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LINEAR SYSTEMS

Output Adaptive Compensation of External Disturbances in MIMO Systems

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Abstract—A problem of output-feedback adaptive compensation of external unknown deterministic disturbances in linear multidimensional plants is considered. The proposed solution is based on an adaptive implementation of the internal model principle together with a special observer. The form of the observer allows us to obtain a regression model of the multidimensional disturbance and design an adaptive controller with a number of tuning parameters equal to the number of unknown coefficients of the characteristic polynomial of the disturbance model. Under certain conditions, the dynamic order of the observer is significantly lower than that of well-known solutions.

Keywords: adaptive compensation of external disturbances, internal model principle, multidimensional systems

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1. INTRODUCTION

The internal model principle is one of an effective and well developed methods for asymptotic compensation of external deterministic disturbances due to many practical applications such as active vibration systems [1–3], internal combustion engines [4, 5], rolling mills [6, 7], continuous casting process [8], computer hard disk drives [9, 10], seismic protection systems [11], offshore cranes [12, 13], systems for ripple reduction in current converters [14] and in synchronous machines [15], etc. According to this principle, an external disturbance is modelled as the output of an autonomous dynamic system (exosystem), and this exosystem is replicated in the structure of the closed-loop system for the disturbance compensation. Initially, the internal model principle was developed for the linear systems and the linear model of external disturbances with known parameters [16; 17; 18, Chapter 4]. However, the assumption about the knowledge of the disturbance model parameters reduces the practical relevance of the method. The case when only the class of disturbances is known in a priori, but not the parameters of a disturbance model, is of great practical significance.

An adaptive version of the internal model principle, when the parameters of an exosystem model are unknown (see [19, Section 1.4; 20] and the references therein) was suggested for the compensation of unknown disturbances. This approach was developed for both systems with the scalar input and the scalar output [21–25] as well as for multidimensional systems with the state-feedback control [19, Section 4.2; 26; 27] or with the output-feedback [20, 28]. A special class is represented by multidimensional systems with the input disturbance [29].

However, well-known solutions of the output-feedback disturbances compensation in multidimensional systems using the internal model principle (both non-adaptive [17; 18, Chapter 4] and adaptive [20, 28]) include construction of a special observer (dynamic filter, precompensator) for

each channel of a multidimensional system. The dimension of this observer is equal to the dimension of the disturbances generator model. Therefore, the total dimension of the used observers is equal to the product of the dimension of disturbances generator model by the number of channels in a control system. It causes increasing the dynamic order of the controller as well as of the number of adjustable parameters.

It is shown in the paper that under additional nonrestrictive assumption about the block-diagonal structure of the disturbance generator it is possible to design a control system with one observer of dimension equaled to dimension of the disturbance generator. As a result, both dynamic order of controller and the number of adjustable parameters are reduced significantly.

The solution proposed is based on the approach reported in [19, 30, 31] and consists in construction of a special external disturbance observer that makes it possible to get the convenient unmixed parametrization of a disturbance.

The paper is organized as follows. In Section 2, the problem statement is described. In Section 3, a disturbance observer is presented, and the problems of its construction for multidimensional systems are discussed. In Section 4 and 5, an adaptive controller is designed. An example with the simulation results is presented in Section 6.

Notations: $|y|$ is the Euclidean norm of a vector y; s is a complex variable; $W(s)[\delta]$ denotes a signal transformation δ (scalar or vector signal) by dynamic block with the transfer function (or the transfer matrix for a vector signal) $W(s)$.

2. PROBLEM STATEMENT

We consider the multidimensional time-invariant plant

$$\begin{cases} \dot{x} = Ax + B(u + \delta), \quad x(0), \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the vector of control signals, $y \in \mathbb{R}^m$ is the vector of output variables, $n \geqslant m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ are known matrices with constant entries, $\delta \in \mathbb{R}^m$ is the vector of unmeasured external disturbances.

Assumption 1. The following assumptions are accepted regarding the plant (1):

A.1.1. the triplet (A, B, C) is fully controllable and observable, and matrices B and C have rank, i.e. rank $B =$ rank $C = m$;

A.1.2. the state-space model (1) is minimum phase;

A.1.3. the vector of output variables y is measurable only, while the state vector x is not accessible for measurements.

Remark 1. Assumption A.1.1 implies that the system (1) is controllable with regard to the vector of the output variables y (see the definition and the criteria in [32]).

Remark 2. In this paper, the minimum phase model (1) is the model that does not have invariant zeros or all its invariant zeros have negative real parts. An *invariant zero* is a complex number s_0, which substitution into the Rosenbrock matrix (that is, for $s = s_0$)

$$P(s) = \begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

reduces the column rank of the matrix [33, p. 237]. For systems with the same number of control signals and the output variables having square Rosenbrock matrix, it means singularity of this matrix for $s = s_0$.

Assumption 2. The external disturbance $\delta = [\delta_1, \ldots, \delta_m]^\top$ is such that:

A.2.1. each its component δ_i $(i = 1, \ldots, m)$ can be represented as the output of the linear generator

$$\begin{cases} \dot{\xi}_i = \Gamma_i \xi_i, \qquad \xi_i(0), \\ \delta_i = h_i^\top \xi_i, \end{cases} \tag{2}$$

where $\xi_i \in \mathbb{R}^{q_i}$ is the state vector, $\Gamma_i \in \mathbb{R}^{q_i \times q_i}$ is the matrix with constant parameters, all eigenvalues of which are on the imaginary axis and simple, $h_i \in \mathbb{R}^{q_i}$ is the vector of constant parameters;

A.2.3. parameters of the matrices Γ_i and the vectors h_i are unknown, while the dimensions q_i are known;

A.2.3. the pairs $(h_i^\top, \Gamma_i)$ are fully observable.

Remark 3. The general exosystem generating the disturbance δ can be represented in the form

$$\begin{cases} \dot{\xi} = \Gamma \xi, \qquad \xi(0), \\ \delta = H \xi, \end{cases} \tag{3}$$

where $\xi = [\xi_1^\top, \ldots, \xi_m^\top]^\top \in \mathbb{R}^q$, matrices $\Gamma = \operatorname{diag}\{\Gamma_i\}$ and $H = \operatorname{diag}\{h_i^\top\}$ $(i = 1, \ldots, m)$ are block-diagonal, and $q = \sum_{i=1}^m q_i$.

The following problem is solved in the paper.

Problem 1. The problem is to design an output-feedback control law providing the boundedness of all the closed-loop signals and the achievement of the limiting equality

$$\lim_{t \to \infty} |y(t)| = 0. \tag{4}$$

In other words, the plant stabilization together with the external disturbance compensation is required simultaneously. Since according to Assumption A.2.3 the external disturbance is unknown a priori, then the classical internal model principle [16–18] is not applicable. Therefore, in order to solve the problem, an adaptive modification of the internal model principle [19, 21, 31] will be used. The main difference from well-known solutions developed earlier for multidimensional systems [19, 26] is that the vector of the output variables y is available for measurements, however the whole state vector x is not. As shown below (see Section 3.2), due to some principal feature the problem of output-feedback compensation of unknown disturbances for multidimensional systems can not be solved by applying approaches developed for the single input - single output systems.

The problem will be solved in three steps. First, by applying a special observer a parameterized model of the external disturbance in the form of a linear regression with a constant matrix of unknown entries and the physically implementable regressor (measured/calculated by the control and the output) will be obtained. Then an adaptation algorithm for the compensating component of the controller will be derived. Finally, a plant state observer and stabilizing component of the controller will be designed.

3. DISTURBANCE OBSERVER

3.1. Canonical Form of an Exosystem and a Disturbance Parameterization

For the design of a scalar disturbance observer in the output-feedback control, the following lemma is useful since it allows us to express the disturbance δ_i via the *filtered disturbance* $\delta_{fi} = w_i(s)[\delta_i]$, where $w_i(s)$ is the minimum phase asymptotically stable transfer function [19, p. 315].

Lemma 1. *The scalar disturbance δ_i can be represented in the form*

$$\delta_i = \psi_{fi}^\top \xi_{fi} + \epsilon_i, \tag{5}$$

where $\psi_{fi} \in \mathbb{R}^{q_i}$ is the vector of constant unknown parameters, ϵ_i exponentially decays, the regressor $\xi_{fi} \in \mathbb{R}^{q_i}$ is the state vector of the observer

$$\dot{\xi}_{fi} = G_i \xi_{fi} + l_i \delta_{fi}, \tag{6}$$

in which $G_i \in \mathbb{R}^{q_i \times q_i}$ is an arbitrary Hurwitz matrix, $l_i \in \mathbb{R}^{q_i}$ is a constant vector selected so that the pair (G_i, l_i) is controllable.

For the methodological purposes, we represent the proof of Lemma 1 since it is different from the proof reported in [19, p. 315]. Based on its analysis in Section 3.2, we will show the main problems that arise in multidimensional systems.

Proof. Assume the matrices (A_{fi}, b_{fi}, c_{fi}) define the minimal realization of the transfer function $w_i(s)$ and

$$\delta_{fi} = c_{fi}^\top \chi_i, \quad \dot{\chi}_i = A_{fi}\chi_i + b_{fi}\delta_i,$$

where χ_i is the state vector of the filter $w(s)$ with the minimal realization. As it is known [18, p. 87], the forced component χ_i^* of the state vector χ_i for the steady-state model (i.e., without taking into account the exponentially decaying transition component) can be represented in the form $\chi_i^* = M_{\xi i}\xi_i$, where the transformation matrix $M_{\xi i}$ is the solution of the Sylvester equation

$$M_{\xi i}\Gamma_i - A_{fi}M_{\xi i} = b_{fi}h_i^\top. \tag{7}$$

Since A_{fi} is Hurwitz, while the eigenvalues of matrix Γ_i are on the imaginary axis, then this matrix equation has a unique solution $M_{\xi i}$ [18, p. 370]. As a result,

$$\delta_{fi} = \bar{h}_i^\top \xi_i + \bar{\epsilon}_i, \quad \dot{\xi}_i = \Gamma_i \xi_i,$$

where $\bar{h}_i^\top = c_{fi}^\top M_{\xi i}$, and $\bar{\epsilon}_i$ exponentially decays. Since the filter $w_i(s)$ is minimum phase, then the pair $(\bar{h}_i^\top, \Gamma_i)$ is observable. For the forced component ξ_{fi}^* of the state vector ξ_{fi} of the filter (6), the equality $\xi_{fi}^* = M_{fi}\xi_i$ holds, where the transformation matrix M_{fi} is the solution of the Sylvester equation

$$M_{fi}\Gamma_i - G_i M_{fi} = l_i \bar{h}_i^\top. \tag{8}$$

Since the matrices Γ_i and G_i have no common eigenvalues, the pair (G_i, l_i) is fully controllable, the pair $(\Gamma_i, \bar{h}_i)$ is fully observable, then there exist a unique nonsingular matrix M_{fi} satisfying the Sylvester equation (8) [34, p. 240]. Then, $\xi_i = M_{fi}^{-1}\xi_{fi}^*$, and taking into account the second equation of (2) we can write (5) with $\psi_{fi}^\top = h_i^\top M_{fi}^{-1}$.

Remark 4. It worth noting that the solution of matrix equations (7) and (8) is not required for design of the filter (6) and the regression model (5). Instead, it is sufficiently to prove the existence of an invertible matrix M_{fi}.

3.2. Filtered Unmixed Disturbance

Let us introduce the dynamic block in the form

$$\dot{\hat{x}} = A\hat{x} + Bu + L_y(y - C\hat{x}), \tag{9}$$

where $\hat{x} \in \mathbb{R}^n$ is the state vector with arbitrary initial condition $\hat{x}(0)$, and matrix L_y is chosen so that the matrix $A_L = A - L_y C$ is Hurwitz and its eigenvalues do not coincide with invariant zeros of the system (1). Then, for the vector $\varepsilon = x - \hat{x}$ we have

$$\dot{\varepsilon} = A_L \varepsilon + B\delta, \qquad \varepsilon(0) = x(0) - \hat{x}(0). \tag{10}$$

We define the *filtered disturbance* as

$$\delta_f = y - C\hat{x}, \tag{11}$$

for which $\delta_f = C\varepsilon$ holds. Then $\delta_f = W_L(s)[\delta] + \epsilon_f$, where $W_L(s) = C(sI - A_L)^{-1}B$, and $\epsilon_f = Ce^{A_L t}\varepsilon(0)$ exponentially decays.

If $m = 1$, then the regressor of model (5) can be formed using the output of observer (9). Let us investigate if it is possible to extend Lemma 1 to the case of multidimensional systems by replacing the filter with the scalar input (6) with a filter with the vector input

$$\dot{\xi}_f = G\xi_f + L\delta_f,$$

where $G = \mathrm{diag}\{G_i\}$, $L = \mathrm{diag}\{l_i\}$. In the case considered, the vector filtered disturbance δ_f is the output of a model (without taking into account the exponentially decaying term)

$$\delta_f = C\varepsilon, \quad \dot{\varepsilon} = A_L\varepsilon + B\delta,$$

and the Sylvester equation (7) takes the form

$$M_\xi \Gamma - A_L M_\xi = BH, \tag{12}$$

and has the unique solution M_ξ for the given conditions. Then, the filtered vector disturbance δ_f can be represented in the form

$$\delta_f = \bar{H}\xi + \epsilon \quad \dot{\xi} = \Gamma\xi,$$

where ϵ exponentially decays, while the matrix $\bar{H} = CM_\xi$ is not generally block-diagonal. Then equation (8) takes the form

$$M_f \Gamma - GM_f = L\bar{H}. \tag{13}$$

As known, for the case of multidimensional systems (see [34, p. 259] and Example 2.9, p. 54 in [26]) the condition, according to which the matrices Γ_i and G_i have no common eigenvalues, while the pairs (G, L) and $(\Gamma, \bar{H})$ are fully controllable and observable, is necessary conditions, however not sufficient for the existence of a nonsingular solution M_f. In the case of singularity of the matrix M_f, the parametrization $\delta = \Psi\xi_f + \epsilon$, where $\Psi = HM_f^{-1}$, does not exist. Therefore, Lemma 1 can not be extended to the case of a vector disturbance.

A possible way to overcome this problem is to reduce the matrix $\bar{H}$ to a block-diagonal form. Then, the equation (13) can be decomposed into m independent ones of the form (8) ensuring nonsingular solutions.

Let us use the following auxiliary statement.

Lemma 2. *Under assumptions A.1.1–A.1.3, then there exists (possibly not unique) physically implementable asymptotically stable $m \times m$ transfer matrix $Q(s)$ such that the transfer matrix $D(s) = Q(s)W_L(s)$ is the asymptotically stable minimum phase and block-diagonal. One of possible realizations of matrix $Q(s)$ is defined by the equation*

$$Q(s) = \mathrm{adj}W_L(s), \tag{14}$$

where $\mathrm{adj}W_L(s)$ *is the adjoint of a matrix $W_L(s)$. In this case, $D(s) = \mathrm{diag}\left\{\frac{\beta(s)}{\alpha(s)}\right\}$, where $\frac{\beta(s)}{\alpha(s)} = \det W_L(s)$.*

Proof. Since the plant (1) is fully controllable and observable, the matrices B and C have full rank, and the matrix L is chosen so that the eigenvalues of the matrix A_L do not coincide with invariant zeros of the system (1),[1] then $\det W_L(s) \neq 0$, and the inversion $W_L^{-1}(s) = \frac{\alpha(s)}{\beta(s)}\mathrm{adj}W_L(s)$ exists. At the same time, due to the asymptotic stability of $W_L(s)$ and the minimum phase property of the plant, the polynomials $\beta(s)$ and $\alpha(s)$ are Hurwitz and $\mathrm{adj}W_L(s)$ is asymptotically stable (1) [35, p. 7]. Then, from the equality $W_L^{-1}(s)W_L(s) = \frac{1}{\det W_L(s)}\mathrm{adj}W_L(s)W_L(s) = I$ we have (14). Straightforward calculations of the matrix $Q(s)$ for other block-diagonal forms $D(s)$ shows that this solution is not unique (see the Example in Section 6 "Simulation results").

Let us choose the matrix of a *successive compensator* $Q(s)$ (for example, in form (14)) and make *the filtered unmixed disturbance*

$$\bar{\delta}_f = Q(s)\,[y - C\hat{x}]\,. \tag{15}$$

For $\bar{\delta}_f$ we can write

$$\bar{\delta}_f = \mathrm{diag}\{w_i(s)\}[\delta], \qquad i = 1, \ldots, m, \tag{16}$$

or

$$\bar{\delta}_{fi} = w_i(s)[\delta_i],$$

where $w_i(s)$ are minimum phase asymptotically stable transfer functions (for example, for $Q(s) = \mathrm{adj}W_L(s)$ we have $w_i(s) = \frac{\beta(s)}{\alpha(s)}$, $i = 1, \ldots, m$), and $\bar{\delta}_{fi}$ is ith coordinate $\bar{\delta}_f$.

3.3. Parameterization of the Initial Multidimensional Disturbance

However, the plant (1) is affected by not the filtered unmixed disturbance $\bar{\delta}_f$, but by the initial disturbance δ. Taking into account Lemma 1 and (16), we can formulate the following lemma.

Lemma 3. *Under assumptions A.1.1–A.1.4 and A.2.1–A.2.3, the disturbance δ acting on the plant* (1) *can be represented in the form of the regression model*

$$\delta = \Theta\bar{\xi}_f + \bar{\epsilon}, \tag{17}$$

where $\bar{\epsilon}$ exponentially decays, the regressor $\bar{\xi}_f$ is the state vector of the observer

$$\dot{\bar{\xi}}_f = G\bar{\xi}_f + LQ(s)\,[y - C\hat{x}] \tag{18}$$

with an arbitrary initial condition $\bar{\xi}_f(0)$, the matrices G, L, and Θ are block-diagonal, i.e.,

$$G = \mathrm{diag}\{G_i\}, \quad L = \mathrm{diag}\{l_i\}, \quad \Theta = \mathrm{diag}\{\psi_{fi}^\top\},$$

and $\psi_{fi} \in \mathbb{R}^{q_i}$ are vectors with unknown constant parameters ($i = 1, \ldots, m$).

Indeed, the dynamic filter (18) with the multidimensional input $\bar{\delta}_f$(15) is splitted into m independent filters (6) with the scalar inputs $\bar{\delta}_i = w_i(s)[\delta_i]$ and the states $\bar{\xi}_{fi}$ what corresponds to the conditions of Lemma 1.

Thus, Lemma 3 reduces the uncertainty of the disturbance δ to the parametric uncertainty of the regression model (17) with the matrix of unknown parameters Θ and the physically realizable regressor $\bar{\xi}_f$. In this case, the plant model (1) can be written as

$$\begin{cases} \dot{x} = Ax + B(u + \Xi\theta + \bar{\epsilon}), \quad x(0), \\ y = Cx, \end{cases} \tag{19}$$

[1] it is known, state feedback – in this case, it is L_yCx – does not affect the zeros of the closed system [34, p. 237].

where $\Xi = \mathrm{diag}\{\bar{\xi}_{fi}^\top\} \in \mathbb{R}^{m\times q}$ is the block diagonal matrix regressor that has m vector blocks $\bar{\xi}_{fi}^\top$ on the main diagonal, $\theta = [\psi_{f1}^\top, \ldots, \psi_{fm}^\top]^\top \in \mathbb{R}^q$ is the vector of unknown parameters constructed with the diagonal blocks $\psi_{fi}^\top$ of the matrix Θ.

Remark 5. As an alternative approach, however not optimal dynamic order of the disturbance observer (and, hence, the number of tuning parameters in the adaptive controller), is the approach, according to which each component δ_{fi} of the filtered disturbance δ_f is considered as the output of a full-dimensional disturbance generator (a similar approach is used in [18, 20, 28]). Indeed, for δ_{fi} we can write

$$\begin{aligned}\delta_{fi} &= \sum_{j=1}^{m} w_{Lij}(s)[\delta_j] = \sum_{j=1}^{m} h_j^\top w_{Lij}(s)[\xi_j] = \sum_{j=1}^{m} h_j^\top M_{ij}\xi_j + \epsilon_{ij}\\ &= \bar{h}_i^\top \xi + \epsilon, \qquad i = 1, \ldots, m,\end{aligned} \tag{20}$$

where $w_{Lij}(s)$ are elements of the transfer matrix $W_L(s)$, M_{ij} is the transformation matrix for the steady-state components, $\bar{h}_i^\top = [h_1^\top M_{i1}, \ldots, h_m^\top M_{im}] \in \mathbb{R}^q$ is the vector of unknown parameters, $\xi \in \mathbb{R}^q$ is the state vector of model (3). The expression (20) motivates us to use a disturbance observer of dimension q in each channel of the control system, which leads to a dynamic order of the observer $m \times q$ in general and to the same number of tuning parameters. At the same time, the proposed model (17) can be obtained using a qth order observer (with a static compensator $Q(s)$ – see Example), the model contains q unknown parameters and, as a result, q adjustable parameters of an adaptive controller.

