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STOCHASTIC SYSTEMS

Trajectory Countermeasures Against a Linear Observer

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Abstract—We consider the controlled dynamics of three objects in n-dimensional space: Attacker (A), Defender (D), and Target (T). Attacker estimates Target's relative position using a Kalman–Bucy filter and constructs a collision trajectory based on this estimate. In response, the Target deploys the Defender, which disrupts the Attacker's estimation process by interfering with its reception channel, thereby preventing interception. This leads to the formulation of a problem: designing an optimal trajectory for the Defender to maximize the time until the Attacker intercepts the Target. Numerical simulations of the dynamics of each object are conducted to evaluate the effectiveness of deploying the Defender.

Keywords: observations control, Kalman–Bucy filter, ADT-game, guidance countermeasures, trajectory optimisation

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1. INTRODUCTION

Multi-player cooperative control problem are of interest to researchers both for practical reasons, such as the development of unmanned vehicle technologies, and for academic ones, as they provide an analytical framework for evaluating solutions and algorithms. Applied multi-player problems, as well as adversarial two-player game-theoretic problems, are divided into two subclasses with opposing objectives: interception [1] and evasion problems [2, 3] or countering interception [4, 5].

As the number of players on either side increases, so do their opportunities to achieve their goals. The one and simple expansion of two-player game (Attacker and Target) is achieved by adding the Defender. It's called ADT (Attacker–Defender–Target) game [6]. However, before formalizing the problem statement, it is necessary to select the policy of each player. The closest to practical applications is the situation when the Attacker bases its decisions on its sensor data. To achieve this, it must estimate its own state and/or that of the target. This is accomplished using the theory of observation control [7], specifically the Kalman filtering method [8, 9], which has been actively developing for a long time [10], including the use of artificial intelligence technologies [11, 12]. Subsequently, in order to successfully intercept a Target, the Attacker needs to choose a guidance algorithm, the most famous of which is the law of proportional navigation [13].

The Defender's objective meanwhile can be interception of the Attcker [14] (in this case it's called "hard" counteraction) as well as influence on its reception channel [15, 16] (in this case it's called "soft" counteraction). Furthermore both the distance between the Target and the Attacker at the terminal point in time and the time required to intercept the Target can be used as a problem criteria.

At the moment, there are a lot of works that provide an analytical solution to the problem of "hard" counteraction in one formalization or another [17]. However, Monte Carlo [18] or artificial intelligence [19] methods are most often used to solve problems of "soft" counteraction, which does not allow to analyze the solution. At the same time, the theory of observational control and the theory of random processes make it possible to study complex dynamic systems, taking into account the decision-making methods used by objects.

Current work amis to formulate the problem of countering the simplest guidance algorithm and its analytical solution using the described mathematical apparatus. The specifics of the setup are such that, due to the use of the Defender, the equation of the observed process — based on which the Attacker builds an estimate of the Target's relative position — does not match the equation that this observed process actually follows.

The structure of the work is as follows. Section 2 formalizes the ADT-game with incomplete information, where every player has a linear dynamic, and the Attacker gets information about the outside world from the linear observation channel, depending on position of the Target and the Defender in the relative coordinate system associated with the Attacker. Acting in a coalition, by choosing their own direction of movement, the Target and Defender must delay or prevent the interseption of the Target, therefore, Section 3 is devoted to the formulation of the optimal control problem. Sections 4 and 5 provide a solution to this problem and numerical simulations showing the effectiveness of using a Defender. In conclusion, the directions of development of the considered task are proposed.

2. THE ADT GAME MODEL WITH INCOMPLETE INFORMATION

2.1. Description of the Studied System

Let's consider a linear system with 3 types of playes — the Attacker (A), the Defender (D) and the Target (T). The equations of motion of such a system can be written in the form of Ito equations

$$\begin{cases} dx_A(t) = Fx_A(t)dt + B_Au(t)dt + \sigma_A dw_A(t), \\ dx_D(t) = Fx_D(t)dt + B_Dv(t)dt + \sigma_D dw_D(t), \\ dx_T(t) = Fx_T(t)dt + \sigma_T dw_T(t) \end{cases} \tag{1}$$

with some initial conditions at time zero.

Let's assume that the Attacker receives information about the system through the observation channel, which is described by the equation

$$dz(t) = \beta_e e(t)dt + \beta_\varepsilon \varepsilon(t)dt + \sigma_z dw_z(t), \tag{2}$$

where

$$\begin{gathered} x_A, x_D, x_T \in \mathbb{R}^n, \quad F \in \mathbb{R}^{n\times n}, \quad B_D \in \mathbb{R}^{m\times n}, \quad v \in \mathbb{R}^m, \\ B_A \in \mathbb{R}^{r\times n}, \quad u \in \mathbb{R}^r, \quad \beta_e, \beta_\varepsilon \in \mathbb{R}^{n\times c}, \\ e(t) = x_A(t) - x_T(t), \quad \varepsilon(t) = x_A(t) - x_D(t). \end{gathered}$$

In equations (1), (2) w_A, w_D, w_T, w_z — are standart n-dimensional Wiener processes, σ_A, σ_D, σ_T, $\sigma_z \in \mathbb{R}^{n\times n}$. Thus, the coordinates of all players have dimension n, the control vectors of the Attacker and Defender have dimensions r and m, respectively, and the measured value z has dimension c.

With this equation of the measurement channel, it is more convenient to switch from the equations (1) to the equations of relative coordinates e, ε dynamics. They have the form of

$$\begin{cases} d\varepsilon(t) = F\varepsilon(t)dt + B_Au(t)dt - B_Dv(t)dt + \sigma_\varepsilon dw_\varepsilon(t), \\ de(t) = Fe(t)dt + B_Au(t)dt + \sigma_e dw_e(t), \end{cases} \tag{3}$$

where

$$\sigma_\varepsilon = \sigma_A + \sigma_D, \quad \sigma_e = \sigma_A + \sigma_T.$$

Let the Attacker estimates the relative position of the Target. It's assumed that that it does not know about the existence of the Defender and uses the Kalman filter [9] as an optimal estimation algorithm. The equations of states and observations, based on which the Attacker makes an estimation, are written as

$$\begin{cases} de(t) = Fe(t)dt + B_A u(t)dt + \sigma_e dw_e(t), \\ dy(t) = \beta_e e(t)dt + \sigma_y dw_y(t). \end{cases} \tag{4}$$

Proposition 1. *To simplify the subsequent discussion, we assume that in system* (4) *pair* (F, σ_e) *is controllable, pair* (F, β_e) *is observable, and* $\det \sigma_q \neq 0$*, where* q *— is any of symbols* A, D, T, z, y.

