|\mathcal{A}_0\| + \max_i\|\mathcal{A}_i\|\sqrt{\frac{3}{\rho}f(k_0)}\right)^2 + 3\max_i\|\mathcal{A}_i\|^2\right]\right.$$

$$\left.\times\left(1+\sqrt{\frac{f(k_0)}{\rho}}\|bc^{\mathrm{T}}\|\right)\|D\|_F + \frac{f(k_0)}{\bar{w}^2\delta}\|bc^{\mathrm{T}}D\|_F\right)\|bc^{\mathrm{T}}D\|_F.$$

The proof of Lemma 4 is complete.

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