The analysis of the model (19) motivates the following choice of the control algorithm:

$$u = u_x + u_\delta, \tag{21}$$

where u_x is the stabilizing control component and u_δ is the compensating control component. The next two sections of the paper are devoted to separate design of the control components.

4. DESIGN OF THE ADAPTATION ALGORITHM

Choosing

$$u_\delta = -\Xi\hat{\theta}, \tag{22}$$

where $\hat{\theta}$ is the vector of adjustable parameters, and substituting (21) and (22) into (19), we derive the model of control error

$$\begin{cases} \dot{x} = Ax + B(u_x + \Xi\tilde{\theta} + \bar{\epsilon}), \quad x(0),\\ y = Cx \end{cases} \tag{23}$$

with the vector of parametric errors $\tilde{\theta} = \theta - \hat{\theta}$.

Since the state vector x is not measurable, and the matrix A can be unstable, for design of an adaptation algorithm generating $\hat{\theta}$ we will form a static regression model defined by the following statement.

Lemma 4. *Let us introduce the extended error*

$$\bar{y} = y - C\hat{x} - \Xi_f\hat{\theta}, \tag{24}$$

where the vector $\hat{x}$ is generated by the filter (9), $\Xi_f \in \mathbb{R}^{m\times qm}$ *is the matrix regressor of the form*

$$\Xi_f = \begin{bmatrix} w_{L\,11}(s)\left[\overline{\xi}_{f1}^{\top}(t)\right] & \dots & w_{L\,1m}(s)\left[\overline{\xi}_{fm}^{\top}(t)\right] \\ \vdots & \ddots & \vdots \\ w_{L\,m1}(s)\left[\overline{\xi}_{f1}^{\top}(t)\right] & \dots & w_{L\,mm}(s)\left[\overline{\xi}_{fm}^{\top}(t)\right] \end{bmatrix}, \tag{25}$$

and $w_{L\,ij}(s)$ are the entries of the asymptotically stable transfer function $W_L(s) = C(sI - A_L)^{-1}B$ $(i = 1, \dots, m,\ j = 1, \dots, m)$. Then, the following equality holds for the signal $\bar{y}$:

$$\bar{y} = \Xi_f\tilde{\theta} + \bar{\epsilon}_f, \tag{26}$$

where $\bar{\epsilon}_f$ exponentially decays.

The proof of Lemma 4 is given in Appendix.

Model (26) is well known in the modern theory of adaptive systems [36, 37] and identification [38] and allows using a wide class of standard adaptation algorithms such as:

— *gradient adaptation algorithm* [19, 36, 37]

$$\dot{\hat{\theta}} = \gamma\Xi_f^{\top}\bar{y}; \tag{27}$$

— *algorithm with improved parametric convergence* [19, 39]

$$\dot{\hat{\theta}} = \gamma\left(d(s)\left[\Xi_f^{\top}\bar{y}\right] - \Omega\hat{\theta}\right), \tag{28}$$

where $\gamma > 0$ is the parameter of adaptation, $\Omega = d(s)[\Xi_f^{\top}\Xi_f]$ is the extended matrix regressor, $d(s)$ is a minimum phase asymptotically stable positive transfer function with real roots of the characteristic polynomial and the unit static gain $(d(0) = 1)$. In the simplest case, $d(s)$ can be selected as the first order block. The convergence properties of the adaptation algorithms (27) and (28) are defined by the following statement with the proof given in Appendix.

Lemma 5. *If assumptions A.1.1–A.1.4 and A.2.1–A.2.3 hold, then*

L.5.1. *adaptation algorithms* (27) *and* (28) *ensure the boundedness of $\bar{y}$ and $\hat{\theta}$, and also the asymptotic tendency of $|\Xi(t)\tilde{\theta}(t)| \to 0$ as $t \to \infty$;*

L.5.2. *if $\lambda(t) \notin \mathcal{L}_1$, where $\lambda(t)$ is the minimum eigenvalue of the matrix $\Omega(t)$, then adaptation algorithm* (28) *provides asymptotic convergence $|\tilde{\theta}(t)| \to 0$ as $t \to \infty$ in addition to property L.5.1;*

L.5.3. *if the regressor ξ satisfies the condition of persistent excitation (in the sense of definition 4.3.1 from* [36] *or definition 3.4 from* [19]*), then adaptation algorithms* (27) *and* (28) *in addition to property L.5.1 provide exponential convergence $|\tilde{\theta}(t)| \to 0$ as $t \to \infty$.*

5. DESIGN OF A STABILIZING COMPONENT AND PROPERTIES OF A CLOSED-LOOP SYSTEM

Since the vector x is not directly measurable, we use a state observer of the form

$$\dot{\bar{x}} = A\bar{x} + Bu_x + L_y(y - C\bar{x}), \tag{29}$$

where $\bar{x}$ is the state vector of the observer with an arbitrary initial condition $\bar{x}(0)$. Let us introduce the vector $\tilde{x} = x - \bar{x}$. Subtracting (29) from (23), we obtain

$$\dot{\tilde{x}} = A_L\tilde{x} + B(\Xi\tilde{\theta} + \bar{\epsilon}).$$

In relation to property L.5.1 of Lemma 5 we have $|\Xi(t)\tilde{\theta}(t)| \to 0$, therefore, $|\tilde{x}(t)| \to 0$ as $t \to \infty$.

A stabilizing control can be represented as

$$u_x = -K\bar{x}, \tag{30}$$

where the feedback matrix is such that the matrix $A_K = A - BK$ is Hurwitz. Substituting (30) into (23), we obtain the model of a closed-loop system.

$$\dot{x} = A_K x + B(\Xi\tilde{\theta} - K\tilde{x} + \bar{\epsilon}).$$

Since $|\tilde{x}(t)| \to 0$ for $t \to \infty$, we immediately get that $|x(t)| \to 0$ as $t \to \infty$, which means that control objective (4) is achieved. Thus, the following statement was proved.

Theorem 1. *Under assumptions A.1.1–A.1.4 and A.2.1–A.2.3, the control law* (21) *containing the compensating component* (22)*, the stabilizing component* (30)*, the disturbance observer* (9), (18)*, the adaptation algorithm* (27) *or* (28) *and the state observer* (29)*, being applied to the plant* (1)*, ensures the boundedness of all signals and the achievement of control objective* (4)*.*

6. EXAMPLE AND THE SIMULATION RESULTS

Let us consider an unstable control plant (1) with matrices

$$A = \begin{bmatrix} 4 & -6 & 0 \\ 4.5 & -7 & 0 \\ 12 & -20 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 2 & 0 \\ 4 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 4 & -1 \\ 3 & -5 & 1 \end{bmatrix}$$

and the transfer matrix

$$W(s) = C(sI - A)^{-1}B = \begin{bmatrix} 0 & \dfrac{-1}{s-1} \\ \dfrac{1}{s^2+3s-1} & \dfrac{1}{s-1} \end{bmatrix}.$$

In this case, the plant has no invariant zeros and hence is minimum phase.

We assume that $\delta = [\delta_1,\ \delta_2]^\top = [\sin t, 1]^\top$ is unmeasurable and a priori unknown disturbance, the first component of which can be considered as the output of the exosystem (2) of the second order, while the second component can be considered as the output of exosystem (2) of the first order. Then, we can write

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} h_1^\top & 0 \\ 0 & h_2^\top \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If we choose

$$L_y = \begin{bmatrix} 6 & 6 \\ 6 & 6 \\ 10 & 12 \end{bmatrix},$$

then,

$$A_L = \begin{bmatrix} -2 & 0 & 0 \\ -1.5 & -1 & 0 \\ -4 & 0 & -1 \end{bmatrix}, \quad W_L(s) = \begin{bmatrix} 0 & \dfrac{-1}{s+1} \\ \dfrac{1}{(s+1)(s+2)} & \dfrac{1}{s+1} \end{bmatrix}.$$

Finally, for the disturbance observer design we choose

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -5 & 0 \\ 0 & 0 & -4 \end{bmatrix}, \quad L = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 0 \\ 0 & 4 \end{bmatrix}.$$

For the methodological purposes, we calculate the matrices M_ξ, $\bar{H}$, and M_f that are solutions of the equations (12) and (13), respectively. Then, we obtain nonblock-diagonal matrix

$$\bar{H} = CM_\xi = \begin{bmatrix} 0 & 0 & -1 \\ 0.1 & -0.3 & 1 \end{bmatrix}$$

and the singular matrix

$$M_f = \begin{bmatrix} 0 & 0 & -0.33 \\ 0 & 0 & 0 \\ -0.04 & -0.28 & 1 \end{bmatrix}.$$

It is shown that the direct extension of Lemma 1 to the case of a vector disturbance is impossible without special methods.

According to the proposed approach, we calculate the transfer matrix of the successive compensator

$$Q(s) = \mathrm{adj} W_L(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} \\ \dfrac{-1}{(s+1)(s+2)} & 0 \end{bmatrix} \tag{31}$$

and form filtered unmixed disturbance (15). Then, we can show that the following model is valid for $\bar{\delta}_f$:

$$\bar{\delta}_f = \begin{bmatrix} \dfrac{1}{(s+1)^2(s+2)} & 0 \\ 0 & \dfrac{1}{(s+1)^2(s+2)} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s+1)^2(s+2)}[\delta_1] \\ \dfrac{1}{(s+1)^2(s+2)}[\delta_2] \end{bmatrix}.$$

However, according to Lemma 2, the compensator (31) that leads the disturbance to an unmixed form is not unique. Indeed, in the example, the simple successive compensator

$$Q(s) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \tag{32}$$

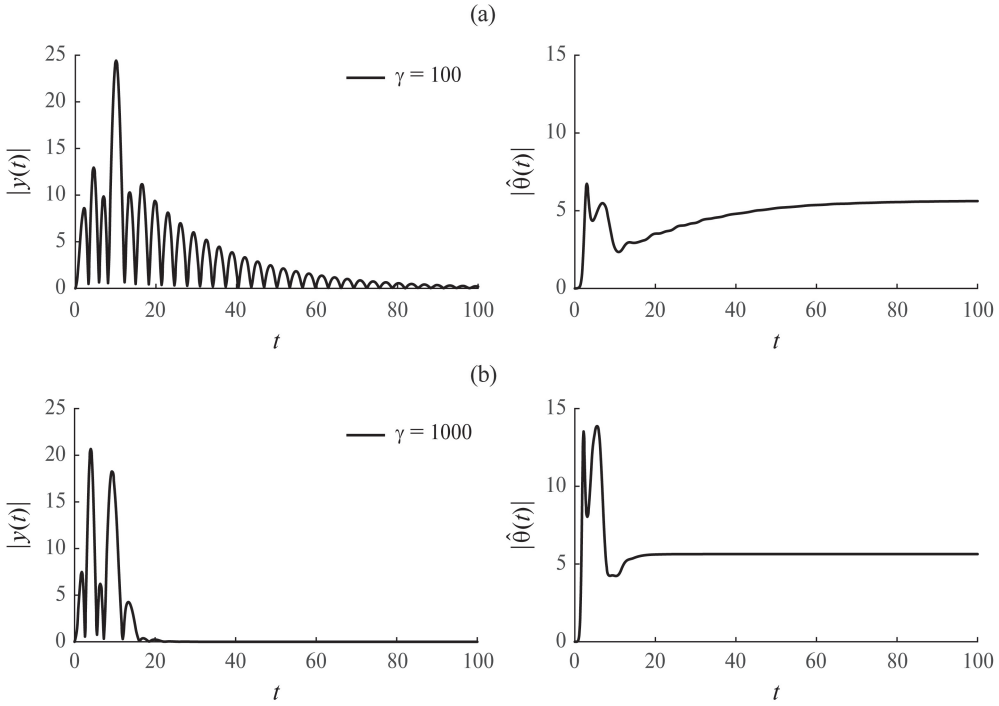
gives the following model of unmixed filtered disturbance:

$$\bar{\delta}_f = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)} & 0 \\ 0 & \dfrac{1}{s+1} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{(s+1)(s+2)}[\delta_1] \\ \dfrac{1}{s+1}[\delta_2] \end{bmatrix}.$$

For simulation, the successive compensator (32) is applied.

Simulation results of the closed-loop system with the adaptation algorithm with improved convergence (28) for $d(s) = \frac{1}{s+1}$ and two different of γ, the stabilizing control (30) with

$$K = \begin{bmatrix} 25.0 & -0.91 & -8.38 \\ 36.25 & -4.83 & -10.67 \end{bmatrix},$$



Transients in the adaptive system closed by the adaptation algorithm with improved parametric convergence for: (a) $\gamma = 100$; (b) $\gamma = 1000$.

for zero initial conditions of the plant and the control algorithm are shown in the Figure. The simulation results demonstrate the achievement of control objective (4) in the presence of external previously unknown disturbance and the opportunity to accelerate the process of tuning controller by increasing the adaptation parameter γ.

7. CONCLUSION

The paper presents the solution to the problem of the output-feedback adaptive compensation of external unknown deterministic disturbances for multidimensional linear systems. The proposed solution is based on the adaptive implementation of the internal model principle and design of a special observer, which makes it possible to obtain an unmixed parametrization of the disturbance. The presented approach allows us to design an adaptive controller with a number of tuning parameters equaled to the number of unknown coefficients of the characteristic polynomial of the disturbance model, thereby, reducing the computational complexity of the algorithm compared to known solutions.

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APPENDIX

Proof of Lemma 4. We rewrite (25) in the state-space form

$$\begin{cases} \dot{\Xi}_x = A_L \Xi_x + B\Xi, \\ \Xi_f = C\Xi_x. \end{cases} \tag{A.1}$$

Then, we introduce an intermediate variable

$$\epsilon_x = x - \hat{x} - \Xi_x\theta \tag{A.2}$$

and calculate its time derivative according to (19), (9) and (A.1):

$$\dot{\epsilon}_x = x - \hat{x} - \Xi_x\theta = Ax + Bu + B\Xi\theta + \bar{\epsilon} - A\hat{x} - Bu - L_y(Cx - C\hat{x}) - A_L\Xi_x\theta - B\Xi\theta = A_L\epsilon_x + \bar{\epsilon}.$$

Due to the Hurwitz property of A_L, the value $\epsilon_x(t)$ exponentially decays.

Then $x - \hat{x} = \Xi_x\theta + \epsilon_x$ or

$$y - C\hat{x} = \Xi_f\theta + \bar{\epsilon}_f, \tag{A.3}$$

where $\bar{\epsilon}_f = C\epsilon_x$. Substituting (A.3) into (24), we obtain (26).

Proof of Lemma 5. We choose the Lyapunov function

$$V = \frac{1}{2\gamma}\tilde{\theta}^\top\tilde{\theta} + \int_t^\infty \bar{\epsilon}_f^\top(\tau)\bar{\epsilon}_f(\tau)d\tau \tag{A.4}$$

and calculate its time derivative in view of (26), the equality $\dot{\tilde{\theta}} = -\dot{\hat{\theta}}$ and the algorithms (27) and (28):

1. The algorithm (27).

$$\dot{V} = -\tilde{\theta}^\top\Xi_f^\top\Xi_f\tilde{\theta} - \tilde{\theta}^\top\Xi_f^\top\bar{\epsilon}_f - \bar{\epsilon}_f^\top\bar{\epsilon}_f = -\frac{1}{2}|\Xi_f\tilde{\theta}|^2 - \frac{1}{2}|\bar{\epsilon}_f| - \frac{1}{2}\left(|\Xi_f\tilde{\theta}| + |\bar{\epsilon}_f|\right)^2 \leqslant -\frac{1}{2}|\Xi_f\tilde{\theta}|^2.$$

From the last expression, it follows that the signals $\hat{\theta}$ and $|\Xi_f\tilde{\theta}| \in \mathcal{L}_2$ are bounded. Due to the boundedness of Ξ and the stability of the filter $W_L(s)$, the output of linear regression (26) $\bar{y}$, and the derivative $\dot{\hat{\theta}}$ are bounded. Since the function $|\Xi_f(t)\tilde{\theta}(t)| \in \mathcal{L}_2$ and it is continuous and bounded, then $|\Xi_f(t)\tilde{\theta}(t)|$, $\dot{\hat{\theta}}(t) \to 0$ as $t \to \infty$.

According to the swapping lemma applied to multichannel systems (see [19, Section D.2]), we have

$$W_L(s)\left[\Xi\tilde{\theta}\right] = \Xi_f\tilde{\theta} + W_L(s)\left[\Xi_f\dot{\tilde{\theta}}\right], \tag{A.5}$$

where Ξ_f is the output of the filter (A.1) from Appendix. From the last expression follows convergence $W_L(s)\left[\Xi(t)\tilde{\theta}(t)\right] \to 0$ as $t \to \infty$ due to the boundedness of $\Xi(t)$, stability of $W_L(s)$, and the convergence $|\Xi_f(t)\tilde{\theta}(t)|$, $\dot{\hat{\theta}}(t) \to 0$. Since $W_L(s)$ is nonsingular and minimum phase, and $\dot{\Xi}$, $\dot{\tilde{\theta}}$ are bounded, then $|\Xi(t)\tilde{\theta}(t)| \to 0$ as $t \to \infty$.

2. The algorithm (28). Let us analyze the derivative of the function V taking into account the Cauchy–Bunyakovsky inequality:

$$\dot{V} = -\tilde{\theta}^\top\Omega\tilde{\theta} - \tilde{\theta}^\top d(s)\left[\Xi_f^\top\bar{\epsilon}_f\right] - \bar{\epsilon}_f^\top\bar{\epsilon}_f = -\tilde{\theta}^\top\Omega\tilde{\theta} - \tilde{\theta}^\top(t)\int_0^t h(t-\tau)\Xi_f^\top(\tau)\bar{\epsilon}_f(\tau)d\tau - |\bar{\epsilon}_f|^2$$

$$\leqslant -\tilde{\theta}^\top\Omega\tilde{\theta} + \left(\int_0^t h(t-\tau)\left(\tilde{\theta}(t)\Xi_f^\top(\tau)\right)^2 d\tau\right)^{\frac{1}{2}}\left(\int_0^t h(t-\tau)\bar{\epsilon}_f^2(\tau)d\tau\right)^{\frac{1}{2}} - |\bar{\epsilon}_f|^2,$$

where $h(t-\tau) \geqslant 0$ is the impulse response of the filter $d(s)$. Taking into account, that

$$\int_0^t h(t-\tau)\left(\tilde{\theta}(t)\Xi_f^\top(\tau)\right)^2 d\tau \leqslant c_d\tilde{\theta}^\top(t)\Omega(t)\tilde{\theta}(t), \quad \int_0^t h(t-\tau)\bar{\epsilon}_f^2(\tau)d\tau \leqslant c_d|\bar{\epsilon}_f|^2,$$

where $c_d = |d(s)|_\infty = 1$, we can continue the analysis $\dot{V}$:

$$\dot{V} \leqslant -\tilde{\theta}^\top\Omega\tilde{\theta} + \left(\tilde{\theta}^\top\Omega\tilde{\theta}\right)^{\frac{1}{2}}|\bar{\epsilon}_f| - |\bar{\epsilon}_f|^2 \leqslant -\frac{1}{2}\tilde{\theta}^\top\Omega\tilde{\theta}. \tag{A.6}$$

From the last expression follows that the signals $\hat{\theta}$ and $|\Omega^{\frac{1}{2}}\tilde{\theta}| \in \mathcal{L}_2$ are bounded. Therefore, due to the stability of the filters $W_L(s)$ and $d(s)$, the functions $\Omega(t)$ and $\dot{\Omega}$ are bounded. Therefore, $|\Omega\tilde{\theta}| \in \mathcal{L}_2$ and $|\Omega(t)\tilde{\theta}(t)| \to 0$ as $t \to \infty$. Since $\dot{\hat{\theta}}(t) \to 0$, then (see the Swapping Lemma in [40]) $d(s)\left[\Xi_f^\top(t)\Xi_f(t)\tilde{\theta}(t)\right] \to 0$ as $t \to \infty$. Since $\dot{\Xi}_f$ and $\dot{\tilde{\theta}}$ are bounded, then from the convergence $d(s)\left[\Xi_f^\top(t)\Xi_f(t)\tilde{\theta}(t)\right] \to 0$ it follows that $\Xi_f(t)\tilde{\theta}(t) \to 0$ as $t \to \infty$. Taking into account the arguments mentioned above for the proof of property L.5.1 for the algorithm (27), from the convergence $\Xi_f(t)\tilde{\theta}(t) \to 0$ we have the convergence $\Xi(t)\tilde{\theta}(t) \to 0$ as $t \to \infty$.

Property L.5.1 is proved.

To prove property L.5.2, we proceed the analysis of the derivative (A.6) taking into account (A.4):

$$\dot{V} \leqslant -\frac{1}{2}\tilde{\theta}^\top\Omega\tilde{\theta} \leqslant -\frac{1}{2}\lambda(t)\tilde{\theta}^\top\tilde{\theta} = -\gamma V + \gamma\int_t^\infty \bar{\epsilon}_f^\top(\tau)\bar{\epsilon}_f(\tau)d\tau.$$

Solving the obtained differential inequality, we have

$$V(t) \leqslant \mathrm{e}^{-\gamma\int_0^t \lambda(\tau_1)d\tau_1}V(0) + \gamma\int_0^t \mathrm{e}^{-\gamma\int_{\tau_3}^t \lambda(\tau_2)d\tau_2}\int_{\tau_3}^\infty \bar{\epsilon}_f^\top(\tau_1)\bar{\epsilon}_f(\tau_1)d\tau_1 d\tau_3,$$

from which it follows property L.5.2.