Remark 1. The second equation of the (4) system does not match the observation equation (2). This is due to the fact that the Attacker is unaware of the Defender's existence when forming the estimation. It is this circumstance that is associated with the accumulation of errors in estimation the relative position of the Target and the subsequent miss of the Attacker.

For brevity of notation, time arguments will be omitted.

Let's assume that the Attacker uses the following control law to approach the Target (it is obtained when solving some linear-quadratic problems, see, for example, [20]):

$$u = \lambda \hat{e}, \quad \lambda \in \mathbb{R}^{n \times r},$$

where $\hat{e}$ — vector e estimation.

In such case random vector e estimation is described in [7] by solving equations

$$d\hat{e} = F\hat{e}\,dt + B_A u\,dt + \gamma\beta_e^T(\sigma_y\sigma_y^T)^{-1}(\beta_e e dt + \beta_\varepsilon \varepsilon dt + \sigma_z dw_z - \beta_e \hat{e}\,dt), \tag{5}$$

$$\frac{d\gamma}{dt} = F\gamma + \gamma F^T + \sigma_e\sigma_e^T - \gamma\beta_e^T(\sigma_y\sigma_y^T)^{-1}\beta_e\gamma^T, \tag{6}$$

where γ — mean square filtering errors.

Then, denoting $\varphi(\gamma) = \gamma\beta_e^T(\sigma_y\sigma_y^T)^{-1}$ and then substituting the expression for control u into the filtration equations (5), (6), we obtain the dynamic equation of vector $\hat{e}$ in the following form

$$d\hat{e} = \hat{F}(\gamma)\hat{e}\,dt + \varphi(\gamma)\beta_\varepsilon\varepsilon\,dt + \varphi(\gamma)\beta_e e\,dt + \hat{\sigma}_e(\gamma)dw_z,$$

where

$$\hat{F}(\gamma) = F + B_A\lambda - \varphi(\gamma)\beta_e,$$
$$\hat{\sigma}_e(\gamma) = \varphi(\gamma)\sigma_z.$$

As a result, the dynamics equations of all the players and the Kalman filter make up the system

$$\begin{cases} d\varepsilon = F\varepsilon dt + B_A\lambda\hat{e}\,dt - B_D v\,dt + \sigma_\varepsilon dw_\varepsilon, \\ de = Fe\,dt + B_A\lambda\hat{e}\,dt + \sigma_e dw_e, \\ d\hat{e} = \hat{F}(\gamma)\hat{e}\,dt + \varphi(\gamma)\beta_\varepsilon\varepsilon\,dt + \varphi(\gamma)\beta_e e\,dt + \hat{\sigma}_e(\gamma)dw_z, \\ \dot{\gamma} = F\gamma + \gamma F^T + \sigma_e\sigma_e^T - \gamma\beta_e^T(\sigma_y\sigma_y^T)^{-1}\beta_e\gamma^T. \end{cases} \tag{7}$$

2.2. Problem Statement

Let's assume that the mathematical expectations of the variables of the (7) system are known at the initial moment of time:

$$\mathbb{E}\, q(0) = \mu_q(0), \tag{8}$$

where q — is any of symbols ε, e, $\hat{e}$. The initial value of the covariance matrices of all vectors will, in turn, be considered unknown.

Remark 2. In fact, setting the initial conditions in this form means that the Target-Defender coalition knows the initial position of the Attacker with accuracy up to a certain measurement error for some reason (for example, according to its own measurements). We also assume that the coalition knows the Attacker's algorithm for estimating the position of the Target. This leads to the following formulation of the countermeasures problem.

Problem 1. For a system whose dynamics is described by the equations (7), the mathematical expectation of random processes at the initial moment of time is described by (8), the travel time is limited by t_*, it is necessary to find the control $v(t)$, limited with bounded absolute value $|v| \leqslant \varkappa$, which delivers the maximum to the criterion

$$J[e] = \mathbb{E}\left(e^T(t_*)e(t_*)\right). \tag{9}$$

Remark 3. At its core, the (9) criterion is a quadratic miss. Let's also pay attention to the type of initial conditions (8) — the variance of random vectors is considered unknown. As will be shown below, due to the type of criterion (9) and the control form, information about the variance will not be needed to solve the problem.

3. REDUCTION OF THE PROBLEM TO A DETERMINISTIC FORM

It is easy to see that the criterion (9) can be transformed to the form

$$J[e] = \mathbb{E}\left(e^T(t_*)e(t_*)\right) = \operatorname{tr}\left(\Sigma_e(t_*)\right) + \mu_e^T(t_*)\mu_e(t_*), \tag{10}$$

where $\Sigma_e(t) = \operatorname{Var} e(t)$ — is a variance of a random vector $e(t)$.

It is also obvious that the dynamic equations of mathematical expectations have the form

$$\begin{cases} \dot{\mu}_\varepsilon = F\mu_\varepsilon + B_A\lambda\hat{\mu}_e - B_D v, \\ \dot{\mu}_e = F\mu_e + B_A\lambda\hat{\mu}_e, \\ \dot{\hat{\mu}}_e = \hat{F}(\gamma)\hat{\mu}_e + \varphi(\gamma)\beta_\varepsilon\mu_\varepsilon + \varphi(\gamma)\beta_e\mu_e, \\ \dot{\gamma} = F\gamma + \gamma F^T + \sigma_e\sigma_e^T - \gamma\beta_e^T(\sigma_y\sigma_y^T)^{-1}\beta_e\gamma^T. \end{cases} \tag{11}$$

If we talk about variance dynamic equations, the following lemma is important, which is given without proof due to its well-known nature.