The proof of property L.5.3 for the algorithm (27) can be found in [37, Section 2.8]. The proof of property L.5.3 for the algorithm (28) can be found in [19, Section 3.2.3].

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NONLINEAR SYSTEMS

Optimal Control of the Coefficients of a Nonlinear Schrödinger-Type Equation

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Abstract—This paper is devoted to the solvability of an optimal control problem for the coefficient at the highest derivative and the quantum potential in a nonlinear and nonstationary Schrödinger-type equation. It generalizes the well-known equation of quantum mechanics. The simultaneous control problem for several coefficients of the state equation is considered, with a performance criterion specified by the residual of the boundary data of the solution. For this problem, well-posedness conditions are established and an existence theorem for its solution is proved. In addition, the optimization problem with a perturbed performance criterion is studied, and an existence and uniqueness theorem for its solution is proved. An explicit form of the first variation of the performance criterion is obtained, and an iterative algorithm for solving these problems is described. The results are novel for the standard Schrödinger equation as well.

Keywords: solvability of optimal control problem, Schrödinger-type equation, control of quantum processes, control of state equation coefficients

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1. INTRODUCTION

The Schrödinger-type equation generalizes the well-known Schrödinger equation of quantum mechanics and is used as a model in the theories of superconductivity and other fields of practice. This equation often arises in the diagnosis of nanostructured materials and atomic-molecular computing, when intra-atomic and intramolecular interaction potentials, laser pulses, and material characteristics need to be controlled. This equation is widely used in quantum information processing, adaptive optics and quasi-optics, bifurcation analysis of nonlinear models, investigation of magnetic quantum phenomena, and other applications; for example, see [1–9] and other publications. Linear and bilinear quantum systems, optimization problems of these systems with one control factor, with real or complex interaction potential, etc. were studied earlier. The finite difference method for numerically solving such control problems was developed in [9–11].

In modern practice it is often necessary to control many coefficients, the most influential quantum characteristics of nonlinear and nonstationary processes [1, 4–10]. Such control problems are most challenging for theoretical analysis and numerical solution: they are nonlinear, have an implicitly defined state operator, and often belong to the class of ill-posed problems [9, 10]; moreover, there are no direct methods for their observation and no formulation of performance criteria. The observation definition form is especially important for constructing performance criteria in quantum control processes since it must correspond to the quantum character of the processes under consideration. In this sense, boundary observation is convenient for measurements as well as for processing of

results. It is crucial to investigate control processes for the coefficients at the higher derivatives of the state equation [1–3, 6–11]. Below, we study a simultaneous optimal control problem for the coefficient at the highest derivative and the quantum potential in a nonlinear nonstationary Schrödinger-type equation. The optimization problem with a "perturbed" performance criterion is also considered. Existence and uniqueness theorems for the solution of these problems are proved, and iterative algorithms for their solution are described. The main results are novel also for the standard Schrödinger equation of quantum mechanics.

2. PROBLEM STATEMENT

Let $l > 0$ and $T > 0$ be given numbers, $0 \leqslant x \leqslant l$, $0 \leqslant t \leqslant T$, $\Omega_t = (0,l) \times (0,t)$, $\Omega = \Omega_T$, and $\psi(x,t)$ be a complex wave function with the spatial coordinate x and time t. The functional spaces used below were introduced, e.g., in [8, 11] and other publications. Accordingly, $L_p(0,l)$ is the Lebesgue space of measurable functions on $(0,l)$ that are integrable with degree $p \geqslant 1$, and $C^k[0,T;B)$ is the Banach space of $k \geqslant 0$ times continuously differentiable functions on $[0,T]$ whose values belong to a Banach space B.

Let $W_p^k(0,l)$ and $W_p^{k,m}(\Omega)$ be the Sobolev spaces of functions with generalized derivatives of orders $k \geqslant 0$ in x and $m \geqslant 0$ in t that are integrable with degree $p \geqslant 1$. We denote by $W_\infty^1(0,l) = \left\{w : w \in L_\infty(0,l), \frac{dw}{dx} \in L_\infty(0,l)\right\}$ the Banach space with the properties mentioned. The symbols $\forall^o$ and $\forall$ mean that the above properties hold for almost all and all, respectively, values of an appropriate variable. Positive constants independent of the values being estimated will be denoted by c_j, $j = 0, 1, 2, \ldots$. From this point onwards, $a_i, b_i, s_i, i = 0, 1, 2, \ldots$, are given positive numbers.

Consider a controlled process described by the following initial boundary-value problem for the Schrödinger-type equation:

$$i\rho^2 \frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}\left(v_0(x)\frac{\partial \psi}{\partial x}\right) - v_1(x)\psi + a_1|\psi|^2\psi = f(x,t), \quad (x,t) \in \Omega, \tag{1}$$

$$\psi(x,0) = \varphi(x), \quad x \in (0,l), \tag{2}$$

$$\frac{\partial \psi(0,t)}{\partial x} = \frac{\partial \psi(l,t)}{\partial x} = 0, \quad t \in (0,T), \tag{3}$$

where $\rho > 0$ and a_1 are given real numbers; $\varphi(x)$ and $f(x,t)$ are given complex measurable functions satisfying the conditions $\varphi \in W_2^2(0,l)$, $\frac{d\varphi(0)}{dx} = \frac{d\varphi(l)}{dx} = 0$, and $f \in W_2^{0,1}(\Omega)$; the coefficients $v_0(x)$ and $v_1(x)$ of this equation are real control functions.

Assume that an additional boundary observation has the form

$$\psi(0,t) = y_0(t), \quad \psi(l,t) = y_1(t), \quad 0 \leqslant t \leqslant T, \tag{4}$$

where $y_0 = y_0(t)$ and $y_1 = y_1(t)$ are given complex functions from the space $L_2(0,T)$. Let the vector function $v = v(x) = (v_0(x), v_1(x))$ be an element of the following set of admissible controls: $V \equiv \{v = (v_0, v_1) :\ v_0 \in W_2^1(0,l),\ v_1 \in L_2(0,l),\ 0 < b_0 \leqslant v_0(x) \leqslant b_1,\ \left|\frac{dv_0(x)}{dx}\right| \leqslant b_2,\ 0 < b_3 \leqslant v_1(x) \leqslant b_4,\ \forall^o x \in (0,l)\}$, where $b_j > 0$, $j = 0, 1, \ldots, 4$, are given numbers. For each element $v \in V$, the wave function $\psi = \psi(x,t) \equiv \psi(x,t;v)$ belonging to the space $B \equiv C^0([0,T]; W_2^1(0,l)) \cap C^1([0,T]; L_2(0,l))$ will be called the solution of the primal problem if it satisfies equation (1) for any $t \in [0,T]$ and almost all $x \in (0,l)$ and conditions (2) and (3) for almost all $x \in (0,l)$ and $t \in (0,T)$. The primal problem is an initial boundary-value problem for the Schrödinger-type equation (1). This problem was studied in [1–11] and other publications. As was established therein, under the above conditions, for each $v \in V$ it has a unique solution in the space B with the a priori upper bound

$$\|\psi(\cdot,t)\|_{W_2^1(0,l)} + \left\|\frac{\partial \psi(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} \leqslant c_0\left(\|\varphi\|_{W_2^1(0,l)} + \|f\|_{L_2(0,T;W_2^1(0,l))}\right). \tag{5}$$

Now consider an optimal control problem for the function $v = v(x) = (v_0(x), v_1(x))$ in equation (1) with the initial boundary conditions (2) and (3): it is required to minimize the performance criterion

$$J_0(v) = \beta_0\|\psi(0,\cdot) - y_0\|^2_{L^2(0,T)} + \beta_1\|\psi(l,\cdot) - y_1\|^2_{L^2(0,T)} \tag{6}$$

on the set V, where the numbers $\beta_0 > 0$ and $\beta_1 > 0$ are given and $\beta_0 + \beta_1 = 1$. Note that problems with controls in state equation coefficients often have unstable solutions (see [9–12], etc.). Therefore, it is reasonable to study the problem with a perturbed performance criterion. Below we will minimize the criterion

$$J_\alpha(v) = \beta_0\|\psi(0,\cdot) - y_0\|^2_{L^2(0,T)} + \beta_1\|\psi(l,\cdot) - y_1\|^2_{L^2(0,T)} + \alpha\|v - \omega\|^2_H \tag{7}$$

on the set V under conditions (1)–(3), where $\alpha \geqslant 0$ is a given number, $H \equiv W_2^1(0,l) \times L_2(0,l)$ is the space of controls, and the element $\omega \in H$ is a given vector function. For the sake of brevity, problems (1)–(3), (6) and (1)–(3), (7) will be called problems (6) and (7), respectively.

3. SOLVABILITY OF THE OPTIMAL CONTROL PROBLEM

Theorem 1. *For any $\alpha \geqslant 0$, problem (7) has at least one solution.*

The proof is provided in the Appendix.

Let $H_\infty \equiv W_\infty^2(0,l) \times W_\infty^1(0,l)$ and $H_1 \equiv W_2^4(0,l) \times W_2^1(0,l)$. We study the optimal control problem on the set $V_1 \equiv \Big\{v = (v_0, v_1); v_0 \in W_2^2(0,l), v_1 \in W_2^1(0,l), 0 < s_0 \leqslant v_0(x) \leqslant s_1, \left|\frac{dv_0(x)}{dx}\right| \leqslant s_2, \left|\frac{d^2v_0(x)}{dx^2}\right| \leqslant s_3, 0 < s_4 \leqslant v_1(x) \leqslant s_5, \left|\frac{dv_1(x)}{dx}\right| \leqslant s_6, \forall x \in (0,l)\Big\}$, where $s_i > 0$, $i = 0,1,2,3,4,5,6$, are given numbers. Assume that the functions $\varphi(x)$ and $f(x,t)$ satisfy the conditions $\frac{d^3\varphi}{dx^3} \in L_2(0,l)$ and $\frac{\partial f}{\partial x} \in W_2^{0,1}(\Omega)$. Under these conditions, for each element $v = v(x)$ from the set V_1, the solution of the primal problem (1)–(3) exists, is unique for each $t \in [0,T]$, belongs to the space $B_1 = C^0([0,T]; W_2^4(0,l)) \cap C^1([0,T]; L_2(0,l))$, and obeys the a priori upper bound

$$\begin{aligned} &\|\psi(\cdot,t)\|_{W_2^1(0,l)} + \left\|\frac{\partial\psi(\cdot,t)}{\partial t}\right\|_{L_2(0,l)} + \left\|\frac{\partial^2\psi(\cdot,t)}{\partial t^2}\right\|_{L_2(0,l)} \\ &\leqslant c_1\left(\|\varphi\|_{W_2^1(0,l)} + \|f\|_{W_2^{1,1}(\Omega)} + \left\|\frac{\partial f}{\partial x}\right\|_{W_2^{1,1}(\Omega)}\right), \quad \forall t \in [0,T]. \end{aligned} \tag{8}$$

For details, see [9, 10].

Consider now an analog of problem (7) on the set $V_1 \subseteq H_1$: it is required to minimize the performance criterion

$$J_\alpha(v) = \beta_0\,\|\psi(0,\cdot) - y_0\|^2_{L_2(0,T)} + \beta_1\,\|\psi(l,\cdot) - y_1\|^2_{L_2(0,T)} + \alpha\,\|v - \omega\|^2_{H_1} \tag{9}$$

under conditions (1)–(3), where $\alpha \geqslant 0$ is a given number, $H_1 = W_2^2(0,l) \times W_2^1(0,l)$ is the space of controls, and $\omega \in H_1$ is a given vector function. It is easy to check that $V_1 \subseteq V$. If $\alpha = 0$, $V = V_1$, and $H = H_1$, problems (7) and (9) will coincide with problem (6). For $V = V_1$ and $H = H_1$, problems (7) and (9) become identical. Examples similar to those in [9, 10] demonstrate that, for $\alpha = 0$, the solution of problem (7) or (9) is unstable and non-unique. However, for $\alpha > 0$, we have the following result.

Theorem 2. *There exists an everywhere dense subset K in the space H_1 such that problem (9) has a unique solution for all $\alpha > 0$ and $\omega \in K$.*

The proof is provided in the Appendix.

4. CONCLUSIONS

The above theorems provide a basis for solving the problems under consideration. According to the existing experience in solving optimal control problems [2, 9, 10, 13, 16, 17], iterative numerical methods are the most effective tools for this purpose. Maple, Matlab, ANSYS, and similar software packages with visualization of the results are preferable and provide a convenient apparatus for numerical solution. Consider an iterative process for solving problem (7) or (9) based on the following numerical scheme of the conditional gradient method:

$$v^{(k+1)}(x) = v^{(k)}(x) + \lambda_k \left(w^{(k)}(x) - v^{(k)}(x) \right), \quad k = 0, 1, 2, \dots,$$

where $v^{(0)}(x)$ is an initial iteration step, which can be any element of the set V (or V_1); the algorithm parameter $\lambda_k \in (0,1)$ is chosen from the condition $J_0\left(v^{(k+1)}\right) < J_0\left(v^{(k)}\right)$; the element $w^{(k)}(x)$, $k = 1, 2, \dots$, is determined by minimizing the linear criterion

$$\delta J_0(v^{(k)}, w - v^{(k)}) + 2a\langle v^{(k)} + v^{(k-1)}, w - v^{(k)} \rangle \to \infty,$$

on the set V (or V_1), where $\delta J_0(v)$ is the first variation of the criterion $J_0(v)$.

Let $h \in H$ be an increment of control $v \in V$ such that $v + h \in V$. The first variation of the criterion $J_0(v)$ has the form

$$\delta J_0(v, h) = \int\limits_\Omega \left[\mathrm{Re} \left(\frac{\partial \psi(x,t)}{\partial x} \frac{\partial \varphi(x,t)}{\partial x} \right) h_0(x,t) + \mathrm{Re} \left(\psi(x,t) \varphi(x,t) \right) h_1(x,t) \right] dxdt,$$

where $\psi(x,t)$ and $\varphi(x,t)$ are the solutions of the primal and conjugate problems, respectively. Following the above scheme, we find a control sequence $(v^{(k)})$. For the problems under consideration, this sequence converges under sufficient conditions presented in [9, 10, 13, 14] and other publications. In addition, the interested reader can find therein wide classes of iterative processes based on different modifications of the gradient, Newton, and other methods for solving extremal problems as well as numerical analysis and regularization of the solution in the case of problem instability.

APPENDIX

Proof of Theorem 1. The existence of a minimizing sequence $(v'(x)) \in V$ for the solution of problem (7) follows from the boundedness from below of the criterion $J_\alpha(v)$. Let us denote $\psi_k = \psi_k(x,t) \equiv \psi(x,t;v^k)$, $k = 1, 2, \dots$, and $H_\infty \equiv W_\infty^1(0,l) \times L_\infty(0,l)$. Since the set V is a closed, bounded, and convex subset in the space H, it is possible to extract a subsequence from the sequence (v^k), denoted again by (v^k) for simplicity, so that $v_0^k \to v_0$ (*) weakly in $L_\infty(0,l)$, $\frac{dv_0^k}{dx} \to \frac{dv_0}{dx}$ (*) weakly in $L_\infty(0,l)$, and $v_1^k \to v_1$ (*) weakly in $L_\infty(0,I)$ as $k \to \infty$. Moreover, by the definition of the set V, it is (*) weakly closed in the space H. Due to the above limit relations and the space $W_\infty^k(0,l)$ embedded into $L_\infty(0,l)$, we obtain $v_0^k \to v_0$ strongly in $L_\infty(0,l)$ as $k \to \infty$.

The a priori bound (5) implies that the sequence $\{\psi_k(x,t)\}$ is uniformly bounded in the norm of B. Then it is possible to extract a subsequence from the sequence $\{\psi_k(x,t)\}$, denoted again by $\{\psi_k(x,t)\}$ for simplicity, so that $\psi_k(\cdot,t) \to \psi(\cdot,t)$ in $W_2^2(0,l)$ and $\frac{\partial \psi_k(\cdot,t)}{\partial t} \to \frac{\partial \psi(\cdot,t)}{\partial t}$ in $L_2(0,l)$ for $t \in [0,T]$ as $k \to \infty$. Since the space B is compactly embedded into $C^0([0,T]; W_0^2(0,l))$ (see [15]), we have $\|\psi_k(\cdot,l) - \pi(\cdot,l)\|_{W_0^2(0,l)} \to 0$ as $k \to \infty$ uniformly in $t \in [0,T]$. As is easily verified, each element of the sequence satisfies the identity

$$\int\limits_0^l \left\{ \frac{i\rho^2 \partial \psi_k(x,t)}{\partial t} + \frac{\partial}{\partial x} \left(v_o^k(x) \frac{\partial \psi_k(x,t)}{\partial x} \right) - v_1^k(x) \psi_k(x,t) \right\} \tag{A.1}$$

$$+ a_1 \|\psi_k(x,t)^2 \| \psi(x,t) - f(x,t) \| \overline{\eta(x)} dx = 0, \quad k = 1, 2, \dots$$

for all $t \in [0,T]$ and an arbitrary function $\eta \in L_2(0,l)$. Furthermore, $\psi_k(x,t) = 1,2,\ldots$, satisfy the initial and boundary conditions (2) and (3). For each $t \in [0,T]$ and any function $\eta \in L_2(0,l)$, passing to the limit on $k = 1,2,\ldots$ in identity (A.1) yields the same identity for the limit function $\psi_k(x,t)$. By analogy, it is straightforward to verify that the limit function $\psi_k(x,t)$ satisfies equation (1) for each $t \in [0,T]$ and almost all $x \in (0,l)$. For $t = 0$ we obtain $\|\psi_k(\cdot,0) - \psi(\cdot,0)\|_{L_2(0,l)} \to 0$ as $k \to \infty$. Therefore, as $k \to \infty$, passing to the limit in

$$\|\psi_k(\cdot,0) - \varphi\|_{L_2(0,l)} \leqslant \|\psi_k(\cdot,0) - \psi_k(\cdot,0)\|_{L_2(0,l)} + \|\psi_k(\cdot,0) - \varphi\|_{L_2(0,l)}$$

shows that the function $\psi(x,t)$ satisfies the initial condition (2) for almost all $x \in (0,l)$. Finally, we prove that the limit function $\psi_k(x,t)$ satisfies the boundary conditions (3). Indeed, for any functions from the space B, the limit relations $\frac{\partial \psi_k(s,\cdot)}{\partial x} \to \frac{\partial \psi(s,\cdot)}{\partial x}$, where $s = 0$ and $s = l$, are valid weakly in $L_2(0,T)$ as $k \to \infty$. With this relation, as $k \to \infty$, passing to the limit in the equality

$$\int\limits_0^T \frac{\partial \psi(s,t)}{\partial x}\eta(t)dt = \int\limits_0^T \left(\frac{\partial \psi(s,t)}{\partial x} - \frac{\partial \psi_k(s,t)}{\partial x}\right)\eta(t)dt + \int\limits_0^T \frac{\partial \psi_k(s,t)}{\partial x}\eta(t)dt, \quad s = 0,l,$$

gives the expression

$$\int\limits_0^T \frac{\partial \psi(s,t)}{\partial x}\eta(t)dt = 0, \quad s = 0,l,$$

for any function $\eta(t)$ from $L_2(0,T)$. Hence, for almost all $t \in (0,T)$, the function $\psi(x,t)$ satisfies the boundary conditions (3). Thus, the limit function $\psi(x,t)$ of the sequence $(\psi_k(x,t))$ is a solution of the primal problem (1)–(3) for each control v from V. In addition, this function is the weak limit of the weakly convergent sequence $\{\psi_k(x,t)\}$ for each $t \in [0,T]$. Since the space B is compactly embedded into $C([0,l];L_2(0,T))$, we have $\|\psi_k(x,t) - \psi(x,t)\|_{L_2(0,T)} \to 0$ as $k \to \infty$ for each $x \in [0,l]$. Consequently, for $x = 0$ and $x = l$, from the weak lower semicontinuity of the norm in $L_2(0,T)$ and the nonnegativity of the numbers β_0, β_l, and α, we arrive at the inequality $J_\alpha * \leqslant J_\alpha(v) \leqslant J_\alpha *$. In other words, the element $v = v(x) \in V$ is a solution of problem (7) for any $\alpha \geqslant 0$. The proof of Theorem 1 is complete.