Lemma 1. *If a random process $x(t)$ follows the Ito equation*

$$dx(t) = A(t)x(t)dt + B(t)v(t)dt + \sigma(t)dw(t)$$

with deterministic matrices $A(t)$, $B(t)$, $\sigma(t)$ and control $v(t)$, then the dynamics of its covariance matrix follows the Ricatti equation

$$\dot{\Sigma}(t) = A(t)\Sigma(t) + \Sigma(t)A^T(t) + \sigma(t)\sigma^T(t).$$

Let's apply Lemma 1 to the first three equations of (7). Then, the variance of the random vector $x = \begin{pmatrix}\varepsilon^T & e^T & \hat{e}^T\end{pmatrix}^T$ is described by the equation

$$\dot{\Sigma} = \Phi(\gamma)\Sigma + \Sigma\Phi^T(\gamma) + \sigma_{\text{total}}\sigma_{\text{total}}^T, \tag{12}$$

where

$$\Phi(\gamma) = \begin{pmatrix} F & 0 & B_A\lambda \\ 0 & F & B_A\lambda \\ \varphi(\gamma)\beta_\varepsilon & \varphi(\gamma)\beta_e & \hat{F}(\gamma) \end{pmatrix}, \quad \sigma_{\text{total}} = \operatorname{diag}\left(\sigma_\varepsilon,\, \sigma_e,\, \hat{\sigma}(\gamma)\right).$$

Lemma 2. $\operatorname{tr}(\Sigma_e(t_*))$ *does not depend on the choice of control* $v(t)$*.*

Proof. It is easy to see that the equations of the (12) system, together with the last equation of the (7) system, form a closed system in which there is no control, which leads to the statement of the lemma. Thus, maximizing the criterion (10) by control is equivalent to maximizing only its second term.

Remark 4. In fact, the criterion (10) contains 2 terms responsible for the observer's "miss". The first of them characterizes the influence of the measurement channel noise of the criterion, which means the influence of the signal emitted by the Defender.

The second term characterizes the effect on the criterion of the Defender's own trajectories. Lemma 2 actually states that the inability to control the signal emitted by the Defender leads to the inability to influence the first term in the criterion (10). This term would be important if it were possible to control the emitted signal, namely the coefficient β_ϵ. However, this is not possible in the formulation under study, and therefore the criterion will be maximized only using the second term responsible for the trajectory of the observer and Defender.

On the other hand, this form of criterion makes it possible to compare the contributions of each component to the overall task criterion at each step or during the evaluation of the entire mission. At the same time, considering each term separately, it is possible to establish the capabilities of the Defender to counteract both by maneuvering and by influencing the measuring channels.

Next, note that the Riccati equation in the (11) system is an independent differential equation. In this case, the remaining three equations of the system, in fact, constitute a linear non-autonomous system of differential equations. This, in turn, means that instead of the initial dynamic system (7), we can consider a system of mathematical expectations described by the equations

$$\begin{cases} \dot{\mu}_\varepsilon = F\mu_\varepsilon + B_A\lambda\hat{\mu}_e - B_D v, \\ \dot{\mu}_e = F\mu_e + B_A\lambda\hat{\mu}_e, \\ \dot{\hat{\mu}}_e = \hat{F}(\gamma)\hat{\mu}_e + \varphi(\gamma)\beta_\varepsilon\mu_\varepsilon + \varphi(\gamma)\beta_e\mu_e, \end{cases}$$

which we will write down more briefly as

$$\dot{\mu} = \Phi(\gamma)\mu + Bv, \tag{13}$$

where

$$\mu = \begin{pmatrix} \mu_\varepsilon \\ \mu_e \\ \hat{\mu}_e \end{pmatrix}, \quad \Phi(\gamma) = \begin{pmatrix} F & 0 & B_A\lambda \\ 0 & F & B_A\lambda \\ \varphi(\gamma)\beta_\varepsilon & \varphi(\gamma)\beta_e & \hat{F}(\gamma) \end{pmatrix}, \quad B = \begin{pmatrix} -B_D \\ 0 \\ 0 \end{pmatrix},$$

and $\gamma(t)$ — is a solution of a differential equation

$$\dot{\gamma} = F\gamma + \gamma F^T + \sigma_e\sigma_e^T - \gamma\beta_e^T(\sigma_y\sigma_y^T)^{-1}\beta_e\gamma^T$$

with an initial condition $\gamma(0) = \gamma_0$.

Instead of the (9) criterion, taking into account the 2 Lemma, we write a new integral criterion for the (13) system

$$J'[e] = -\mu_e^T(t_*)\mu_e(t_*) = -\frac{1}{2}\int\limits_0^{t_*}\langle\dot{\mu}, Q\mu\rangle\,dt = -\frac{1}{2}\int\limits_0^{t_*}\langle A\mu, Q\mu\rangle\,dt,$$

where

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & E_n & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and formulate the following problem.

Problem 2. For a system whose dynamics is described by the equations (13), the initial conditions are set by the equations (8), the travel time is limited by t_*, it is necessary to find the control $v(t)$, limited modulo $|v| \leqslant \varkappa$, which delivers minimum of the criterion

$$J'[\mu] = -\frac{1}{2}\int\limits_0^{t_*}\langle A\mu, Q\mu\rangle\,dt. \tag{14}$$

Proposition 2. *The solution to the Problem 2 is the solution to the Problem 1.*

Proof. The proof, in fact, is the reasoning carried out above.

4. SOLVING A DETERMINISTIC OPTIMAL CONTROL PROBLEM

To solve the Problem 2, let's use the Pontryagin maximum principle [21]. To do this, we will write down the Pontryagin function

$$H = \langle\psi, \Phi(\gamma)\mu + Bv\rangle + \frac{1}{2}\langle\Phi(\gamma)\mu, Q\mu\rangle.$$

The maximum condition

$$\langle\psi, Bv\rangle = \langle B^T\psi, v\rangle \longrightarrow \max_{|v|\leqslant\varkappa}$$

gives the following form of optimal control almost everywhere:

$$v^*(t) = -\varkappa\left|B_D^T\psi_1\right|^{-1} B_D^T\psi_1, \tag{15}$$

where ψ_i — are components of the vector $\psi = \begin{pmatrix}\psi_1^T & \psi_2^T & \psi_3^T\end{pmatrix}^T$.

The equations of the conjugate system are written as

$$\dot{\psi} = -\Phi^T(\gamma)\psi - \frac{1}{2}\left(\Phi^T(\gamma)Q + Q\Phi(\gamma)\right)\mu. \tag{16}$$

The initial conditions for the (13) system are set as (8). At the same time, for conjugate variables from the transversality conditions [21], values are known at the right end of the trajectory in the form of $\psi(t_*) = 0$.

The solution of such two-point problem is found by the sequential approximation method described in [22], having previously solved the Ricatti differential equation separately. Its essence is as follows: any valid control $v_1(t)$ is chosen as an initial approximation. Next, at the kth iteration of the method, it is required:

(1) to solve the Cauchy problem for the equation (13) with initial conditions (8) and control $v_k(t)$. Thus, we obtain the trajectory $\mu_k(t)$ by $[0, t_*]$;

(2) to solve the conjugate system (16) with terminal conditions $\psi(t_*) = 0$ from t_* to 0 for $v(t) = v_k(t)$, $\mu(t) = \mu_k(t)$. Thus, we obtain the conjugate variables $\psi_k(t)$ by $[0, t_*]$;

(3) to define the control of $v_{k+1}(t)$ on $[0, t_*]$, according to the gather (15).

The algorithm stops when the value

$$\mathrm{Err} = \sum_{t=0}^{t_*} |v_{k+1}(t) - v_k(t)| \tag{17}$$

becomes less than the pre-selected number $\bar{\delta}$: $\mathrm{Err} < \bar{\delta}$. The amount in (17) is taken from all points of the time grid entered by the user on the segment $[0, t_*]$.

Remark 5. Solving such a boundary value problem can take a considerable amount of time. Namely, executing a function written in MATLAB to calculate the control in the Section 6 takes an average of 0.0967 s on an Apple M1 processor. In the case of high object speeds, such a time to solve the boundary value problem will not allow successful Target protection.

5. LINEARIZED SYSTEM RESEARCH

Note that for linear systems, it is known that in the case of observability and controllability of the system, the Kalman filter error γ, and therefore the feedback coefficient itself $\varphi(\gamma)$ converge at $t \longrightarrow \infty$ for any initial matrix $\gamma(0)$ [23]. In this case, given the terminal form of the (10) functional, it would be reasonable to consider the (13) system starting from some large point in time t with a constant matrix γ.

According to the Proposition 1, the conditions of Theorem 3.7 [23, p. 237] on the convergence of the solution of the Riccati differential equation are fulfilled. Therefore, the limiting value of the covariance matrix satisfies the algebraic Riccati equation

$$F\mathcal{P} + \mathcal{P}F^T + \sigma_e\sigma_e^T - \mathcal{P}\beta_e^T(\sigma_y\sigma_y^T)^{-1}\beta_e\mathcal{P}^T = 0.$$

The limiting value of the feedback coefficient is found according to the equation

$$\varphi = \mathcal{P}\beta_e^T\left(\sigma_y\sigma_y^T\right)^{-1}.$$

In this case, for large t, instead of a dynamic system (13) with a nonlinear Riccati equation, we can consider a linear system of mathematical expectations described by the equations

$$\begin{cases} \dot{\mu}_\varepsilon = F\mu_\varepsilon + B_A\lambda\hat{\mu}_e - B_D v, \\ \dot{\mu}_e = F\mu_e + B_A\lambda\hat{\mu}_e, \\ \dot{\hat{\mu}}_e = \hat{F}\hat{\mu}_e + \varphi\beta_\varepsilon\mu_\varepsilon + \varphi\beta_e\mu_e, \end{cases}$$

which we will write down more briefly as

$$\dot{\mu} = A\mu + Bv, \tag{18}$$

where

$$\mu = \begin{pmatrix} \mu_\varepsilon \\ \mu_e \\ \hat{\mu}_e \end{pmatrix}, \quad A = \begin{pmatrix} F & 0 & B_A\lambda \\ 0 & F & B_A\lambda \\ \varphi\beta_\varepsilon & \varphi\beta_e & \hat{F} \end{pmatrix}, \quad B = \begin{pmatrix} -B_D \\ 0 \\ 0 \end{pmatrix}.$$

The optimal control problem for the (18) system is formulated using the (14) criterion as follows:

Problem 3. For a system whose dynamics is described by the equations (18), the initial conditions are set by the equations (8), the travel time is limited by t_*, it is necessary to find the control $v(t)$, limited by module $|v| \leqslant \varkappa$, which delivers minimum to the criterion

$$J'[\mu] = -\frac{1}{2}\int\limits_0^{t_*} \langle A\mu, Q\mu\rangle \, dt.$$

In the Problem 3 the law of optimal control coincides with (15), however, the equations of the conjugate system, unlike the equations of (16), are written as

$$\dot{\psi} = -A^T\psi - \frac{1}{2}\left(A^TQ + QA\right)\mu.$$

6. NUMERICAL SIMULATION

6.1. Simulation Parameters

Let's consider a system that is represented by a double integrator. The dynamics of such a system is described in simple movements. We will denote by the symbol E_k a unit matrix of size $k \times k$, and by the symbol 0_k the matrix $k \times k$, each element of which is equal to 0. Next, let's assume that the matrices of the (1) system have the form

$$F = \begin{pmatrix} 0_2 & E_2 \\ 0_2 & 0_2 \end{pmatrix}, \quad B_A = B_D = \begin{pmatrix} 0_2 \\ E_2 \end{pmatrix}, \quad n = 4, \quad m = r = 2. \tag{19}$$

Other constants are set as follows:

$$\lambda = -6\begin{pmatrix} E_2 & E_2 \end{pmatrix}, \quad \varkappa = 6 \times 10^{-3}, \quad \beta_e = \beta_\varepsilon = E_4, \quad t_* = 50, \tag{20}$$

$$\sigma_A = \sigma_D = \sigma_T = \sigma_z = \sigma_y = 10^{-3} \times E_4, \tag{21}$$

$$\mathbb{E}\, x_A(0) = x_A^0 = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbb{E}\, x_T(0) = \mathbb{E}\, x_D(0) = x_D^0 = x_T^0 = \begin{pmatrix} 0 \\ 0 \\ -0.1 \\ 0.2 \end{pmatrix}. \tag{22}$$

Remark 6. For the (4) system with matrices selected according to (19)–(22), proposition 1 is fulfilled.