Proof of Theorem 2. We check the continuity of the criterion $J_0(v)$ on the set V_1. Let $\delta\psi = \delta\psi(x,t) = \psi(x,t;\, v+\delta v) - \psi(x,t;\, v)$, where $\delta v \in H_1$ is an increment of an element $v \in V_1$ such that $v + \delta v \in V_1$, and let $\psi(x,t;\, v)$ be the solution of the primal problem (1)–(3) for element $v \in V_1$. Due to (1)–(3), the function $\delta\psi = \delta\psi(x,t)$ satisfies the equation

$$\begin{gathered} i\rho^2\frac{\partial \delta\psi}{\partial t} + \frac{\partial}{\partial x}\Big((v_0(x) + \delta v_0(x))\frac{\partial \delta\psi}{\partial x}\Big) - (v_1(x) + \delta v_1(x))\delta\psi \\ + a_1\left(|\psi_\delta|^2 + |\psi|^2\right)\delta\psi + a_1\psi_\delta\psi\overline{\delta\psi} = -\frac{\partial}{\partial x}\Big(\delta v_0(x)\frac{\partial \psi}{\partial x}\Big) + \delta v_1(x)\psi, \quad (x,t) \in \Omega, \end{gathered} \tag{A.2}$$

with homogeneous initial boundary conditions. To estimate the solution of this initial boundary-value problem, we multiply both sides of equation (A.2) by the function $\delta\psi(x,t)$ and integrate the result over the domain Ω_t. In view of the homogeneity of the initial condition, subtracting from the resulting equality its complex conjugate yields

$$\|\rho\delta\psi(\cdot,t)\|^2_{L_2(0,l)} = 2a_1\int \mathrm{Im}(\psi\psi_\delta, (\delta\overline{\psi})^2)\,dxd\tau$$

$$+ 2\int \mathrm{Im}\left[\left(-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial \psi}{\partial x}\right) + \delta v_1(x)\psi\right)\delta\overline{\psi}\right]dxdt, \quad \forall t \in [0,T].$$

Applying the Cauchy–Bunyakovsky–Schwarz inequality to this identity, we obtain

$$\begin{aligned}\|\rho\delta\psi(\cdot,t)\|^2_{L_2(0,l)} &\leqslant 2|a_1|\int\limits_{\Omega_t}|\psi_\delta|\,|\psi|\,\delta\psi^2\,dx\,d\tau\\ +\int\limits_{\Omega_t}\left[-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)+\delta v_1(x)\psi\right]^2 dx\,d\tau &+\iint\limits_{\Omega_t}|\delta\psi|^2\,dx\,d\tau,\ \forall\, t\in[0,T].\end{aligned}\tag{A.3}$$

As is known [15], if a function $\varphi(.,t)\in W_2^1(0,l)$ is nonzero at the ends of the interval $[0,l]$, it satisfies the inequality

$$\|\varphi(\cdot,t)\|_{L_2(0,l)}\leqslant c\left\|\frac{\partial\varphi(\cdot,t)}{\partial x}\right\|^{\frac12}_{L_2(0,l)}\|\varphi(.,t)\|^2_{L_2(0,l)}+d\,\|\varphi\|_{L_2(0,l)},$$

where $c>0$, a, and $d>0$ are some constants. From this inequality and the a priori bound (8) for the solution of the primal problem (1)–(3), for each $V\in V_1$, it follows that

$$\|\psi(\cdot,t)\|^2_{L_2(0,l)}\leqslant c_4;\quad \|\psi^2(\cdot,t)\|^2_{L_2(0,l)}\leqslant c_4.\tag{A.4}$$

By these inequalities in equation (A.2), we have

$$\begin{aligned}\|\delta\psi(\cdot,t)\|^2_{L_2(0,l)}&\leqslant\left(2|a_1|c_4^2+1\right)\int\limits_0^t\|\delta\psi(\cdot,\tau)\|^2_{L_2(0,l)}\,d\tau\\ +\int\limits_\Omega\left|-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)+\delta v_1(x)\psi\right|^2 dxd\tau,&\quad\forall\, t\in[0,T].\end{aligned}$$

Estimating the left-hand side of this inequalities gives

$$\int\limits_{\Omega_t}\left|-\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)+\delta v_1(x)\psi\right|^2 dxd\tau\leqslant c_5\int\limits_0^t\|\psi(\cdot,\tau)\|^2_{W_2^2(0,l)}\,d\tau\|\delta v\|^2_{B^1}.$$

Considering this inequality in equation (A.2), we obtain

$$\|\delta\psi(\cdot,t)\|^2_{L_2(0,l)}\leqslant c_6\int\limits_0^t\|\delta\psi(\cdot,\tau)\|^2_{L_2(0,l)}\,d\tau+c_7\|\delta v\|^2_H,\quad\forall\, t\in[0,T].$$

With Gronwall's lemma applied to this inequality, it follows that

$$\|\delta\psi(\cdot,t)\|^2_{L_2(0,l)}\leqslant c_8\|\delta v\|^2_{B_1},\quad\forall\, t\in[0,T].\tag{A.5}$$

Now it is necessary to estimate the function $\frac{\partial\delta\psi}{\partial x}(x,t)$. For this purpose, we multiply both sides of equation (A.2) by $L(\delta\overline{\psi})=-\frac{\partial}{\partial x}\left((v_0(x)+\delta v_0(x))\frac{\partial\delta\overline{\psi}}{\partial x}\right)$ and integrate the resulting expression over the domain Ω_t :

$$\begin{aligned}\int\limits_{\Omega_t}\Big(i\rho^2\frac{\partial\delta\psi}{\partial t}L(\delta\psi)-|L(\delta\psi)|^2-(v_0(x)+\delta v_0(x))\delta\psi L(\delta\psi)&\\ +a_1\left(|\psi_\delta|^2+|\psi|^2\right)\delta\psi L(\delta\overline{\psi})+a_1\psi_\delta\psi\delta\overline{\psi}L(\delta\overline{\psi})\Big)dxd\tau&\\ =\int\limits_\Omega\left(\frac{\partial}{\partial x}\left(\delta v_0(x)\frac{\partial\psi}{\partial x}\right)-\delta v_1(x)\psi\right)L(\delta\overline{\psi})dxd\tau,\quad\forall\, t\in[0,T].&\end{aligned}$$

We integrate on x in both sides of this equation and subtract from the resulting equality its complex conjugate:

$$\int\limits_{\partial_t} \rho^2 \frac{\partial}{\partial t}\Big(v_o(x)+\delta v_o(x)\Big)\Big|\frac{\partial \delta\psi}{\partial x}\Big|^2\,dxd\tau$$
$$= 2\int\limits_{\Omega t}\left[\mathrm{Im}\Big(v_o(x)+\delta v_o(x)\Big)\left(\frac{dv_1(x)}{dx}+\frac{d\delta v_1(x)}{dx}\right)\delta\psi\frac{\partial\delta\overline{\psi}}{\partial x}\right]dxd\tau$$
$$-2a_1\int\limits_{\Omega_t}\left[\mathrm{Im}\Big(v_o(x)+\delta v_o(x)\Big)\frac{\partial}{\partial x}\Big(\Big|\psi_\delta\Big|^2+|\psi|^2\Big)\delta\psi\frac{\partial\delta\overline{\psi}}{\partial x}\right]dxd\tau$$
$$-2a_1\int\limits_{\Omega_t}\left[\mathrm{Im}\Big(v_o(x)+\delta v_o(x)\Big)\frac{\partial}{\partial x}\Big(\psi_\delta\psi\Big)\delta\overline{\psi}\frac{\partial\delta\overline{\psi}}{\partial x}\right]dxd\tau$$
$$-2a_1\int\limits_{\Omega_t}\left[\mathrm{Im}\Big(v_o(x)+\delta v_o(x)\Big)\psi_\delta\psi\left(\frac{\partial\delta\overline{\psi}}{\partial x}\right)^2\right]dxd\tau$$
$$-2\int\limits_{\Omega_t}\left[\mathrm{Im}\Big(v_o(x)+\delta v_o(x)\Big)\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\Big(\delta v_o(x)\frac{\partial\psi}{\partial x}\Big)-\delta v_1(x)\psi\right]\frac{\partial\delta\overline{\psi}}{\partial x}\right]dxd\tau,\quad\forall\,t\in[0,T].$$

Note that $v+\delta v\in V_1$ and $\delta\psi\,(x,0)=0,\ x\in(0,l)$. Due to the Cauchy–Bunyakovsky–Schwarz inequality and the bounds (A.4), we have

$$\begin{aligned}
s_0\left\|\frac{\partial\delta\psi(\cdot,t)}{\partial x}\right\|^2_{L_2(0,l)} &\leqslant c_9\int\limits_0^t\left\|\frac{\partial\delta\psi(\cdot,\tau)}{\partial x}\right\|^2_{L_2(0,l)}d\tau + c_{10}\int\limits_{\Omega_t}|\delta\psi|^2\,dxd\tau\\
&+4s_1\int\limits_{\Omega_t}|\delta v_0(x)|^2\left|\frac{\partial^3\psi}{\partial x^3}\right|^2dxd\tau+8s_1\int\limits_{\Omega_t}\left|\frac{d\delta v_0(x)}{dx}\right|^2\left|\frac{\partial^2\psi}{\partial x^2}\right|^2dxd\tau\\
&+4s_1\int\limits_{\Omega_t}\left|\frac{d\delta v_1(x)}{dx}\right|^2|\psi|^2\,dxd\tau+4s_1\int\limits_{\Omega_t}\left(\left|\frac{d^2\delta v_0(x)}{dx^2}\right|+|\delta v_1(x)|\right)^2\left|\frac{\partial\psi}{\partial x}\right|^2dxd\tau\\
&+c_8\,|a_1|\,(2+s_1)\int\limits_{\Omega_t}\left(\left|\frac{\partial\psi_\delta}{\partial x}\right|+\left|\frac{\partial\psi}{\partial x}\right|\right)^2|\delta\psi|^2\,dxd\tau,\ \forall t\in[0,T],
\end{aligned}\tag{A.6}$$

where $c_9=s_1(|a_1|c_4+|a_1|c_4^2+1+s_5)$ and $c_{10}=s(|a_1|c_4^2+s_5)$. Now it is necessary to estimate the last term on the right-hand side of this inequality. As is known [15], an arbitrary function $\varphi(\cdot,t)\in W_\lambda(0,l),\ \forall\,t\in[0,T]$ satisfies the inequality

$$\left\|\frac{\partial\varphi(\cdot,t)}{\partial x}\right\|_{L_\infty(0,l)}\leqslant c_0\left\|\frac{\partial\varphi(\cdot,t)}{\partial x}\right\|_{L_2(0,l)}\|\varphi(\cdot,t)\|_{L_2(0,l)},\quad\forall\,t\in[0,T],\tag{A.7}$$

where $c_0>0$ is some constant. With the functions $\frac{\partial\psi_\delta(x,t)}{\partial x}$ and $\frac{\partial\psi(x,t)}{\partial x}$ taken instead of the function $\varphi(x,t)$ in this inequality, we utilize (A.5) and (A.6) to obtain

$$\left.\frac{\partial\psi_\delta(\cdot,t)}{\partial x}\right|_{L_\infty(0,l)}\leqslant c_{11},\qquad\left.\left|\frac{\partial\psi(x,t)}{\partial x}\right|\right|_{L_\infty(0,l)}\leqslant c_{11}.\tag{A.8}$$

Using these bounds in (7) and inequality (A.5), we have

$$\left\|\frac{\partial\delta\psi(\cdot,t)}{\partial x}\right\|^2_{L_2(0,l)}\leqslant c_{12}\int\limits_0^t\left\|\frac{\partial\psi(\cdot,t)}{\partial x}\right\|^2_{L_2(0,l)}d\tau+c_{13}\left\|\delta v\right\|^2_{B_1}.$$

With Gronwall's lemma applied to this inequality, it follows that

$$\left\|\frac{\partial\psi(x,t)}{\partial x}\right\|^2_{L_2(0,l)}\leqslant c_{15}\|\delta v\|^2_{B_1},\quad\forall\,t\in[0,T].$$

Combining this bound with (A.5) gives

$$\left\|\delta\psi(\cdot,t)\right\|^2_{W_2^1(0,l)}\leqslant c_{15}\|\delta v\|^2_{B_1},\quad\forall\,t\in[0,T]. \tag{A.9}$$

Let us transform the increment of the criterion $J_0(v)$ on an arbitrary element $v\in V_1$:

$$\begin{gathered}\Delta J_0(\nu)=J_0(\nu+\delta\nu)-J_0(\nu)=2\beta_0\int\limits_0^T\mathcal{R}\epsilon\left[\left(\psi(0,t)-y_0(t)\right)\delta\psi(0,t)\right]\,dt\\+2\beta_1\int\limits_0^t\mathcal{R}\epsilon\left[\left(\psi(l,t)-y_1(t)\right)\delta\psi(l,t)\right]\,dt+\beta_0\left\|\delta\psi(x,t)\right\|^2_{L_2(0,T)}+\beta_1\left\|\delta\psi(x,t)\right\|^2_{L_2(0,T)}.\end{gathered} \tag{A.10}$$

By the embedding theorem, for the trace of functions from the space $W_2^{1,0}(\Omega)$, we have

$$\begin{gathered}\|\psi(0,\cdot)\|^2_{L_2(0,T)}+\|\psi(l,\cdot)\|^2_{L_2(0,T)}\leqslant c_{16}\|\psi\|^2_{W_2^{1,0}(\Omega)},\\\|\delta\psi(0,\cdot)\|^2_{L_2(0,T)}+\|\delta\psi(l,\cdot)\|^2_{L_2(0,T)}\leqslant c_{16}\|\delta\psi\|^2_{W_2^{1,0}(\Omega)}.\end{gathered}$$

These inequalities imply

$$\|\psi(0,\cdot)\|^2_{L_2(0,T)}+\|\psi(l,\cdot)\|^2_{L_2(0,T)}\leqslant c_{17}, \tag{A.11}$$

$$\|\delta\psi(0,\cdot)\|^2_{L_2(0,T)}+\|\delta\psi(l,\cdot)\|^2_{L_2(0,T)}\leqslant c_{17}. \tag{A.12}$$

Let us apply the Cauchy–Bunyakovsky–Schwarz inequality to the expression (A.10) and use (A.11) and (A.12). In view of the condition $y_0,y_1\in L_2(0,l)$, we finally derive the upper bound

$$|\Delta J_0(v)|\leqslant c_{19}(\|\delta v\|_{B^1}+\|\delta v\|^2_{B_1}),\quad v\in V_1.$$

According to this inequality, the criterion $J_0(v)$ is continuous on the set V_1. The criterion $J_0(v)$ is positive: $J_0(v)\geqslant 0$ for $v\in V_1$. Furthermore, the set V_1 is a closed, bounded, and convex set in the Hilbert space H_1. Hilbert spaces are uniformly convex. If a functional $I_0(v)$ is lower semicontinuous and bounded from below on a closed bounded set $U\subset X$ of a uniformly convex Banach space X, then there exists an everywhere dense subset $K\subset X$ such that, for any $w\in K$ and any $\alpha>0$, the functional $I_0(v)+\alpha\|v-w\|^2_X$ achieves the minimum value on a single element of the set $K\cap U$ [16]. For problem (9), all conditions and statements of this theorem are valid, and its proof is complete.

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STOCHASTIC SYSTEMS

A Decomposition–Autocompensation Method for Signal Recognition Based on the Principles of Continuity, Invariance, Multiplication, and Ranking with Regular and Irregular Interferences

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Abstract—Considering the principles of continuity, invariance, multiplication, and ranking, we develop a novel optimal signal recognition method under essential a priori uncertainty, with application to real-time information-measuring systems. By assumption, in addition to random noise with an unknown distribution law but a given correlation matrix, the observation equation may contain a regular interference with an analytical finite-spectral representation and an irregular interference without any effective probabilistic model. The latter interference can be described only by introducing some numerical characteristics and constraints confirmed by the operation practice of a particular system. This method is invariant to the above interferences, does not require traditional state-space expansion, and ensures the decomposition of the computational procedure. We analyze random and methodological errors as well as the computational effect achieved. An illustrative example is given.

Keywords: essential a priori uncertainty, regular interference, irregular interference, sample clogging factor, continuity principle, invariance principle, multiplication principle, ranking principle, spectral coefficients, optimality criterion, decomposition, recognition algorithms

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1. INTRODUCTION

For a modern information-measuring system (hereinafter referred to as a/the system), signal recognition is one of the most important problems; it includes signal detection, discrimination, resolution, smoothing, parameter estimation, etc. [1–18].

In the sense of applications, of major interest are real-time systems under essential uncertainty and strict requirements to recognition quality, not only on the average (over an entire ensemble of realizations) but also in a single case (for one fixed sample of measurements). The matter concerns a system with quite high risks (losses) from an incorrect decision in such a case.

Essential uncertainty arises, e.g., when real measurements are subjected to random noise with an unknown distribution and, moreover, to regular and irregular interferences. In this case, the noise information is limited only by its correlation matrix; the regular interference information, by a given finite set of its basis functions; the irregular interference information, by some numerical characteristics and constraints for a particular system. For example, the sample clogging factor can be one such characteristic.

Essential uncertainty is often eliminated using the least squares (LS) method or one of its modifications in a simple or extended version [5, 7–9, 14–18]. By the well-known Gauss–Markov

theorem, this method yields the best estimates on average; for example, see [5, p. 34]. Less common methods include maximum posterior probability density, maximum likelihood, linear and nonlinear filtering, and some others [1–7, 11, 14]. As a rule, these methods require a rather large amount of a priori statistical information and are difficult to implement in real time (especially when expanding the state space). In addition, they are often unstable (in computational terms) and have poor convergence in iterative calculations, e.g., in the case of unsuccessfully assigned initial conditions or ravine-type objective functions. Such methods are most often used at stages related to mathematical modeling when justifying the potential capabilities of the system. In the single cases mentioned (when one fixed sample is involved), the estimation results can cause significant losses, especially under an irregular interference.

The method of generalized invariant-unbiased estimation (GIUE) [18, 19] was developed for any linear numerical characteristics of useful signals (e.g., spectral coefficients, derivatives of different orders, integrals, smoothed values, etc.) under a regular interference (sometimes called signal-like, systematic, singular, or dynamic) without state space expansion. GIUE autocompensates for a regular interference, smoothens random noise, and yields optimal estimates with the minimum trace of the correlation matrix of estimation errors; no a priori information about the distribution law is required, and it suffices to know only the correlation matrix for a given noise. The method ensures the maximum possible decomposition of the computational procedure, which leads to the inversion of matrices of significantly lower dimensions compared to the extended least squares (XLS) method.

However, the GIUE capabilities are significantly limited if measurements contain both regular and irregular interferences (e.g., in the form of separate single pulses or packets of different-shape pulses with unknown parameters). An irregular interference can occur both on the entire observation interval and on separate, a priori unknown, parts of this interval and is poorly formalized (e.g., within finite-analytical spectral analysis). As a rule, an attempt to introduce additional basis functions for describing an irregular interference sharply increases dimension and causes significant errors. Under essential uncertainty and real observational conditions, expanding the spectral composition of the total interference (regular plus irregular interferences) requires inverting ill-conditioned matrices of too high dimensions, which strongly distorts the results of measurement processing within GIUE.

Below we develop a novel signal recognition method inheriting all the GIUE advantages without state space expansion. Therefore, it is possible to implement the corresponding algorithms in real time for the system under uncertainty. This method provides a trade-off between average estimation (based on the conditions of unbiasedness, invariance to a regular interference, and the minimum trace of the correlation matrix of errors) and partial estimation (for a fixed sample) considering the minimum influence of an irregular interference on the resulting estimate.

To avoid rather cumbersome calculations, the presentation below will mainly focus on the problem of estimating the linear parameters of signals. However, we will also discuss appropriate generalizations and practical recommendations for extending the results to other problems related to signal recognition, including those with nonlinear parameters (by analogy with [19]).

2. MATHEMATICAL FORMULATION OF THE RECOGNITION PROBLEM IN THE CASE OF ESSENTIAL A PRIORI UNCERTAINTY

On a given time interval $[0, T]$, we introduce a basic grid $\mathrm{C}_0 = \left\{t_n, n = \overline{1, N_0}\right\}$ of nodes and write the following observation equation on this grid:

$$\mathbf{H}_0 = \mathbf{S}_0 + \mathbf{\Theta}_0 + \mathbf{D}_0 + \mathbf{\Xi}_0, \tag{1}$$

where $\mathbf{H}_0 = \left[h_n, n = \overline{1, N_0}\right]^{\mathrm{T}}$, $\mathbf{S}_0 = \left[s_n(\mathbf{A}), n = \overline{1, N_0}\right]^{\mathrm{T}}$, $\mathbf{\Theta}_0 = \left[\theta_n(\mathbf{B}_\theta), n = \overline{1, N_0}\right]^{\mathrm{T}}$, $\mathbf{D}_0 = \left[q_n d_n(\mathbf{B}_d), n = \overline{1, N_0}\right]^{\mathrm{T}}$, and $\mathbf{\Xi}_0 = \left[\xi_n, n = \overline{1, N_0}\right]^{\mathrm{T}}$ are the vectors of samples of an input observation $h(t)$, a useful signal $s(t, \mathbf{A})$, a regular interference $\theta(t, \mathbf{B}_\theta)$, an irregular interference $d(t, \mathbf{B}_d)$, and noise $\xi(t)$, respectively, $h_n = h(t_n)$, $s_n(\mathbf{A}) = s(t_n, \mathbf{A})$, $\theta_n(\mathbf{B}_\theta) = \theta(t_n, \mathbf{B}_\theta)$, $d_n(\mathbf{B}_d) = d(t_n, \mathbf{B}_d)$, and $\xi_n = \xi(t_n)$; $\mathbf{A} = \left[a_m, m = \overline{1, M_s}\right]^{\mathrm{T}}$, $\mathbf{B}_\theta = \left[b_{\theta r}, r = \overline{1, M_\theta}\right]^{\mathrm{T}}$, and $\mathbf{B}_d = \left[b_{dr}, r = \overline{1, M_d}\right]^{\mathrm{T}}$ are the unknown vector spectral coefficients of the linear decompositions of the signal, regular, and irregular interference, respectively; finally, $q_n \in \{0, 1\}$ is the indicator of zero and nonzero irregular interference samples, where $\sum_{n=1}^{N_0} q_n = M_d$ and the parameter M_d corresponds to the real number of nonzero irregular interference samples in the observation equation (1).