6.2. Simulation Results for Successful Defender Operation over the Entire Time Period

In the following we will talk about the mathematical expectations of the corresponding random processes only, since this characteristic fully reflects the essence of the problem.

Let's consider the case when the Defender has an expected effect on the receiving channel, affecting the trajectory of the Attacker as a whole. At the same time, up to the terminal moment of time t_*, we believe that the Attacker does not change his targeting tactics, even though the estimation in his receiving channel may deteriorate over time. The trajectories along which objects move with control calculated according to the equality (15) and the equations (18) in the plane of mathematical expectations of their coordinates are shown in Fig. 1 in the case of using a Defender, as well as in Fig. 2 without using a Defender.

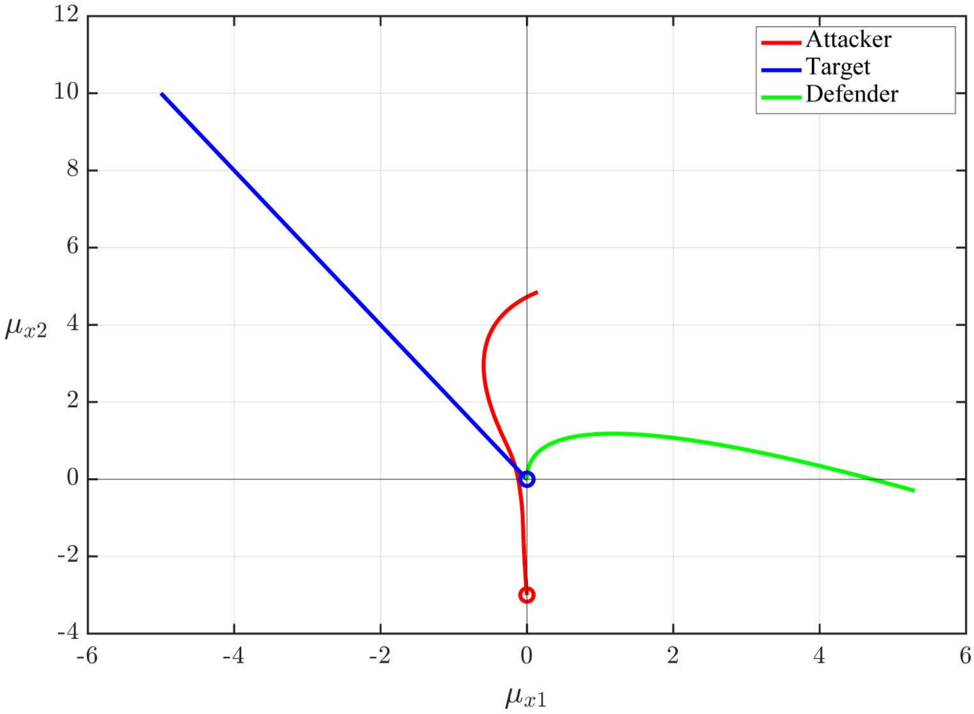

Fig. 1. Object movement trajectories when using the Defender.

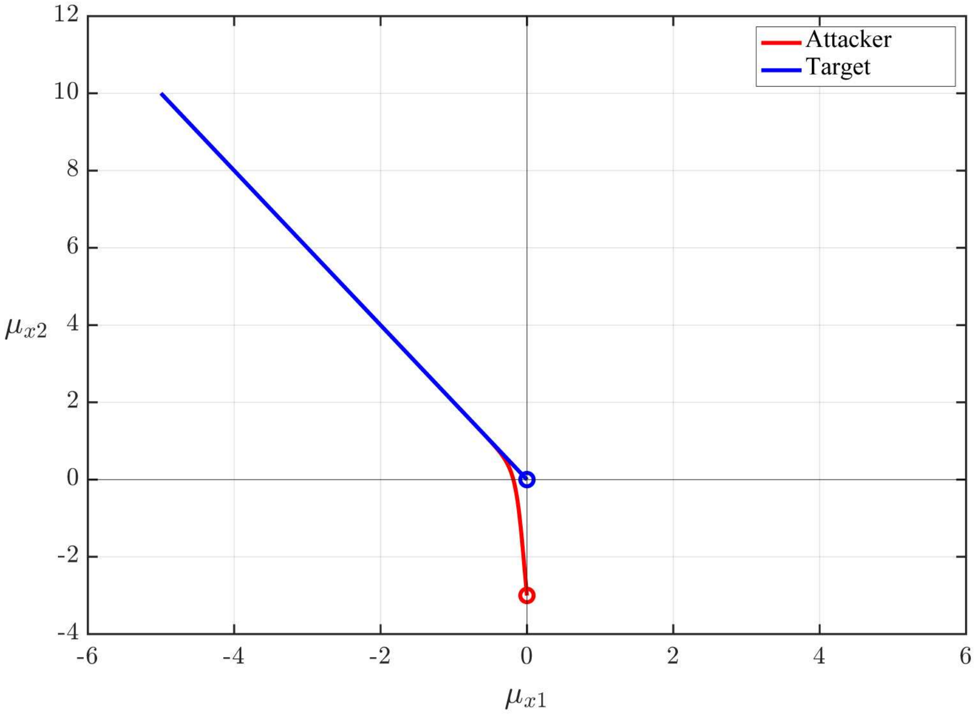

Fig. 2. Object movement trajectories without using a Defender.

In the first case, the criterion value is

$$J'[\mu_1] = -53.06,$$

while in the second, it is

$$J'[\mu_2] \approx 0.$$

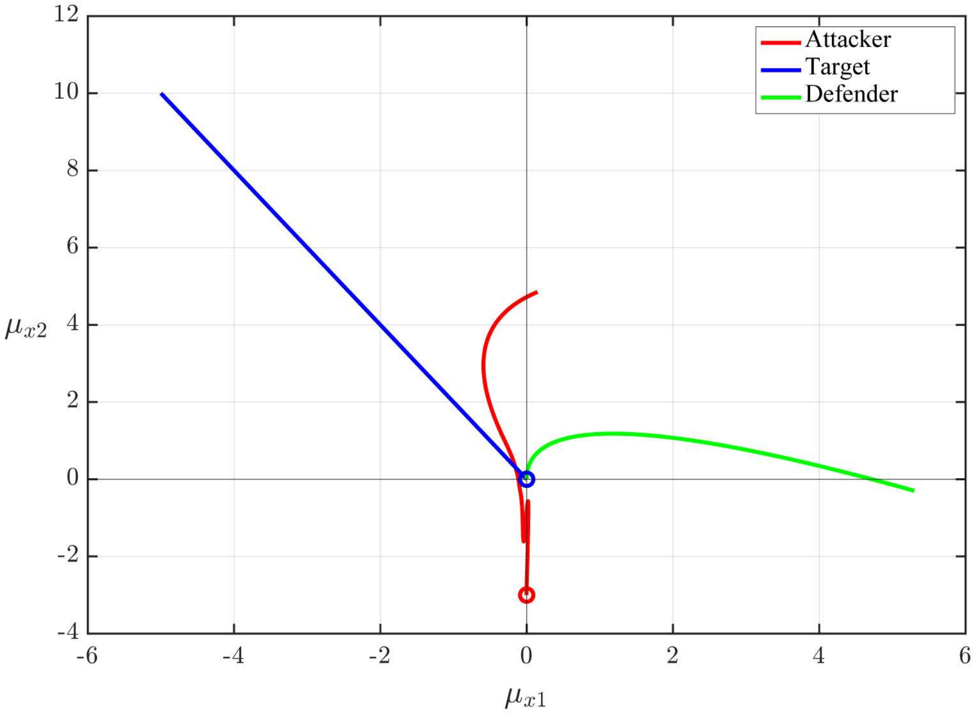

Fig. 3. Object movement trajectories when using the Defender, calculated according to (13).

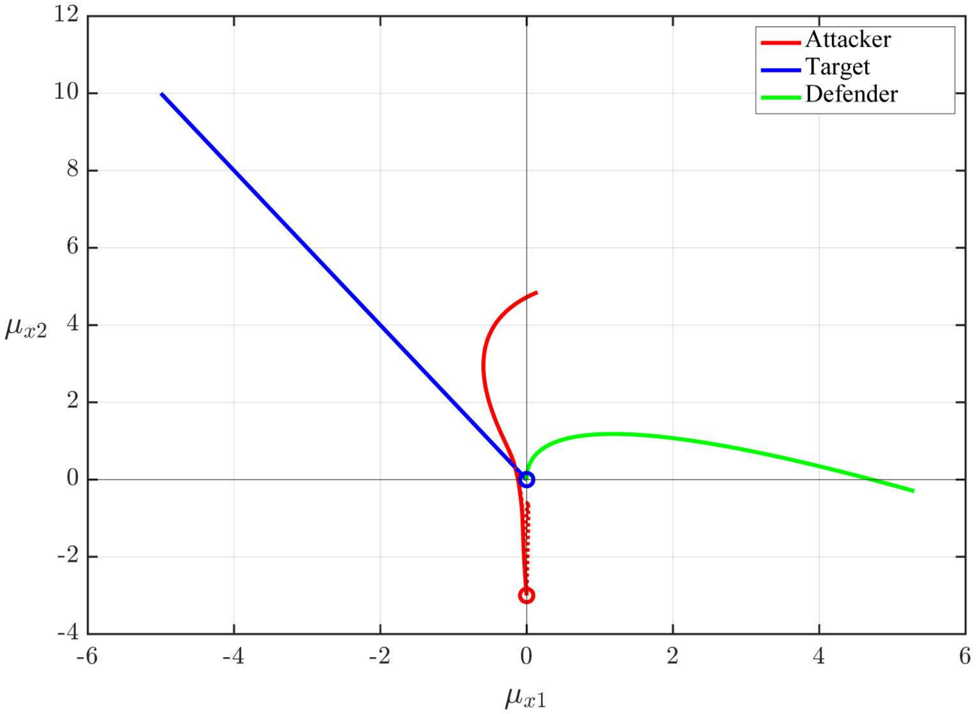

Fig. 4. Object movement trajectories when using the Defender, calculated according to (18) and (13).

At the same time, the trajectories along which objects move with control calculated according to the equality (15) and the equations (13) in the plane of mathematical expectations of their coordinates are shown in Fig. 3 (for $\gamma_0 = 4 \times 10^{-4}\, E_4$). As one can see, after some time, which is required for the convergence of the solution of the Riccati equation, the mathematical expectations of the coordinates of the objects practically do not differ. This is especially clearly seen in Fig. 4,