Let $\{\overline{t_1}, \overline{t_2}, \ldots, \overline{t_{M_d}}\}$ denote the set of nodes associated with nonzero irregular interference samples, where $\overline{t_m} \in \{t_1, t_2, \ldots, t_{N_0}\}$ and $\overline{t_{m+1}} > \overline{t_m}$. To describe the irregular interference, we use the following approach. Let $K_{\max} \in \{0, 1, \ldots\}$ be the maximum possible number of nodes associated with the irregular interference, i.e., $M_d \leqslant K_{\max}$. We require the condition

$$N_{\min} + K_{\max} \leqslant N_0, \tag{2}$$

where $N_{\min}$ is the minimum number of input observation samples sufficient to qualitatively recognize the useful signal in the absence of regular and irregular interferences.

We will consider different time grids of nodes of length N for which

$$N_{\min} \leqslant N \leqslant N_0. \tag{3}$$

With this descriptive approach to irregular interferences, their nonzero samples can take arbitrary values (including anomalous ones) as well as be scattered (single) or concentrated (in packets). In physical channels, such interferences may correspond, e.g., to pulses of various shapes, durations, and intensities. These interferences are often due to various transients, switching, hashing, natural and artificial disturbance, etc., and may be masked by signal and noise. Nonzero irregular interference samples can be arbitrarily located on the segment $[0, T]$, and there exists no universal and satisfactory model to describe them. The only way to consider irregular interferences is to impose some quantitative constraints (like (2) and (3)) matching the operation practice of a particular system.

For arbitrary values of $\mathbf{A}$, $\mathbf{B}_\theta$, and $\mathbf{B}_d$, we use the following linear finite-dimensional combinations (the signal and regular interference models widespread in practice):

$$s(t, \mathbf{A}) = \mathbf{A}^{\mathrm{T}} \mathbf{\Psi}(t), \tag{4}$$

$$\theta(t, \mathbf{B}_\theta) = \mathbf{B}_\theta^{\mathrm{T}} \mathbf{\Omega}_\theta(t), \tag{5}$$

$$d(t, \mathbf{B}_d) = \mathbf{B}_d^{\mathrm{T}} \mathbf{\Omega}_d(t), \tag{6}$$

where $\mathbf{\Psi}(t) = \left[\psi_m(t), m = \overline{1, M_s}\right]^{\mathrm{T}}$, $\mathbf{\Omega}_\theta(t) = \left[\omega_{\theta r}(t), r = \overline{1, M_\theta}\right]^{\mathrm{T}}$ and $\mathbf{\Omega}_d(t) = \left[\omega_{dr}(t), r = \overline{1, M_d}\right]^{\mathrm{T}}$ are given basis functions of the signal, regular, and irregular interference, respectively, $\omega_{dr}\left(\overline{t_r}\right) = 1$, and $\omega_{dr}(t) = 0$ for all $t \neq \overline{t_r}$.

Suppose that the extended functional basis $\{\mathbf{\Psi}(t), \mathbf{\Omega}_\theta(t), \mathbf{\Omega}_d(t)\}$ is linearly independent on the grid C_0 (by analogy with [18, 19]). The noise $\mathbf{\Xi}_0$ is characterized by zero mean and a correlation matrix $\mathbf{K}_{\mathbf{\Xi}_0}$.

Note that quite rare anomalous outliers of the noise $\xi(t)$ are further associated with irregular interferences, i.e., they are combined with the corresponding nonzero coordinates of the vector $\mathbf{D}_0$. Thus, in any sample $\mathbf{\Xi}_0$, all coordinates can be enclosed in gates of a size determined by one of the well-known rules (e.g., by the three-sigma rule).

For the recognition problem, we will consider two main subproblems as follows.

Subproblem 1 (the classical problem of estimating the useful signal parameters) is to construct an optimal estimate $\mathbf{A}^*$ of the vector $\mathbf{A}^*$ based on the algorithm that autocompensates for the regular and irregular interferences. *Subproblem 2* is to construct estimates $\mathbf{A}^*$ and $\mathbf{B}^*$ for the vectors $\mathbf{A}$ and $\mathbf{B}^*$, respectively, invariant to the irregular interference. In this case, the estimation of the signal parameters shall autocompensate for the regular interference and, vice versa, the estimation of the regular interference parameters shall autocompensate for the signal.

Based on (1)–(6), it is required to develop an optimal recognition method (for the two main subproblems above) under essential a priori uncertainty (using only the correlation matrix $\mathbf{K}_{\mathbf{\Xi}_0}$ as statistical information) without state space expansion. In both subproblems, optimality means that the estimates shall be unbiased, have the minimum trace of the correlation matrices of estimation errors, and provide a significant computational advantage over XLS and GIUE, in terms of decomposition and reduction in the amount of calculations and a gain in accuracy due to inverting matrices of smaller dimensions.

3. A GENERAL APPROACH TO SIGNAL RECOGNITION BASED ON THE PRINCIPLES OF CONTINUITY, MULTIPLICATION, AND RANKING

The approach proposed below involves the principle of a continuous dependence of recognition quality on the parameters of the time grid. In particular, the successive reduction of the volume of a sufficiently large grid (by removing certain nodes) leads to a smooth evolution of the accuracy of the resulting estimate. This effect also concerns a continuous change in the position of grid nodes on a given time interval.

Assume that a set of different reduced grids can be formed from the basic grid C_0 and its variants obtained by removing some nodes: $\left\{\mathrm{C}_j, j=\overline{1,J-1}\right\}$ (where $J \geqslant 2$, $\mathrm{C}_j = \left\{t_{jn}, n=\overline{1,N_j}\right\}$, $N_j < N_0$, $t_{jn} \in \mathrm{C}_0$, $t_{jn} \neq t_{jk}$ $\forall n \neq k$, $n,k \in \overline{1,N_j}$, $t_{j,n+1} > t_{jn}$).

We represent the entire family of grids $\mathrm{C} = \{\mathrm{C}_0, \mathrm{C}_1, \ldots, \mathrm{C}_{J-1}\}$, including C_0, as

$$\mathrm{C} = \mathrm{C}^{\#} \cup \mathrm{C}^{\wedge} \cup \mathrm{C}^{\&}. \tag{7}$$

The symbol $\mathrm{C}^{\#}$ indicates the set of grids without nodes associated with nonzero irregular interference samples. Next, the symbol $\mathrm{C}^{\wedge}$ stands for the set of grids that may contain nodes associated with normal irregular interference samples. These are the irregular interference samples that, all together, weakly affect the estimation results. Finally, the symbol $\mathrm{C}^{\&}$ denotes the set of grids that may contain nodes associated with both normal and abnormal irregular interference samples that, all together, devalue the estimation results. The sets $\mathrm{C}^{\#}$ and $\mathrm{C}^{\wedge}$ will be called admissible and the set $\mathrm{C}^{\&}$ inadmissible.

The main idea of the method is that by handling $\mathrm{C}^{\#}$ and $\mathrm{C}^{\wedge}$, one can obtain estimates invariant (or almost invariant) to regular and irregular interferences (the invariance principle). For this purpose, we assign to each grid C_j, $j = \{0, 1, \ldots, J-1\}$, the observation equation

$$\mathbf{H}_j = \mathbf{S}_j + \mathbf{\Theta}_j + \mathbf{D}_j + \mathbf{\Xi}_j \tag{8}$$

and the GIUE-optimal estimates

$$\begin{cases} \mathbf{A}_j^* = \mathbf{P}_j^{\mathbf{A}} \mathbf{H}_j, \quad s_j^*\left(t, \mathbf{A}_j^*\right) = \left(\mathbf{A}_j^*\right)^{\mathrm{T}} \mathbf{\Psi}(t), \\ \mathbf{B}_{\theta j}^* = \mathbf{P}_{\theta j}^{\mathbf{B}} \mathbf{H}_j, \quad \theta_j^*\left(t, \mathbf{B}_{\theta j}^*\right) = \left(\mathbf{B}_{\theta j}^*\right)^{\mathrm{T}} \mathbf{\Omega}_\theta(t), \end{cases} \tag{9}$$

where $\mathbf{H}_j$ is the vector of the input observation samples $\mathbf{H}_0$ corresponding to the reduced grid C_j, and $\mathbf{P}_j^{\mathbf{A}}$ and $\mathbf{P}_{\theta j}^{\mathbf{B}}$ are the linear decomposition estimation matrices:

$$\begin{cases} \mathbf{P}_j^{\mathbf{A}} = \left[\mathbf{\Lambda}_{\theta j}^{\mathbf{A}}\mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Psi}_j\left(\mathbf{\Psi}_j^{\mathrm{T}}\mathbf{\Lambda}_{\theta j}^{\mathbf{A}}\mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Psi}_j\right)^{-1}\right]^{\mathrm{T}}, \\ \mathbf{P}_{\theta j}^{\mathbf{B}} = \left[\mathbf{\Lambda}_{\theta j}^{\mathbf{B}}\mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Omega}_{\theta j}\left(\mathbf{\Omega}_{\theta j}^{\mathrm{T}}\mathbf{\Lambda}_{\theta j}^{\mathbf{B}}\mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Omega}_{\theta j}\right)^{-1}\right]^{\mathrm{T}}, \end{cases} \tag{10}$$

$$\begin{cases} \mathbf{\Lambda}_{\theta j}^{\mathbf{A}} = \mathbf{E}_{Nj} - \mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Omega}_{\theta j}\left(\mathbf{\Omega}_{\theta j}^{\mathrm{T}}\mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Omega}_{\theta j}\right)^{-1}\mathbf{\Omega}_{\theta j}^{\mathrm{T}}, \\ \mathbf{\Lambda}_{\theta j}^{\mathbf{B}} = \mathbf{E}_{Nj} - \mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Psi}_j\left(\mathbf{\Psi}_j^{\mathrm{T}}\mathbf{K}_{\mathbf{\Xi}j}^{-1}\mathbf{\Psi}_j\right)^{-1}\mathbf{\Psi}_j^{\mathrm{T}}. \end{cases} \tag{11}$$

In formulas (9)–(11), $\mathbf{\Psi}_j = [\psi_{jnm}, n = \overline{1,N_j}, m = \overline{1,M_s}]$ and $\mathbf{\Omega}_{\theta j} = [q_{jn}\omega_{\theta jnr}, n = \overline{1,N_j}, r = \overline{1,M_{\theta j}}]$ are the basis matrices of the signal $\mathbf{S}_j$ and interference $\boldsymbol{\theta}_j$, respectively, $\psi_{jnm} = \psi_m(t_{jn})$ and $\omega_{\theta jnr} = \omega_{\theta r}(t_{jn})$; $\mathbf{E}_{Nj}$ is an identity matrix corresponding to the dimension of the vector $\mathbf{H}_j$.

If $\mathrm{C}_j \in \mathrm{C}^{\#}$, these estimates will match the corresponding conditions of minimality, unbiasedness, and invariance [18, 19] (i.e., will be optimal). If $\mathrm{C}_j \in \mathrm{C}^{\wedge}$, we obtain quasi-optimal estimates; in the case $\mathrm{C}_j \in \mathrm{C}^{\&}$, the estimation errors may become unacceptably large.

The multiplication principle consists in the possibility of forming a set of partial estimates corresponding to the family of grids $\mathrm{C} = \{\mathrm{C}_0, \mathrm{C}_1, \ldots, \mathrm{C}_{J-1}\}$ satisfying conditions (2) and (3).

Consider subproblem 1. For each pair of grids C_j and C_m, we form the scalar residuals

$$\Delta_{\mathbf{A}jm}(t) = s(t, \mathbf{A}_j^*) - s(t, \mathbf{A}_m^*) = [(\mathbf{A}_j^*) - (\mathbf{A}_m^*)]^{\mathrm{T}}\mathbf{\Psi}(t), \quad j,m = \overline{0, J-1},\ j > m,$$

and their norms $\Delta_{\mathbf{A}jm} = \|\Delta_{\mathbf{A}jm}(t)\|$, $j,m = \overline{0, J-1}$ (any norm of the corresponding functional space). Based on the norms, we construct a variational series $\Delta_{\mathbf{A}[v]}$, $v = \overline{1, (J^2-J)/2}$ (a monotonically increasing sequence of scalar residuals), which fully characterizes the quality of all grids used. If at least one grid in a pair C_j, C_m belongs to the set $\mathrm{C}^{\&}$, the corresponding element $\Delta_{\mathbf{A}[v]}$ will be in the tail of the variational series. The initial elements of this series will be associated with the pairs C_j, C_m corresponding to the sets $\mathrm{C}^{\#}$ and $\mathrm{C}^{\wedge}$. Thus, under the above constraints, guaranteed clustering is performed for some elements $\Delta_{\mathbf{A}[v]}$ of the variational series in a sufficiently small neighborhood $(0, \delta_{\mathbf{A}})$, where $\delta_{\mathbf{A}} > 0$ denotes the truncation parameter of the series. For a given grid C_0, the constant $\delta_{\mathbf{A}}$ is chosen in advance for a particular system, considering the analytical expressions for the random and methodological errors of the GIUE procedure [18, 19], when planning a measurement experiment [5].

All the pairs of grids not satisfying the condition

$$\Delta_{\mathbf{A}[v]} < \delta_{\mathbf{A}} \tag{12}$$

are rejected. This is the ranking principle mentioned above.

We introduce several important notions.

Definition 1. A pair of grids C_j, C_m is said to be admissible if condition (12) holds.

Definition 2. An arbitrary irregular interference sample is said to be anomalous if its corresponding node, present in any grid of the set C, violates condition (12). Otherwise, the irregular interference sample is said to be normal.

Definition 3. A group of two or more samples is said to be anomalous (even if each separate sample is normal) when the corresponding nodes, present in any grid of the set C, violate condition (12). Otherwise, the group of irregular interference samples is said to be normal.

Definition 4. The sample clogging factor $\mathbf{H}_0$ is the number $k_d = 100\,(K_{\max}/N_0)$, expressed as a percentage.

Since all grids of the set $\mathrm{C}^{\&}$ violate condition (12), they are automatically rejected. A strict criterion for the anomalousness of input observation samples $\mathbf{H}_0$ sounds as follows: a sample is anomalous if it never occurs in pairs of grids satisfying condition (12). This criterion can be somewhat relaxed by considering the rare presence of potentially anomalous samples in the pairs specified. This is especially relevant in cases of ambiguity about an appropriate class (normal or anomalous) for a particular sample. Anyway, the criterion allows rejecting both anomalous and potentially anomalous samples of the input observation. However, the false rejection of some normal samples is possible.

The anomalousness criterion for a group of samples sounds as follows: a group is anomalous if it repeatedly occurs in rejected (inadmissible) pairs of grids and never occurs in admissible pairs of grids (those that have passed rejection).

In general, the ranking principle may lead to situations when the number of good grids satisfying condition (12) may increase, i.e., some redundancy of private estimates. Consequently, the following legitimate question arises: how should this redundancy be considered in order to construct a more reliable resulting estimate?

To construct the resulting estimate $\mathbf{A}^*$ of the vector $\mathbf{A}$, we proceed as follows. Let C_j^*, $j = \overline{1, J^*}$, denote all the grids included in the admissible pairs. (These grids will be called competing.) For each fixed grid C_j^* and all other competing grids C_m^*, we form the total scalar residual

$$\Delta_{\mathbf{A}j}^* = \sum_{\substack{m=1 \\ m \neq j}}^{J^*} \Delta_{\mathbf{A}jm}. \tag{13}$$

This residual shows the efficiency of the grid C_j^* compared to all other competing grids C_m^*, $m \neq j$, $m \in \overline{1, J^*}$, included in admissible pairs.

Obviously, within the models and constraints adopted, the criterion for selecting the optimal number $j^* \in \{1, \ldots, J^*\}$ of the optimal grid $\mathrm{C}_{j^*}^* \in \{\mathrm{C}_1^*, \ldots, \mathrm{C}_{J^*}^*\}$ is as follows:

$$j^* = \arg\min_j \Delta_{\mathbf{A}j}^*. \tag{14}$$

This criterion can be easily realized in practice if the matrix of scalar residuals is constructed for all admissible pairs of time grids. Among the input observation samples corresponding to the optimal grid C_{j*}^*, there are no anomalous samples and anomalous groups of irregular interference samples; in addition, the estimates

$$\begin{cases} \mathbf{A}_{j^*}^* = \mathbf{P}_{j^*}^{\mathbf{A}} \mathbf{H}_{j^*}, \\ s\left(t, \mathbf{A}_j^*\right) = \left(\mathbf{A}_{j^*}^*\right)^{\mathrm{T}} \boldsymbol{\Psi}(t) \end{cases} \tag{15}$$

will be GIUE-optimal for the grid $\mathrm{C}_{j^*}^*$.

Let us pass to subproblem 2. It is easily reduced to subproblem 1 by introducing the single vector of estimated parameters $\mathbf{Z} = \left[\mathbf{A}^{\mathrm{T}}, \mathbf{B}_\theta^{\mathrm{T}}\right]^{\mathrm{T}}$. Due to (9), we can assign to each grid C_j the GIUE-optimal estimate $\mathbf{Z}_j^* = \left[\left(\mathbf{A}_j^*\right)^{\mathrm{T}}, \left(\mathbf{B}_{\theta j}^*\right)^{\mathrm{T}}\right]^{\mathrm{T}}$. Finding the estimate $\mathbf{A}_j^*$ requires invariance with respect to the regular interference (i.e., the parameter $\boldsymbol{\Theta}_0$ in (1) is treated as a disturbing factor.) In turn, finding the estimate $\mathbf{B}_{\theta j}^*$ requires invariance with respect to the signal $\mathbf{S}_0$ (i.e., the signal now treated as a disturbing factor for the regular interference). The matrices $\mathbf{P}_j^{\mathbf{A}}$ and $\mathbf{P}_{\theta j}^{\mathbf{B}}$ are used to find the estimates $\mathbf{A}_j^*$ and $\mathbf{B}_{\theta j}^*$, respectively; they are formed by (10) and (11). Obviously, to solve subproblem 2, it suffices to replace $\Delta_{\mathbf{A}[v]}$, $\delta_{\mathbf{A}}$, $\Delta_{\mathbf{A}jm}$, and $\Delta_{\mathbf{A}j}^*$ in (12)–(14) with $\Delta_{\mathbf{Z}[v]}$, $\delta_{\mathbf{Z}}$, $\Delta_{\mathbf{Z}jm}$, and $\Delta_{\mathbf{Z}j}^*$, respectively.

For the grid $\mathrm{C}^*_{j^*}$, we obtain the following solution of the optimal regular interference identification problem:

$$\begin{cases} \mathbf{B}^*_{\theta j^*} = \mathbf{P}^{\mathbf{B}}_{\theta j^*}\mathbf{H}_{j^*}, \\ \theta\left(t, \mathbf{B}^*_{\theta j^*}\right) = \left(\mathbf{B}^*_{\theta j^*}\right)^{\mathrm{T}} \mathbf{\Omega}_\theta(t). \end{cases} \tag{16}$$

In view of (15) and (16), the final estimate for subproblem 2 is $\mathbf{Z}^*_{j^*} = \left[\left(\mathbf{A}^*_{j^*}\right)^{\mathrm{T}}, \left(\mathbf{B}^*_{\theta j^*}\right)^{\mathrm{T}}\right]^{\mathrm{T}}$.

For the method under consideration, it is fundamental to select competing grids C^*_j, $j = \overline{1, J^*}$, satisfying condition (12) and choose the optimal grid $\mathrm{C}^*_{j^*}$ among them. Obviously, this is directly related to the initial set of grids $\mathrm{C} = \{\mathrm{C}_0, \mathrm{C}_1, \ldots, \mathrm{C}_{J-1}\}$ satisfying conditions (2) and (3). Modern advances in the field of parallel computers (especially those on new principles [21–24]) give hope that the multiplication principle of grids and partial estimates will not become an obstacle for prospective real-time systems. However, this approach is not possible for all the existing systems, as it may require a huge number of channels for parallel data processing and significant computational costs. Therefore, along with the optimal solution, it is necessary to consider quasi-optimal approaches to the design of reduced grids.

4. SOME RULES FOR BUILDING REDUCED TIME GRIDS

According to the principle of grid multiplication, a set of reduced grids is formed on a given time interval by some rule χ,

$$\chi : \mathrm{C}_0 \to \{\mathrm{C}_j\}_{j=0}^{J-1}, \tag{17}$$

so that, under some constraints on irregular interferences, the resulting partial estimates $\left\{\mathbf{A}^*_j\right\}_{j=0}^{J-1}$ and $\left\{\mathbf{Z}^*_j\right\}_{j=0}^{J-1}$ based on such grids surely include not only bad but also good estimates for which condition (12) is valid. Obviously, in the general case, the choice of an appropriate rule χ is ambiguous and depends on a particular system and observation conditions.

Rule 1. Let the system operate a small sample. Using combinatorics and the basic grid C_0, we form all possible reduced grids with the number of nodes not less than $N_{\min}$. By the complete enumeration method, the number of such grids is equal to

$$J = \sum_{m=0}^{N_0 - N_{\min}} C_{N_0}^{N_{\min}+m}, \tag{18}$$

where $C_{N_0}^{N_{\min}+m}$ denotes the corresponding binomial coefficient (the number of combinations of $N_{\min} + m$ elements in a total of N_0 elements).

For a fixed M_d, the number of grids not associated with nonzero irregular interference samples is given by

$$\overline{J} = \sum_{m=0}^{\overline{N}} C_{N_0 - M_d}^{N_{\min}+m}, \tag{19}$$

where $\overline{N} = N_0 - N_{\min} - M_d$.

In the special case when $M_d = K_{\max}$, we can form the minimum number of such grids, $\overline{J}_{\min} = \sum_{m=0}^{\overline{N}_{\min}} C_{N_0 - K_{\max}}^{N_{\min}+m}$, where $\overline{N}_{\min} = N_0 - N_{\min} - K_{\max}$.

For example, let $N_0 = 5$, $N_{\min} = 3$, $K_{\max} = 2$, and $M_d = 1$; then we have $J = 16$, $\overline{J}_{\min} = 1$, and $\overline{J} = 5$. If an irregular interference sample (for the case $M_d = 1$) is anomalous, the boundedness condition (12) will hold for five grids and fail for eleven. Among the good grids, four will have a length of $N_j = 3$, $j = \overline{1,4}$, and the fifth grid a length of $N_5 = 4$.

Rule 2. For large samples, a more suitable rule χ is based on representing the grid C_0 as L elementary grids $\overline{C}_l$ adjacent to each other (the full cover method):

$$\mathrm{C}_0 = \bigcup_{l=1}^{L} \overline{\mathrm{C}}_l, \quad \overline{\mathrm{C}}_l \cap \overline{\mathrm{C}}_d = \varnothing, \quad \forall l \neq d \quad l, d \in \overline{1,L}, \tag{20}$$

where $\overline{\mathrm{C}}_l = \left\{t_{l_m}, m = \overline{1,M_l}\right\}$ is the elementary grid of nodes $t_{l_m} \in \mathrm{C}_0$, $l_m \in \overline{1,N_0}$, $\sum_{l=1}^{L} M_l = N_0$.

One should keep in mind that the grid $\overline{\mathrm{C}}_l$ matches only the neighbor nodes from the grid C_0, i.e., those following each other: $\left(t_{l_{m+1}} = t_{l_1+m}, m = \overline{0, M_l - 1}\right)$. It is easiest to consider the elementary grids $\overline{\mathrm{C}}_l$ of the same length by taking $M_l = M$ and $L \cdot M = N_0$.

From L elementary grids, using combinatorics, we can also form the desired family of different reduced grids $\left\{\mathrm{C}_j, j = \overline{1, J-1}\right\}$, with the number of nodes not less than $N_{\min}$. In contrast to Rule 1, now potential anomalousness is associated with individual elementary grids. Given the maximum possible number of elementary potentially anomalous grids, it is easy to find the total number of reduced grids not containing these regions and other characteristics of the rule χ.

Rule 3. Along with (20), the partial cover method can be used:

$$\mathrm{C}_0 \supset \bigcup_{l=1}^{L} \overline{\mathrm{C}}_l. \tag{21}$$

This method can minimize computations but involves possible errors in decision-making.

Rule 4. To avoid such errors, it is possible to realize the rule χ step by step using different sets of elementary grids at each step. In this case, we propose the following step-by-step methods of full and partial cover by elementary grids of length M_{li}:

$$\begin{cases} \mathrm{C}_{0i} = \bigcup\limits_{l=1}^{L_i} \overline{\mathrm{C}}_{li}, & i = \overline{1,I}, \\ \mathrm{C}_{0i} \supset \bigcup\limits_{l=1}^{L_i} \overline{\mathrm{C}}_{li}, & i = \overline{1,I}, \end{cases} \tag{22}$$

where i is the step number and $L_i > L$.

According to (22), at each step i we construct a family of grids C_i instead of C and check condition (12) for all its elements. If this condition completely fails, it is necessary to proceed to the next step. If this condition is valid at least for one reduced grid from C_I, this grid will be considered optimal. Obviously, the step-by-step method suits well for large grids C_0 owing to insignificant computational costs (compared to the complete enumeration method), but it may lose in terms of efficiency for large I. However, under conditions (2) and (3), such a situation is quite rare.

Rule 5. An even simpler way of building reduced grids is possible when the system includes an input observation analyzer to identify potentially anomalous regions on the segment $[0, T]$ (in terms of the presence of irregular interferences in them). By excluding the nodes corresponding to these regions from C_0, we can form the desired family of reduced grids satisfying conditions (2) and (3).

The choice of an appropriate rule to form reduced grids entirely depends on the system under consideration and the requirements imposed on it. Formulas (17)–(22) are quite convenient for the quantitative justification of reduced grids.

5. RECOGNITION ALGORITHMS UNDER UNCERTAINTY

The recognition algorithm for subproblem 1 includes the following steps.

Step 1.1. Based on C_0, construct the set of grids $\left\{\mathrm{C}_j, j=\overline{0,J-1}\right\}$.

Step 1.2. For each grid C_j, find the optimal estimation matrix $\mathbf{P}_j^{\mathbf{A}}$.

Step 1.3. For each grid C_j, find the estimate $\mathbf{A}_j^*=\mathbf{P}_j^{\mathbf{A}}\mathbf{H}_j$.

Step 1.4. For each pair C_j, C_m, form the scalar residual $\Delta_{\mathbf{A}jm}$.

Step 1.5. Based on all residuals $\Delta_{\mathbf{A}jm}$, construct the variational series $\Delta_{\mathbf{A}[v]}$, $v=\overline{1,(J^2-J)/2}$.

Step 1.6. Based on the variational series and condition (12), form the set of competing grids C_j^*, $j=\overline{1,J^*}$.

Step 1.7. For the competing grids, calculate the total residuals $\Delta_{\mathbf{A}j}^*$, $j=\overline{1,J^*}$.

Step 1.8. Based on the criterion (14), find the optimal number $j^*\in\{1,\ldots,J^*\}$ of the optimal grid $\mathrm{C}_{j^*}^*$.

Step 1.9. Using the matrix $\mathbf{P}_{j^*}^{\mathbf{A}}$ and the grid $\mathrm{C}_{j^*}^*$, calculate the estimate $\mathbf{A}_{j^*}^*$ for the vector $\mathbf{A}$ and the estimate $s\left(t,\mathbf{A}_{j^*}^*\right)=\left(\mathbf{A}_{j^*}^*\right)^{\mathrm{T}}\mathbf{\Psi}(t)$ for the useful signal $s\left(t,\mathbf{A}\right)$.

The recognition algorithm for subproblem 2 includes the following steps.

Step 2.1. Based on C_0, construct the set of grids $\left\{\mathrm{C}_j, j=\overline{0,J-1}\right\}$.

Step 2.2. For each grid C_j, find the optimal estimation matrices $\mathbf{P}_j^{\mathbf{A}}$ and $\mathbf{P}_{\theta j}^{\mathbf{B}}$.

Step 2.3. For each grid C_j, find the estimates $\mathbf{A}_j^*$ and $\mathbf{B}_{\theta j}^*$.

Step 2.4. Form the unified vector $\mathbf{Z}_j^*=\left[\left(\mathbf{A}_j^*\right)^{\mathrm{T}},\left(\mathbf{B}_{\theta j}^*\right)^{\mathrm{T}}\right]^{\mathrm{T}}$.

Step 2.5. For each pair C_j, C_m, form the scalar residual $\Delta_{\mathbf{Z}jm}$.

Step 2.6. Based on all residuals $\Delta_{\mathbf{Z}jm}$, construct the variational series $\Delta_{\mathbf{Z}j}^*$, $j=\overline{1,J^*}$.

Step 2.7. Based on the variational series and condition (12), form the set of competing grids C_j^*, $j=\overline{1,J^*}$.

Step 2.8. For the competing grids, calculate the total residuals $\Delta_{\mathbf{Z}j}^*$, $j=\overline{1,J^*}$.

Step 2.9. Based on the criterion (14), find the optimal number $j^*\in\{1,\ldots,J^*\}$ of the optimal grid $\mathrm{C}_{j^*}^*$.

Step 2.10. Using the matrices $\mathbf{P}_{j^*}^{\mathbf{A}}$ and $\mathbf{P}_{\theta j^*}^{\mathbf{B}}$ and the grid $\mathrm{C}_{j^*}^*$, calculate the estimate $\mathbf{A}_{j^*}^*$ for the vector $\mathbf{A}$ and the estimate $s\left(t,\mathbf{A}_{j^*}^*\right)=\left(\mathbf{A}_{j^*}^*\right)^{\mathrm{T}}\mathbf{\Psi}(t)$ for the useful signal $s(t,\mathbf{A})$, as well as the estimate $\mathbf{B}_{\theta j^*}^*$ for the vector $\mathbf{B}_\theta$ and the estimate $\theta\left(t,\mathbf{B}_{\theta j^*}^*\right)=\left(\mathbf{B}_{\theta j^*}^*\right)^{\mathrm{T}}\mathbf{\Omega}_\theta(t)$ for the regular interference $\theta(t,\mathbf{B}_\theta)$.

6. A COMPARATIVE ANALYSIS OF THE METHOD

The correlation matrices of the estimation errors on the grid $\mathrm{C}_{j^*}^*$ are found by the rule

$$\begin{cases}\mathbf{K}_{j^*}^{\mathbf{A}}=\mathbf{P}_{j^*}^{\mathbf{A}}\mathbf{K}_{\mathbf{\Xi}_{j^*}}\left(\mathbf{P}_{j^*}^{\mathbf{A}}\right)^{\mathrm{T}},\\ \mathbf{K}_{\theta j^*}^{\mathbf{B}}=\mathbf{P}_{\theta j^*}^{\mathbf{B}}\mathbf{K}_{\mathbf{\Xi}_{j^*}}\left(\mathbf{P}_{\theta j^*}^{\mathbf{B}}\right)^{\mathrm{T}}.\end{cases}\tag{23}$$

The expression (23) allows one to assess the potential capabilities of the method in each particular case, considering the requirements for the system.

According to the optimality criterion used, these matrices have minimum traces, i.e., $Sp\mathbf{K}_{j^*}^{\mathbf{A}}\to\min$ and $Sp\mathbf{K}_{\theta j^*}^{\mathbf{B}}\to\min$ in the class of all linear estimates. The methodological error

due to the finiteness of the representations (4) and (5) can be considered as follows. Let the observation equation on the optimal grid $\mathrm{C}^*_{j^*}$ have the form (with irregular interference compensation)

$$\mathbf{H}_{j^*} = (\mathbf{S}_{j^*} + \boldsymbol{\Delta}\mathbf{S}_{j^*}) + (\boldsymbol{\Theta}_{j^*} + \boldsymbol{\Delta}\boldsymbol{\Theta}_{j^*}) + \boldsymbol{\Xi}_{j^*}, \tag{24}$$

where $\boldsymbol{\Delta}\mathbf{S}_{j^*}$ and $\boldsymbol{\Delta}\boldsymbol{\Theta}_{j^*}$ are the additive corrections to the signal and regular interference due to the tails of the functional series used.

In this case, the estimate $\mathbf{A}^*_{j^*}$ of the vector $\mathbf{A}$ (calculated without the corrections) is given by

$$\mathbf{A}^*_{j^*} = \mathbf{P}^{\mathbf{A}}_{j^*}(\mathbf{S}_{j^*} + \boldsymbol{\Delta}\mathbf{S}_{j^*}) + \mathbf{P}^{\mathbf{A}}_{j^*}(\boldsymbol{\Theta}_{j^*} + \boldsymbol{\Delta}\boldsymbol{\Theta}_{j^*}) + \mathbf{P}^{\mathbf{A}}_{j^*}\boldsymbol{\Xi}_{j^*}, \tag{25}$$

where the optimal estimation matrix $\mathbf{P}^{\mathbf{A}}_{j^*}$ is found using the finite representations (4) and (5) by rejecting the tails.

Therefore, assuming $\boldsymbol{\Xi}_{j^*} = \mathbf{0}$, we have the following representation for the the true value of $\mathbf{A}$:

$$\mathbf{A} = (\mathbf{P}^{\mathbf{A}}_{j^*} + \boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{j^*})(\mathbf{S}_{j^*} + \boldsymbol{\Delta}\mathbf{S}_{j^*}) + (\mathbf{P}^{\mathbf{A}}_{j^*} + \boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{j^*})(\boldsymbol{\Theta}_{j^*} + \boldsymbol{\Delta}\boldsymbol{\Theta}_{j^*}). \tag{26}$$

The average value of the methodological error is

$$\overline{\Delta\mathbf{A}_{j^*}} = \mathrm{M}\left\{\mathbf{A} - \mathbf{A}^*_{j^*}\right\} = \boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{j^*}(\mathbf{S}_{j^*} + \boldsymbol{\Delta}\mathbf{S}_{j^*}) + \boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{j^*}(\boldsymbol{\Theta}_{j^*} + \boldsymbol{\Delta}\boldsymbol{\Theta}_{j^*}), \tag{27}$$

where $\mathrm{M}\{\cdot\}$ denotes the expectation operator under $\mathrm{M}\{\boldsymbol{\Xi}_{j^*}\} = \mathbf{0}$.

Formulas (23)–(27) can be used to select the necessary parameters of the method that minimize the resulting estimation error in each particular case. Necessary and sufficient conditions for the existence of a unique solution of the estimation problem within the method require nonsingularity and some constraints for the ranks of several matrices (by analogy with [18, 19]). In practice, the given conditions are satisfied by rationally choosing the functional bases used and the number of degrees of freedom in the signal and regular interference models as well as by setting up appropriate observation conditions. All these issues concern the planning of a computational experiment and are not considered further, as separate studies are required in each particular case.

To perform a comparative analysis, we consider several methods: $\mathrm{M}_{(1)}$ (XLS), $\mathrm{M}_{(2)}$ (GIUE), and $\mathrm{M}_{(3)}$ (the novel method). Each possible irregular interference can be assigned one of the hypotheses Γ_l, $l = \overline{1, L}$. Obviously, $L = J$ and, for fixed l, by analogy with (1) and (8), the model observation is

$$\mathbf{H}_{0l} = \mathbf{S}_0 + \boldsymbol{\Theta}_0 + \mathbf{D}_{0l} + \boldsymbol{\Xi}_0.$$

The recognition problem based on $\mathrm{M}_{(1)}$ is solved by minimizing the quadratic form $\chi(\mathbf{Z}_{dl})$:

$$(l^*, \mathbf{Z}^*_{dl}) = \arg\min_{l,\mathbf{Z}_{dl}} \chi(\mathbf{Z}_{dl}) = \arg\min_{l,\mathbf{Z}_{dl}} \left[\Delta(l, \mathbf{Z}_{dl})\right]^{\mathrm{T}} (\mathbf{K}_{\boldsymbol{\Xi}_0})^{-1} \left[\Delta(l, \mathbf{Z}_{dl})\right], \quad i \in \{1, 2, 3\}, \tag{28}$$

where $\Delta(l, \mathbf{Z}_{dl}) = \mathbf{H}_0 - \mathbf{H}_{0l}$, $\mathbf{H}_{0l} = \mathbf{H}_{0l}(l, \mathbf{Z}_{dl})$, and $\mathbf{Z}_{dl} = \left[\mathbf{Z}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}_{dl}\right]^{\mathrm{T}} = \left[\mathbf{A}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}_{\theta}, \mathbf{B}^{\mathrm{T}}_{dl}\right]^{\mathrm{T}}$.

One measure of the efficiency of $\mathrm{M}_{(i)}$ is the dimension of the matrices under inversion. Clearly, for a fixed l, $\mathrm{M}_{(1)}$ requires to invert a matrix of order $\rho_{(1)l} = M_s + M_\theta + M_{dl}$, where M_{dl} is the number of nonzero coordinates in the vector $\mathbf{D}_{0l}$. In turn, $\mathrm{M}_{(2)}$ implies the formation of a joint basis matrix for the regular and irregular interferences; for a fixed l, this leads to the inversion of two matrices of order $\rho_{(2)s} = M_s$ and $\rho_{(2)\theta dl} = M_\theta + M_{dl}$, respectively. Regardless of the number of the hypothesis under consideration, $\mathrm{M}_{(3)}$ requires the inversion of two matrices of orders $\rho_{(3)s} = \rho_{(2)s} = M_s$ and $\rho_{(3)\theta} = M_\theta$. Therefore, we can choose the most preferable method by the criterion

$$i^* = \arg\min_i \rho_{(i)},$$

where $i, i^* \in \{1, 2, 3\}$, $\rho_{(1)} = \max\limits_l\{M_s + M_\theta + M_{dl}\}$, $\rho_{(2)} = \max\limits_l\{M_s, M_\theta + M_{dl}\}$, and $\rho_{(3)} = \max\limits_l\{M_s, M_\theta\}$.

As can be seen, in all the cases, $\rho_{(3)} \leqslant \rho_{(1)}$ and $\rho_{(3)} \leqslant \rho_{(2)}$; in the absence of irregular interference (but under regular interference), we have $\rho_{(3)} < \rho_{(1)}$ and $\rho_{(3)} = \rho_{(2)}$. Without regular and irregular interferences, $\rho_{(1)} = \rho_{(2)} = \rho_{(3)}$.

The analysis shows that if the dimension of the regular and irregular interferences is high, the condition $M_\theta + M_{dl} > M_s$ is valid, and the matrices under inversion are ill-conditioned, $\mathrm{M}_{(3)}$ will be much preferable to $\mathrm{M}_{(1)}$ and $\mathrm{M}_{(2)}$ in terms of computational stability. (In this case, we always have $\rho_{(3)} < \rho_{(1)}$ and $\rho_{(3)} < \rho_{(2)}$). For these conditions, along with $\rho_{(1)}$, $\rho_{(2)}$, and $\rho_{(3)}$, the conditionality numbers $\mu_{(1)}$, $\mu_{(2)}$, and $\mu_{(3)}$ must be used, which are the stability characteristics of the methods compared.

To comparatively assess the computational complexity of the methods, we will use the following characteristics: $V_{(i)}^{\Sigma}$ (the total number of nodes of the original grid C_0 that are used to test all hypotheses); $Q_{(i)}^{\Sigma}$ and $T_{(i)}^{\Sigma}$ (the total number of operations (addition and multiplication) and time, respectively, required to realize $\mathrm{M}_{(i)}$.) The following characteristics are adopted for comparative accuracy assessment: $\Delta s_{(i)}$ and $\Delta\theta_{(i)}$ (the resulting estimation errors of the signal and regular interference, respectively). They are given by $\Delta s_{(i)} = \max\limits_t \left| s(t, \mathbf{A}) - s\left(t, \mathbf{A}_{(i)}^*\right)\right|$ and $\Delta\theta_{(i)} = \max\limits_t \left|\theta(t, B_\theta) - \theta\left(t, B_{\theta(i)}^*\right)\right|$.