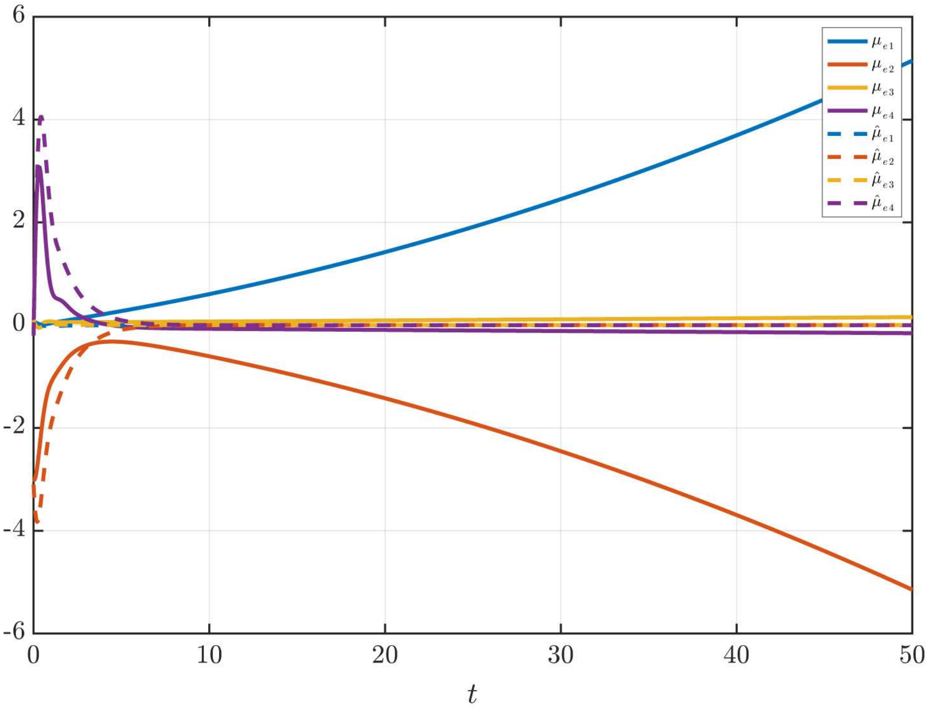

Fig. 5. Change of the vectors μ_e and $\hat{\mu}_e$ components over time.

on which solid lines represent the trajectories of objects calculated according to the equations (18), and dotted lines — according to (13).

We also show in Fig. 5 graphs of changes in the components of mathematical expectations of the vector of relative coordinates and its estimates.

6.3. Simulation Results for the Successful Operation of the Defender in a Part of the Time Period

Now let's assume that at some point in time $t = \tau$ the signal emitted by the Defender stopped influencing the choice of the Attacker's direction of movement. In practice, such a situation may be associated with a failure of the signal-generating element on board the Defender or a correction of the Attacker's guidance algorithms. To describe such a scenario, which we will call a scenario with correction of guidance algorithms, we introduce the concept of the moment (time) of interception of t^* according to the equality

$$t^* = \min\{t : |\mu_e(t)| < r_0\}.$$

Let's choose $r_0 = 10^{-4}$. The trajectories of all objects in the plane of mathematical expectations up to the moment of interception, if the Defender is not in use, are shown in Fig. 6. The interception time is equal to $t^* = 5.0$.

Let's assume that when using the Defender, the correction of the guidance algorithms occurred at $t = \tau = 20$. The trajectories of objects moving in the plane of mathematical expectations up to the moment of interception in this case are shown in Fig. 7. The interception time is equal to $t^* = 25.05$.

A significant increase in interception time makes the use of even one Defender advisable if the Attacker's targeting algorithms involve correcting the work taking into account the use of the Defender by the Target. In practice, this allows you to provide the Target with a temporary reserve to perform an evasive maneuver, build a subsequent defense strategy, or release other Defenders.

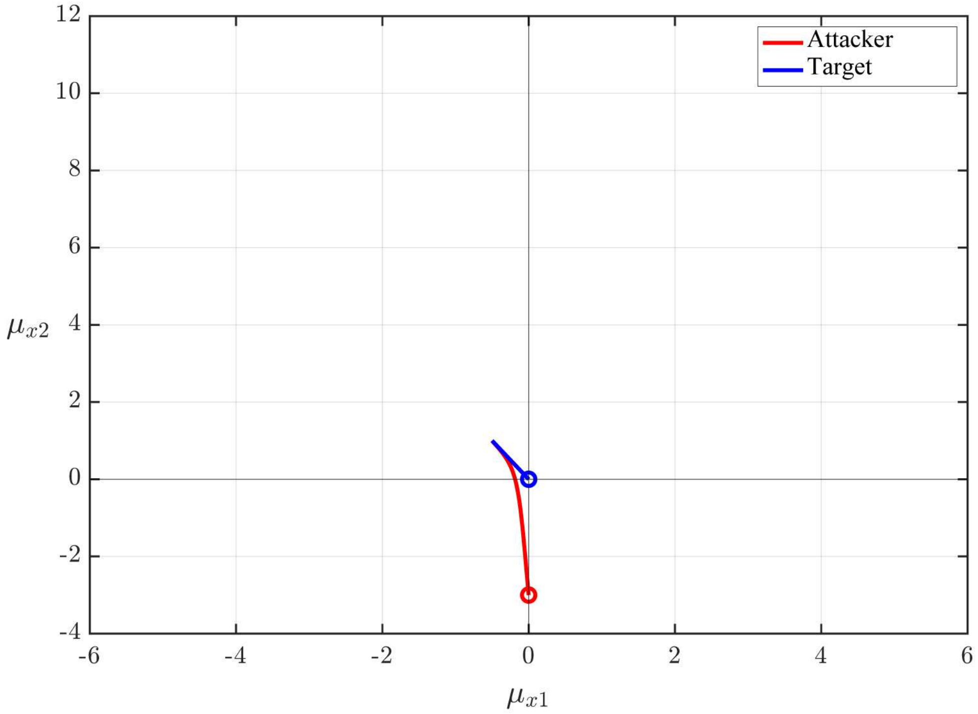

Fig. 6. Object movement trajectories without using a Defender until the moment of interception.