Omitting intermediate calculations, we present the final results:

— for the characteristic $V_{(i)}^{\Sigma}$,

$$V_{(1)}^{\Sigma} = V_{(2)}^{\Sigma} = LN_0, \quad V_{(3)}^{\Sigma} = \sum_{m=0}^{N_0 - N_{\min}} C_{N_0}^{N_{\min}+m}\left(N_0 - m\right);$$

— for the characteristic $Q_{(i)}^{\Sigma}$,

$$Q_{(1)}^{\Sigma} = \sum_{m=0}^{N_0 - N_{\min}} \left\{ C_{N_0}^{N_{\min}+m}\left[2N_0^3 + 6N_0^2\gamma_m + N_0\left(4\gamma_m^2 - 3\gamma_m\right) + 2\gamma_m^3 - \gamma_m^2\right]\right\},$$

$$\gamma_m = M_s + M_\theta + m,$$

$$Q_{(2)}^{\Sigma} = \sum_{m=0}^{N_0 - N_{\min}} \left\{ C_{N_0}^{N_{\min}+m}\left[2(N_0^3 + N_0^2\varphi_m) + N_0\left(4\beta_m - \varphi_m\right) + 2(\alpha_m - \varphi_m)\right]\right\},$$

$$\beta_m = \eta_m^2 + M_s^2, \quad \alpha_m = \eta_m^3 + M_s^3, \quad \varphi_m = 2\eta_m + 3M_s, \quad \eta_m = M_\theta + m,$$

$$Q_{(3)}^{\Sigma} = \sum_{m=0}^{N_0 - N_{\min}} \left\{ C_{N_0}^{N_{\min}+m}\left[2\left(N_0 - m\right)^3 + 2\left(N_0 - m\right)^2\varphi_0 + \left(N_0 - m\right)\left(4\beta_0 - \varphi_0\right) + 2\alpha_0 - \beta_0\right]\right\},$$

$$\alpha_0 = (M_\theta + M_d)^3 + M_s^3, \quad \beta_0 = M_\theta^2 + M_s^2, \quad \varphi_0 = 2M_\theta + 3M_s, \quad \eta_0 = M_\theta.$$

According to the analysis of these expressions, a significant computational gain $\mathrm{M}_{(3)}$ over $\mathrm{M}_{(1)}$ and $\mathrm{M}_{(2)}$ can be achieved by processing the data on reduced grids of smaller volume (compared to the original grid C_0). This gain also grows with increasing the dimension of the irregular interference.

The calculation of the characteristics $T_{(i)}^{\Sigma}$ and $\Delta\theta_{(i)}^{\Sigma}$ depends on both the initial data of the problem and the capabilities of the computing environment used.

To test the family of hypotheses, L parallel data processing channels can be organized to perform a large amount of vector-matrix computations. However, significant optimization of such computations is possible since all vectors and matrices used in different channels are obtained by

an appropriate reduction of the vectors and matrices on the basic grid C_0. In other words, most operations in different channels are repeated.

The novel method can be easily extended to other problems (e.g., detection and discrimination) related to signal recognition (including those with nonlinear parameters [19]) and solved within the theory of hypotheses. In this case, the number of parallel data processing channels may increase significantly, as the reduced grids should be constructed directly for each hypothesis and each node of the definitional domain of a nonlinear parameter. Here, the most effective approaches to deal with irregular interferences are the full (20) and partial (21) cover methods, e.g., step-by-step (22).

7. AN ILLUSTRATIVE EXAMPLE

We apply the novel method to the single-position passive ranging of a radiating target with partially known motion parameters only by energy measurements (the energy method [24, 25]). Consider a target moving in the direction toward a rangefinder so that the slant range varies according to the law $R(t) = R_0 + R_0^{(1)}t + 2^{-1}R_0^{(2)}t^2$, where R_0, $R_0^{(1)}$, and $R_0^{(2)}$ are the initial range, radial velocity, and radial acceleration, respectively, before the time instant $t = 0$. Let the measured signal (energy parameter) be $s(t) = 1 - q_0^{-1}(t)$, where $q_0(t) = \left[p_0^{-1}p(t)\right]^{1/2}$ and $p(t)$ is the power of the electromagnetic wave at the rangefinder's input, $p_0 = p(0)$.

The well-known passive location equation for a stationary channel has the form

$$p = \zeta_0 R^{-2},$$

where $p = p(t)$, $R = R(t)$, and $\zeta_0 = \mathrm{const}$ is a generalized coefficient relating power and range.

Such a simplified model is encountered in practice under several constraints on the observation conditions of a radiating target [24, 25]. In the case considered here, it will demonstrate well the effectiveness of the novel method without resorting to complex ranging algorithms for the nonstationary case.

The required range is given by

$$R(t) = s^{-1}(t)D_0(t)[1 - s(t)], \quad t > 0, \tag{29}$$

where $D_0(t) = \left(R_0^{(1)}t + 2^{-1}R_0^{(2)}t^2\right)$ means the distance traveled by the target in the time t.

Formula (29) serves to calculate the range based on estimates of the parameter $s(t) = 1 - q_0^{-1}(t)$. Obviously, $s(t)$ can be represented as

$$s(t) = -\left(R_0^{-1}R_0^{(1)}t + 2^{-1}R_0^{-1}R_0^{(2)}t^2\right), \quad t > 0.$$

In view of (4), it follows that $s(t) = s(t, a_1, a_2) = a_1\psi_1(t) + a_2\psi_2(t)$, i.e., $M_s = 2$, $a_1 = -R_0^{-1}R_0^{(1)}$, $a_2 = -2^{-1}R_0^{-1}R_0^{(2)}$, $\psi_1(t) = t$, and $\psi_2(t) = t^2$. Thus, after estimating the coefficients a_1 and a_2 from measurements of the energy parameter $s(t, a_1, a_2)$, we can find the desired range by formula (29). Next, $\mathrm{M}_{(3)}$ will be used for estimation whereas $\mathrm{M}_{(1)}$ and $\mathrm{M}_{(2)}$ for comparative analysis. Note that all time parameters below are measured in seconds, distances in meters, radial velocity in meters per second, radial acceleration in meters per second squared, and the sample clogging factor in percent. The energy parameter is a dimensionless quantity.

For the experiment, we took $R_0 = 3 \times 10^4$, $R_0^{(1)} = 2 \times 10^3$, $R_0^{(2)} = 2 \times 10^2$, $T = 10$, $N_0 = 10$, $t_n - t_{n-1} = \Delta t = 1$, $t_1 = 1$, $t_{10} = 10$, $M_d = K_{\max} = 2$, $N_{\min} = 6$, $k_d = 20$, and $M_\theta = 0$. (In other words, the influence of the regular interference was neglected.) The measurement noise was modeled using a random number generator of the Gaussian distribution with the diagonal matrix $\mathbf{K}_{\boldsymbol{\Xi}_0}$ with the nonzero element $\sigma_0^2 = 10^{-10}$. For the cases when $N \geqslant N_{\min}$, and in the absence of an

irregular interference, the average root-mean-square error of range estimation (over an ensemble of 500 realizations) by any method must satisfy the condition $\overline{\Delta R} \leqslant 5 \times 10^2$.

The computations were carried out in MATLAB ver. R20119b on a PC with a 2.6 GHz quad-core CPU and 8 GB DDR3 RAM.

Table 1 shows the exact values of the energy parameter on the grid C_0.

Table 1

t_n	1	2	3	4	5	6	7	8	9	10
s_n	0.067	0.147	0.230	0.320	0.417	0.520	0.630	0.747	0.870	1

The following intermediate values were calculated for the initial data: $L = J = 386$, $\gamma_m = 2 + m$, $\beta_m = m^2 + 4$, $\alpha_m = m^3 + 8$, $\varphi_m = 2m + 6$, and $\eta_m = m$, $0 \leqslant m \leqslant 4$.

The final results of the computational experiment (after rounding) are combined in Table 2.

Table 2

Characteristics / $\mathrm{M}_{(i)}$	$\rho_{(i)}$	$\mu_{(i)}$	$V^{\Sigma}_{(i)}$	$Q^{\Sigma}_{(i)}$	$T^{\Sigma}_{(i)}$	$\overline{\Delta R_{(i)}}$
$\mathrm{M}_{(1)}$	6	544	3.9×10^3	2.5×10^6	1.3×10^{-3}	1.3×10^3
$\mathrm{M}_{(2)}$	4	148	3.9×10^3	1.9×10^6	9.9×10^{-4}	9.3×10^2
$\mathrm{M}_{(3)}$	2	103	2.6×10^3	4.7×10^5	2.4×10^{-4}	8.2×10^2

According to the data of this table, XLS and GIUE lose to the novel method in all their characteristics even under a small dimension of the matrices inverted. Using the result of [25], we can consider the nonstationary case ($\zeta_0 \neq \mathrm{const}$) with not only irregular interferences but also regular ones. In such conditions, the vector of estimated parameters expands, which requires increasing the volume of the original grid C_0. The effect achieved grows significantly with increasing the dimension of the problem and this volume.

The ranging method considered in this example, in combination with the algorithm proposed for estimating the energy parameter (based on the novel method), can be used both independently and with other known passive location methods. The former option can be appropriate when the energy channel is the only source of reliable information in the rangefinder (e.g., under either the single-position energy design of the rangefinder or the structural degradation of the two- or multi-position system: failures of separate positions, the abnormal operation of angular measurement channels, communication lines, etc.). The capability to solve the ranging problem autonomously with both regular and irregular interferences can find a wide application in various fields of passive location systems.

The latter option allows treating the passive-energy method as an alternative method when constructing a complex algorithm for the reliable operation of an intelligent system under various, even unfavorable, conditions. In addition to the novel method, such an algorithm can be based on the angular-measuring energy method and passive methods widely used in practice (difference ranging, triangulation, etc. as well as their different modifications).

Energy measurements traditionally do not belong to the class of reliable measurements. Hence, the novel signal recognition method under regular and irregular interferences may improve the operating efficiency of energy channels with the required accuracy of amplitude and power measurements.

8. CONCLUSIONS

The signal recognition method developed in this paper realizes the principles of continuity, invariance, multiplication, and ranking using a family of reduced grids. It can be effectively combined

with algorithms for orthogonal decomposition and solving ill-posed problems [27–29]. The possibility of decomposing computational procedures, reducing the dimension of matrices inverted, and decreasing the amount of computations allows solving more efficiently a whole range of applications-oriented problems with measurement processing in various fields. The signal recognition algorithms under regular and irregular interferences are not difficult to implement in special-purpose multichannel computers for systems operating in real time.

Compact analytical expressions have been derived to select in advance, for a particular application, the necessary models of signals and interferences as well as the values of their parameters to achieve the potential capabilities of the novel method. All computational procedures in each channel are reduced to the simplest mathematical operations over vectors and matrices; it is possible to combine this method with traditional approaches to solving applied problems on the optimal and quasi-optimal processing of measurements.

Advances in parallel computing give hope that any problems related to signal recognition can soon be solved using the novel method.

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STOCHASTIC SYSTEMS

Multivariate Continuous Distributions and Copulas Generating Nontransitive Tuples of Dependent Random Variables

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Abstract—This paper continues the author's research series on the nontransitivity phenomenon of the stochastic precedence relation in probability theory. Based on the Condorcet paradox, examples of trivariate continuous distributions and copulas generating nontransitive tuples of dependent random variables are constructed. Limit theorems for multivariate mixtures are proven.

Keywords: nontransitivity, stochastic precedence, copulas, multivariate mixtures, limit theorems

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1. INTRODUCTION

In theory and practice, various superiority relations between objects often have the property of transitivity: if A is superior to B and B is superior to C, then A is superior to C. However, it happens the other way around. For example, in the classical rock–paper–scissors game (also known as Rochambeau, Roshambo, Ro-sham-bo, Bato Bato Pik, and Jak-en-poy in some countries), rock beats scissors, scissors beats paper but, at the same time, paper beats rock.

Several articles by A.N. Poddiakov, particularly [1, 2], were devoted to various aspects and numerous examples of the nontransitivity of superiority relations in nature, technology, and society.

Among recent publications, note the interesting studies of nontransitivity in biology, from the life of ants [3] and soil bacteria [4].

Below, we will address the nontransitivity of the stochastic precedence relation between random variables, i.e., when one variable is more likely to be smaller than the other. This relation was applied in statistical analysis problems [5, 6]. Methods for comparing probability distributions based on statistical data (histograms) were the subject of the recent work [7], where the author considered various stochastic orders.

H. Steinhaus and S. Trybula first mentioned the problem of nontransitivity from the probabilistic point of view [8, 9], with the strength of materials as an application. Let the strength of iron bars from three different factories be compared pairwise in laboratory conditions. Theoretically, there may be a situation that bars from the first factory are worse than those from the second (they have lower strength more often) and, simultaneously, bars from the second factory are worse than those from the third and bars from the third factory are worse than those from the first.

Recently, such a phenomenon has been discovered in medical statistics when comparing life expectancy in some groups of people [10].

The issue of nontransitivity has become widespread on an example of *nontransitive dice*. In such dice sets, the numbers on faces are appropriately drawn to create the nontransitive relations of the corresponding random variables. Nontransitive dice were popularized by M. Gardner [11, 12]; see the extensive literature in this field.

In this paper, we continue the studies of nontransitivity initiated in [13–17]. (The interested reader can find more details about the range of problems, historical background, related literature, and applications therein.) In contrast to the earlier publications focused on the case of independent random variables, below we consider dependent random variables.

In this sense, a classical example is the famous Condorcet paradox of voting. Let us present its simplest formulation. There are three candidates for some post and three voters who have individual preferences for candidates (in points) according to the following row vectors:

$$\begin{gathered}(1,2,3),\\(3,1,2),\\(2,3,1).\end{gathered} \tag{1}$$

Then, when choosing between the first and second candidates, the second will win. Indeed, the first and third voters have higher points for the second candidate than for the first ($1<2$ and $2<3$, respectively), and the second voter has lower points ($3>1$); as a result, the second candidate will win by two votes out of three. Similarly, when choosing between the second and third candidates, the third will win; between the first and third, the first.

Such situations often arise when experts assess, e.g., the quality of goods. In this case, difficulties are because many characteristics cannot be directly measured in quantitative terms (by technical means) and are assessed intuitively in conventional units (points). In the case of food products, an important role is played by organoleptic indicators: taste, color, smell, etc. It is necessary to exclude nontransitive subsets in expert measurements, and various methods are used for this purpose [18–20].

As is believed, increasing the number of experts (interviewees) solves the problem; however, sometimes this approach does not work. For example, a mass survey on the choice of a territory development project (green zone, family recreation park, business center) revealed nontransitivity in answers [21]. It is assumed that people assess projects considering certain factors (environmental, social, financial), but the weights (importance) of these factors vary for different people. If the joint distribution of weights were similar to those studied below, this would explain the results. In addition, a more detailed analysis of the data using computer statistical methods (cluster analysis, factor analysis, etc.) would show the division of people into groups based on their views, even if these groups are not formally positioned or realized by the participants.

Note also the works [22, 23].

The properties of decision-making methods in multicriteria individual choice problems were extensively studied in [24]. Under nontransitivity conditions, a "best" alternative can always be found when comparing some current alternative with others, which leads to an infinite Markov random walk in formal terms. In the modern development of artificial intelligence, automatic decision systems, and robots, it is necessary to consider the nontransitivity phenomenon. A human can realize such a situation and make a willed decision whereas a machine can get stuck.

2. DEFINITIONS AND KNOWN RESULTS

We begin with strict definitions.

Definition 1. A relation $\prec$ is said to be nontransitive if, for any objects A, B, and C, the relations $A\prec B$ and $B\prec C$ DO NOT imply the relation $A\prec C$ but, on the contrary, it is possible that $C\prec A$.

Definition 2. A set of objects A, B, and C is said to be nontransitive if the nontransitivity of the superiority relation is realized on it, i.e., $A \prec B \prec C \prec A$ (or in the reverse order).

Definition 3. X stochastically precedes Y $(X \prec Y)$ if

$$\mathbf{E}\,\mathrm{sgn}(Y - X) > 0$$

or

$$\mathbf{P}(X < Y) > \mathbf{P}(X > Y).$$

If $\mathbf{P}(X = Y) = 0$ (e.g., the random variables are independent and continuous or have nonoverlapping value sets), then $X \prec Y$ is equivalent to

$$\mathbf{P}(X < Y) > \frac{1}{2}.$$

Note that if random variables are independent and continuous, and obey an identical distribution, then the equality $\mathbf{P}(X < Y) = 1/2$ always holds. For continuous random variables, this equality can be violated due to different distributions or dependence; both cases require separate consideration.

Well, let random variables X_1, X_2, and X_3 be such that

$$\mathbf{P}(X_1 < X_2) > \frac{1}{2}, \quad \mathbf{P}(X_2 < X_3) > \frac{1}{2}, \quad \mathbf{P}(X_3 < X_1) > \frac{1}{2},$$

then $X_1 \prec X_2$, $X_2 \prec X_3$, but $X_3 \prec X_1$. Thus, a situation of nontransitivity arises when

$$P_{X_1X_2X_3} = \min\{\mathbf{P}(X_1 < X_2), \mathbf{P}(X_2 < X_3), \mathbf{P}(X_3 < X_1)\} > \frac{1}{2},$$

and not only the fact but also strength of nontransitivity is of interest. The latter can be measured by the value of $P_{X_1X_2X_3}$.

According to [9], for independent random variables,

$$\max_{X_1,X_2,X_3} P_{X_1X_2X_3} = \frac{\sqrt{5}-1}{2} \approx 0.618. \tag{2}$$

A combination of $n \geqslant 3$ independent random variables was studied in [26, 27]. As was subsequently shown in [28] (and later reproved geometrically in [29]), the maximum of the probabilities

$$P_{X_1\ldots X_n} = \min\{\mathbf{P}(X_1 < X_2), \ldots, \mathbf{P}(X_{n-1} < X_n), \mathbf{P}(X_n < X_1)\}$$

is

$$\max_{X_1,\ldots,X_n} P_{X_1\ldots X_n} = 1 - \left(4\cos^2\frac{\pi}{n+2}\right)^{-1}, \quad n \geqslant 3.$$

This result coincides with (2) for $n = 3$ since

$$\cos\frac{\pi}{5} = \frac{1+\sqrt{5}}{4}.$$

In the case of dependent variables [27],

$$\max_{X_1,\ldots,X_n} P_{X_1\ldots X_n} = \frac{n-1}{n}, \quad n \geqslant 3;$$

particularly for $n = 3$, we obtain $P_{X_1X_2X_3} = 2/3$.

For three dependent random variables, a game interpretation can be as follows: a random vector (X_1, X_2, X_3) is played after two players have sequentially chosen a component of this vector. The winner is the player whose component takes a higher value. Nontransitivity means that whatever component the first player chooses, the second can choose his/her component so that the probability of win be $P_{X_1X_2X_3} > 1/2$, i.e., get an advantage.

Consider a probabilistic-statistical modification of the Condorcet paradox corresponding to an arbitrarily large number of voters, each having one of the three opinions about the candidates described by (1) equiprobably.

Let the set of dependent random variables (X, Y, Z) take values from (1) with probability 1/3 each. Then

$$\mathbf{P}(X < Y) = \mathbf{P}(Y < Z) = \mathbf{P}(Z < X) = \frac{2}{3}.$$

In this case, each of the variables is uniformly distributed on the set $\{1, 2, 3\}$.

The question arises: how can we pass from a discrete distribution to its continuous counterpart? Consider different examples.

Recall the concept of a copula [30].

Definition 4. An (m-dimensional) copula C is a multivariate distribution function on $[0, 1]^m$ with the uniform partial (marginal) distributions.

A copula of a distribution F in R^m is a copula C that satisfies the expression

$$F(x_1, \ldots, x_m) = C(F_1(x_1), \ldots, F_m(x_m)),$$

where $F_1, \ldots, F_m$ are marginal distribution functions.

Such a representation exists by Sklar's theorem and is unique in the case of continuous marginal distributions. From this point onwards, we assume continuity.

If $X_1, \ldots, X_m$ are given and $U_i = F_i(X_i)$, $1 \leqslant i \leqslant n$, then all U_i are uniformly distributed on $[0, 1]$, with the joint distribution C, i.e.,

$$\mathbf{P}(U_1 \leqslant u_1, \ldots, U_m \leqslant u_m) = C(u_1, \ldots, u_m).$$

In the continuous case with all X_i identically distributed, we have

$$\mathbf{P}(X_i < X_j) = \mathbf{P}(U_i < U_j), \quad i \neq j.$$

3. THREE-DIMENSIONAL CONTINUOUS COPULAS

Consider first some natural generalizations of the Condorcet example with uniform marginal distributions on $[0, 1]$. (The resulting trivariate distributions can be used as copulas.)

Example 1. Let

$$X = T, \quad Y = \left\{T + \frac{1}{3}\right\}, \quad Z = \left\{T + \frac{2}{3}\right\},$$

where $\{.\}$ denotes the fractional part of a number and T is uniformly distributed on $[0, 1]$. Then X, Y, and Z obey the uniform distribution on $[0, 1]$ and

$$\mathbf{P}(X < Y) = \mathbf{P}(Y < Z) = \mathbf{P}(Z < X) = \frac{2}{3}. \tag{3}$$

The graphs of X, Y, and Z depending on T are shown in Fig. 1. The probabilities correspond to the fractions of the closed interval $[0, 1]$ on which the corresponding inequalities are true.

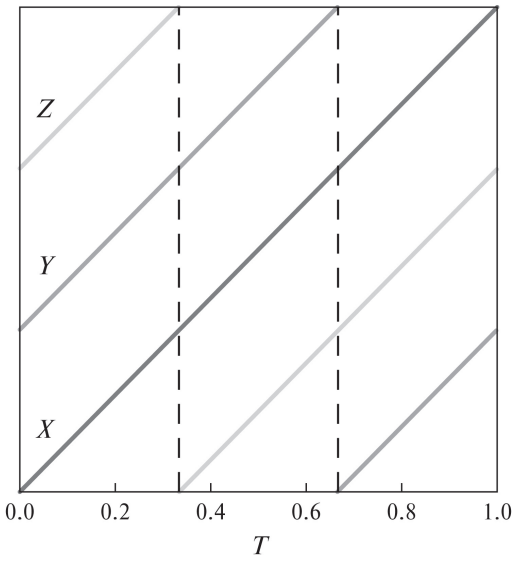

Fig. 1. Example 1: X, Y, and Z depending on T.