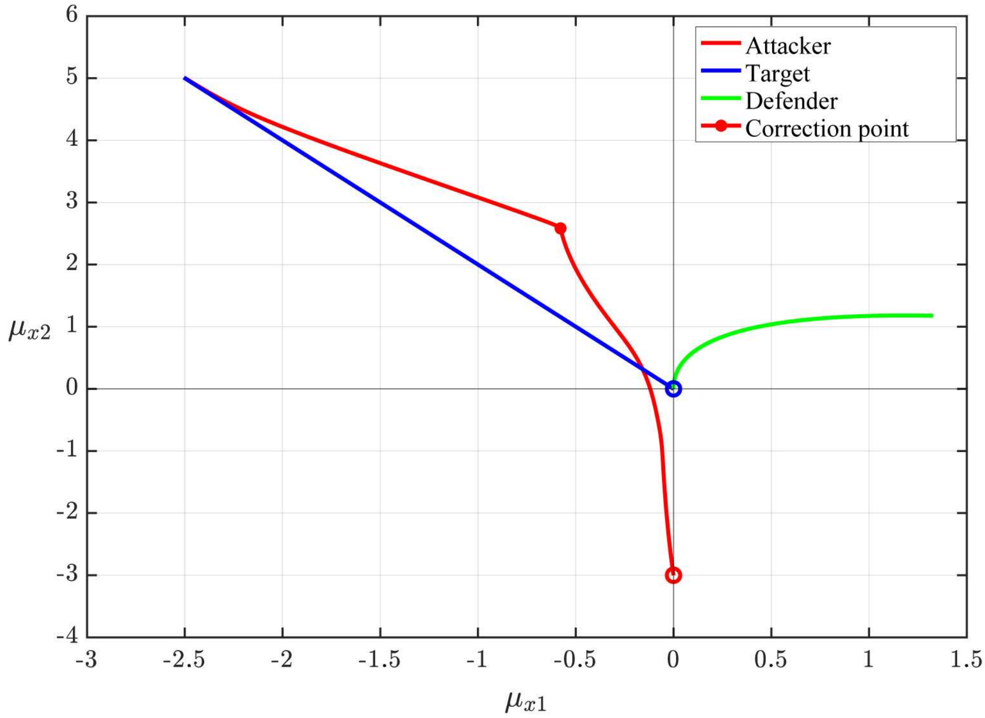

Fig. 7. Object movement trajectories when using a Defender until the moment of interception. The moment of correction of the Attacker's targeting algorithms is indicated by a solid red dot.

Remark 7. The proposed method for constructing the Defender's trajectory also involves using it if the known equations are not the equations of motion of objects in the laboratory report system (i.e., the equations (1)), but the equations of motion of objects in the coordinate system associated with the Attacker (i.e., the equations (3)).

6.4. Comparison of Proposed and Alternative Strategies

Let's compare the value of the criterion on the optimal trajectory obtained in Section 6.2 and on the trajectories obtained using simpler heuristic strategies:

(1) moving in the direction opposite to the Target;
(2) maximum proximity to the Attacker.

The first strategy is described by the control law

$$\tilde{v}(t) = -\varkappa \left|\left(x_T^0\right)_{3,4}\right|^{-1} \left(x_T^0\right)_{3,4} = \text{const}, \tag{23}$$

and the second one, by –

$$\bar{v}(t) = -0.02\left(\left(x_D(t)\right)_{1,2} - \left(x_A(t)\right)_{1,2}\right). \tag{24}$$

The trajectories of all objects in the plane of mathematical expectations when using the Defender of the law of control (23) are shown in Fig. 8. The value of the criterion J' in this case is

$$J'[\mu_3] = -51.6962 > J'[\mu_1].$$

The trajectories of the movement of objects in the case of using the Defender of the law of control (24) are shown in Fig. 9. The value of the criterion J' in this case is

$$J'[\mu_4] = -0.93297 > J'[\mu_1].$$

Thus, the proposed optimal solution allows you to obtain a trajectory that significantly reduces the criterion compared to simpler strategies and, as a result, improves the tactical situation for the Target.

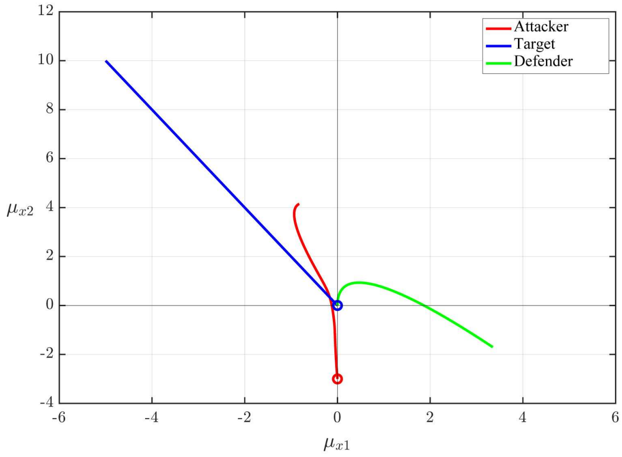

Fig. 8. Object movement trajectories when the Law of Control (23) is used by the Defender.

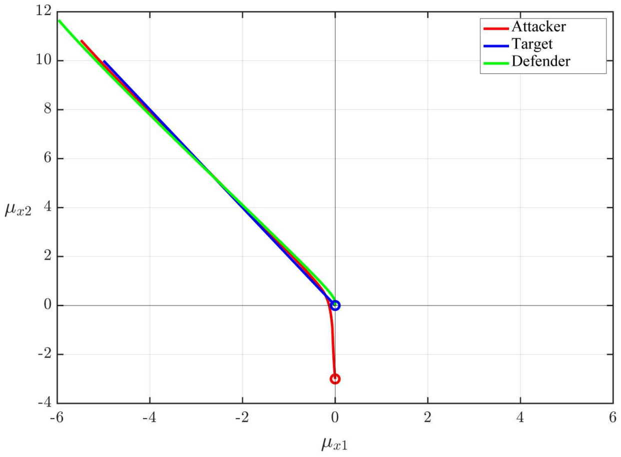

Fig. 9. Object movement trajectories when the Law of Control (24) is used by the Defender.

7. CONCLUSIONS

The work showed the effectiveness of using a Defender in the problem of distracting an Attacker from intercepting a target. The proposed model of using a Defender when influencing an Attacker's receiving channel makes it possible to qualitatively model and predict changes in the Attacker's guidance algorithms, as well as formalize the formulation of problems for optimizing the Target's trajectory evasion from interception in a game with incomplete information of three players.

Further work will be aimed at formalizing and researching interception problems with more complex guidance algorithms, including those that take into account the possibility of using one or more Defenders and determining how to use them.

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