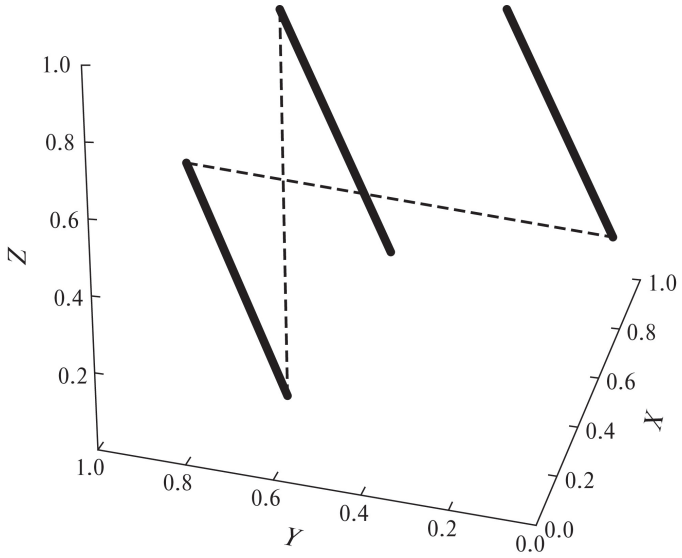

Fig. 2. Example 1: the distribution in the unit cube.

In this case, the distribution of the vector turns out to be continuous, albeit not absolutely continuous (having a density) but singular (centered on a manifold of lower dimension, i.e., three segments) in the unit cube; see Fig. 2.

Now we proceed to absolutely continuous distributions. Let $p(x, y, z)$, where $x, y, z \in [0, 1]$, denote the joint distribution density of random variables X, Y, and Z.

Example 2. Let $p(x, y, z) = 2$ if $x < y < z$ or $z < x < y$ or $y < z < x$, and $p(x, y, z) = 0$ otherwise. Then X, Y, and Z are also uniformly distributed on $[0, 1]$ and formula (3) is valid. The above density can be represented as

$$p(x, y, z) = 1 - \operatorname{sgn}\{(y - x)(z - y)(x - z)\}. \tag{4}$$

This distribution is centered in three pyramids inside the unit cube (Fig. 3).

In this case, the trivariate distribution has a density, but this density is discontinuous. However, the form of (4) suggests to pass to polynomial density.

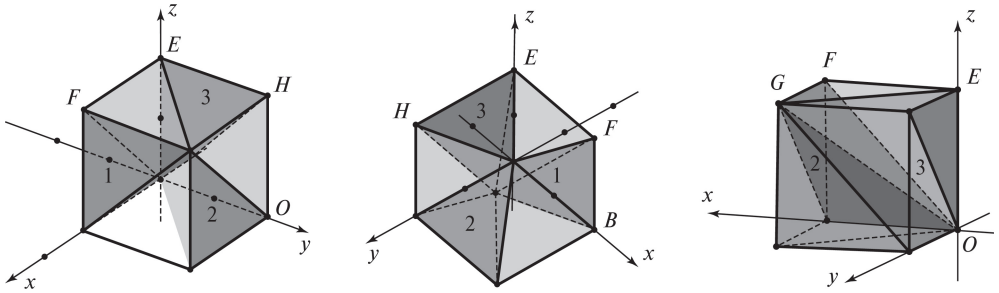

Fig. 3. Example 2: the distribution in the unit cube.

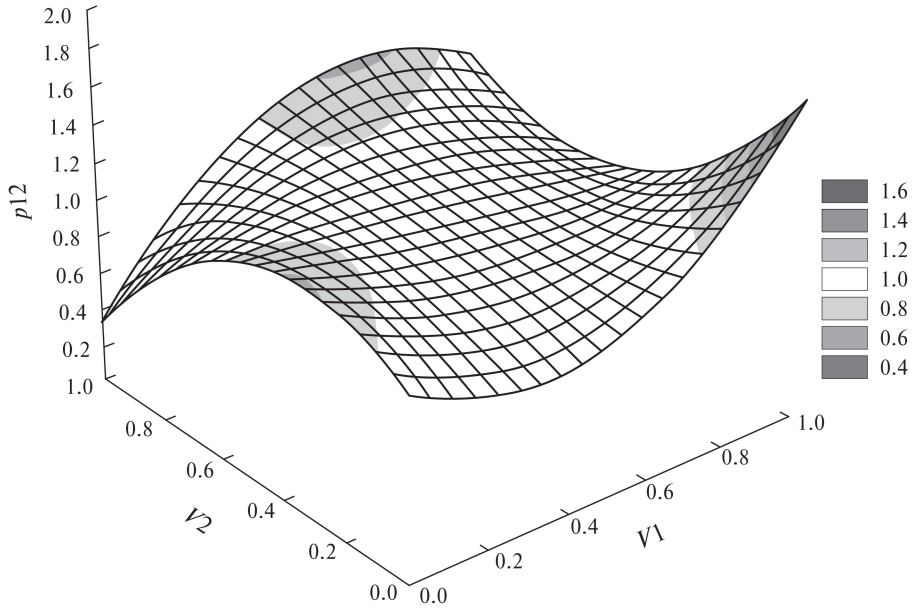

Fig. 4. Example 3: the bivariate distribution density.

Example 3. Let

$$p(x,y,z) = 1 - K(y-x)(z-y)(x-z). \tag{5}$$

As can be verified, in this case, the marginal distributions remain uniform on $[0,1]$. The question is to choose a constant K so that the density be nonnegative.

We have

$$\max_{[0,1]^3}\{(y-x)(z-y)(x-z)\} = \frac{1}{4},$$

and this maximum is achieved at the points $(1/2,1,0)$, $(0,1/2,1)$, and $(1,0,1/2)$.

Hence, the maximal admissible value is $K=4$; with

$$p(x,y,z) = 1 - 4(y-x)(z-y)(x-z),$$

integration yields the joint distribution function

$$F(x,y,z) = xyz\left(1 + \frac{2}{3}(xy(y-x) + yz(z-y) + xz(x-z))\right).$$

The joint density of X and Y takes the form

$$p_{12}(x,y) = 1 + \frac{2}{3}(y-x)(2-3x-3y+6xy)$$

(see Fig. 4); therefore,

$$\mathbf{P}(X<Y) = \mathbf{P}(Y<Z) = \mathbf{P}(Z<X) = \frac{47}{90} = 0.522\ldots.$$

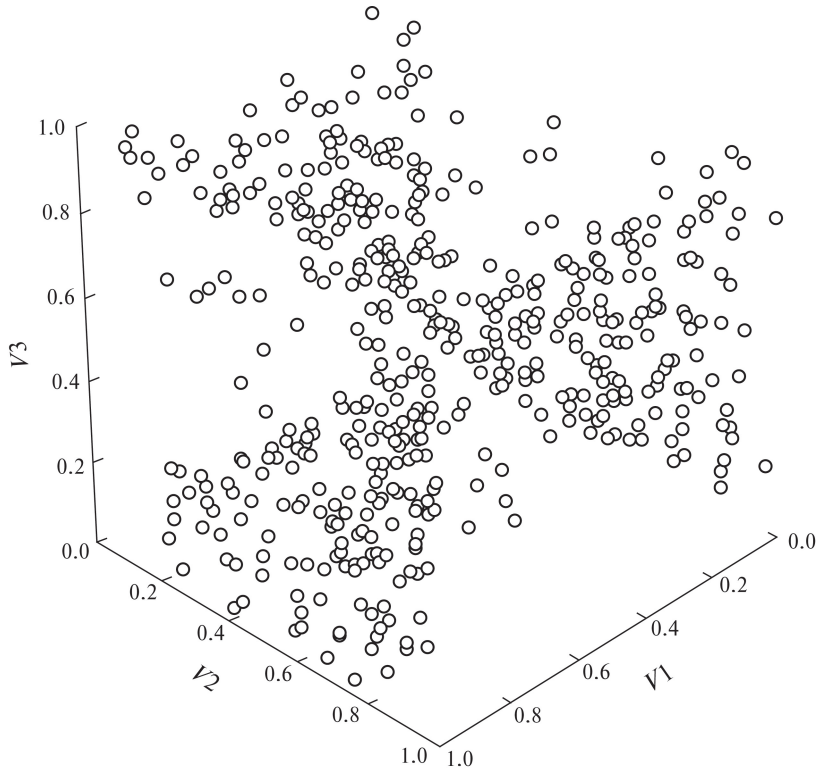

Fig. 5. Example 4: the simulation result.

Example 4. Consider an intermediate density between (4) and (5):

$$p(x,y,z) = 1 - \sqrt[k]{4(y-x)(z-y)(x-z)}, \tag{6}$$

where $k \geqslant 3$ is an odd natural number. Obviously, as $k \to \infty$, we have convergence to the situation described in Example 2. The probability P_{XYZ} was estimated by Monte Carlo simulations with 10^6 points; see the table below.

k	P_{XYZ}
3	0.569
5	0.595
7	0.610
9	0.621
11	0.628

Figure 5 presents the simulation result for $k = 11$ and 10^3 points. The convergence to the three pyramids from Example 2 can be observed.

4. MULTIVARIATE MIXTURES

Now consider another approach, with noise introduced into discrete data, as in [31] as applied to nontransitive Efron dice (independent random variables).

Example 5. Let a random vector (X_1, X_2, X_3) be formed as follows: independent Gaussian variables with zero mean and a variance $\sigma^2 > 0$ are added coordinate-wise to the values of (1), taken with equal probabilities 1/3.

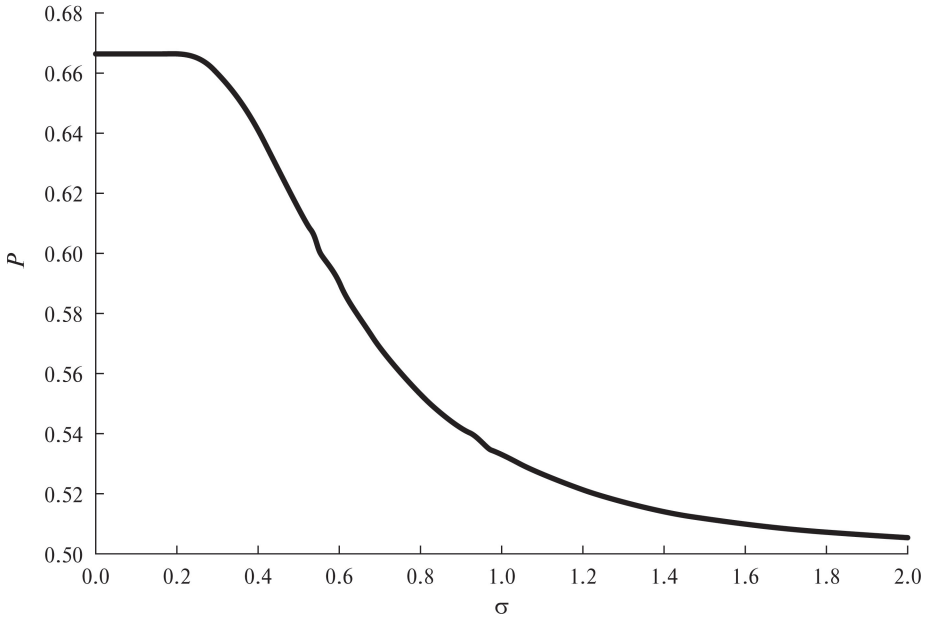

Fig. 6. Example 5: $P_{X_1X_2X_3}$ depending on σ.

Proposition 1. *In Example 5,*

$$P_{X_1X_2X_3} = \mathbf{P}(X_1 < X_2) = \mathbf{P}(X_2 < X_3) = \mathbf{P}(X_3 < X_1) = \frac{1}{3}\left(2\Phi\left(\frac{1}{\sigma\sqrt{2}}\right) + \Phi\left(-\frac{\sqrt{2}}{\sigma}\right)\right) > \frac{1}{2},$$

where $P_{X_1X_2X_3} \to 2/3$ *as* $\sigma \to 0$ *and* $P_{X_1X_2X_3} \to 1/2$ *as* $\sigma \to \infty$.

The proof is given in the Appendix.

Here, nontransitivity is observed for *any* $\sigma^2 > 0$.

The graph of the probability $P_{X_1X_2X_3}$ depending on σ is shown in Fig. 6.

The following question arises: as $\sigma \to 0$, the distribution converges to a discrete (three-point) distribution, and what happens to the copula?

Proposition 2. *In Example 5, as* $\sigma \to 0$, *the distribution of the copula converges to the uniform distribution on the union of cubes*

$$[0,1/3] \times [1/3,2/3] \times [2/3,1],$$
$$[2/3,1] \times [0,1/3] \times [1/3,2/3],$$
$$[1/3,2/3] \times [2/3,1] \times [0,1/3],$$

with the density $p(x_1,x_2,x_3) = 9$ *on this set and* 0 *otherwise.*

The proof is given in the Appendix.

With the standard uniform distribution function denoted by

$$R(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leqslant x < 1 \\ 1, & x \geqslant 1, \end{cases} \tag{7}$$

the limit copula from Proposition 2 can be represented as

$$C_0(u_1,u_2,u_3) = \frac{1}{3}\Big(R(3u_1)R(3u_2-1)R(3u_3-2)$$
$$+ R(3u_1-2)R(3u_2)R(3u_3-1) + R(3u_1-1)R(3u_2-2)R(3u_3)\Big).$$

For the limit copula, we have

$$\mathbf{P}(U_1 < U_2) = \mathbf{P}(U_2 < U_3) = \mathbf{P}(U_3 < U_1) = 2/3.$$

Let us proceed to the more general case.

Recall that a (discrete) *multivariate mixture* is a distribution of a random vector that takes, with some probabilities (weights), values of random vectors with given distributions. (Moreover, the vector is chosen independently of the values.)

A mixture of Gaussian vectors is called a *Gaussian* mixture. Thus, Example 5 illustrates a Gaussian mixture.

Let a random vector (X_1, X_2, X_3) be distributed as a mixture, with weights p_i, $i = 1, 2, 3$, of random vectors

$$(a_{i1}, a_{i2}, a_{i3}) + \lambda(\eta_{i1}, \eta_{i2}, \eta_{i3}),$$

where $A = (a_{ij})_{i,j=1}^n$ is some matrix, $(\eta_{i1}, \eta_{i2}, \eta_{i3})$, $i = 1, 2, 3$, are random vectors, and $\lambda > 0$.

In addition, let $0 < p_i < 1/2$, $i = 1, 2, 3$.

Theorem 1. *Assume that $\eta_{i1} - \eta_{i2}$, $\eta_{i2} - \eta_{i3}$, and $\eta_{i3} - \eta_{i1}$ are continuous for all $i = 1, 2, 3$ and a_{ik} in the rows of A have the same order as in (1). Then there exists a number $\lambda_1 > 0$ such that nontransitivity holds for $\lambda < \lambda_1$, i.e., $P_{X_1X_2X_3} > 1/2$, and*

$$\lim_{\lambda \to 0} P_{X_1X_2X_3} = 1 - \max\{p_1, p_2, p_3\}.$$

The proof is given in the Appendix.

Theorem 2. *Assume that $\eta_{ij} = \eta_j$, where η_j are independent and obey an identical continuous distribution, $i, j = 1, 2, 3$, and a_{ik} in the columns of A have the same order as in (1). Then there exists a number $\lambda_2 > 0$ such that the nontransitivity of (U_1, U_2, U_3) holds for $\lambda < \lambda_2$, i.e., $P_{U_1U_2U_3} > 1/2$; moreover, as $\lambda \to 0$, the distribution of the copula converges to a mixture of uniform distributions with weights p_i, $i = 1, 2, 3$, on the parallelepipeds*

$$\begin{aligned}
&[0, p_1] \times [p_1, p_1 + p_2] \times [p_1 + p_2, 1],\\
&[p_1 + p_2, 1] \times [0, p_1] \times [p_1, p_1 + p_2],\\
&[p_1, p_1 + p_2] \times [p_1 + p_2, 1] \times [0, p_1],
\end{aligned}$$

so that

$$\lim_{\lambda \to 0} P_{U_1U_2U_3} = 1 - \max\{p_1, p_2, p_3\}.$$

The proof is given in the Appendix.

In this case, using the notation (7), the limit copula can be written as

$$\begin{aligned}
&C_0(u_1, u_2, u_3)\\
&= p_1 R\left(\frac{u_1}{p_1}\right) R\left(\frac{u_2 - p_1}{p_2}\right) R\left(\frac{u_3 - (p_1 + p_2)}{p_3}\right)\\
&+ p_2 R\left(\frac{u_1 - (p_1 + p_2)}{p_3}\right) R\left(\frac{u_2}{p_1}\right) R\left(\frac{u_3 - p_1}{p_2}\right)\\
&+ p_3 R\left(\frac{u_1 - p_1}{p_2}\right) R\left(\frac{u_2 - (p_1 + p_2)}{p_3}\right) R\left(\frac{u_3}{p_1}\right).
\end{aligned}$$

5. CONCLUSIONS

In this paper, we have established new results regarding the nontransitivity of the stochastic precedence relation in probability theory. Based on the Condorcet paradox, several examples of trivariate continuous distributions and copulas generating nontransitive sets of dependent random variables have been constructed. The cases of singular distributions and distributions with a discontinuous density or polynomial density, as well as the case of multivariate mixtures, have been considered. Limit theorems for multivariate mixtures have been proven. Such multivariate continuous distributions can describe and explain the manifestations of nontransitivity in psychology, economics, biology, and other disciplines. This phenomenon should be taken into account when designing automatic decision systems, artificial intelligence systems, robots, etc.

APPENDIX

Proof of Proposition 1. We use the representation

$$X_i = X_i^0 + \varepsilon_i, \quad i = 1, 2, 3,$$

where the vector (X_1^0, X_2^0, X_3^0) takes the values (1) equiprobably and the random variables $\varepsilon_i \sim N(0, \sigma^2)$ are independent.

Let us calculate the probability $\mathbf{P}(X_1 < X_2)$; the rest can be found by analogy. We have

$$\mathbf{P}(X_1 < X_2) = \mathbf{P}(X_1^0 + \varepsilon_1 < X_2^0 + \varepsilon_2) = \mathbf{P}(\varepsilon_1 - \varepsilon_2 < X_2^0 - X_1^0).$$

With the notation $\varepsilon = \varepsilon_1 - \varepsilon_2$, it follows that $\varepsilon \sim N(0, 2\sigma^2)$ and

$$\begin{aligned}\mathbf{P}(X_1 < X_2) &= \frac{1}{3}\mathbf{P}(\varepsilon < 2-1) + \frac{1}{3}\mathbf{P}(\varepsilon < 1-3) + \frac{1}{3}\mathbf{P}(\varepsilon < 3-2)\\ &= \frac{2}{3}\mathbf{P}(\varepsilon < 1) + \frac{1}{3}\mathbf{P}(\varepsilon < -2) = \frac{1}{3}\left(2\Phi\left(\frac{1}{\sigma\sqrt{2}}\right) + \Phi\left(-\frac{\sqrt{2}}{\sigma}\right)\right).\end{aligned}$$

Proof of Proposition 2. Let ν denote the number of a vector from (1) chosen as the value (X_1^0, X_2^0, X_3^0). Under the condition $\nu = 1$, we have

$$\begin{aligned}U_1 &= \frac{1}{3}\left(\Phi\left(\frac{1+\varepsilon_1-1}{\sigma}\right) + \Phi\left(\frac{1+\varepsilon_1-2}{\sigma}\right) + \Phi\left(\frac{1+\varepsilon_1-3}{\sigma}\right)\right),\\ U_2 &= \frac{1}{3}\left(\Phi\left(\frac{2+\varepsilon_2-1}{\sigma}\right) + \Phi\left(\frac{2+\varepsilon_2-2}{\sigma}\right) + \Phi\left(\frac{2+\varepsilon_2-3}{\sigma}\right)\right),\\ U_3 &= \frac{1}{3}\left(\Phi\left(\frac{3+\varepsilon_3-1}{\sigma}\right) + \Phi\left(\frac{3+\varepsilon_3-2}{\sigma}\right) + \Phi\left(\frac{3+\varepsilon_3-3}{\sigma}\right)\right).\end{aligned}$$

Note that $\varepsilon_i/\sigma \sim N(0,1)$. We define the random variables $U_i^0 = \Phi(\varepsilon_i/\sigma)$; they are uniformly distributed on $[0,1]$ and independent. In addition,

$$\Phi\left(\frac{\varepsilon_i + c}{\sigma}\right) \xrightarrow{P} \begin{cases} 0, & c < 0\\ U_i^0, & c = 0\\ 1, & c > 0,\end{cases} \qquad \sigma \to 0.$$

Consequently,

$$(U_1, U_2, U_3) \xrightarrow{P} \left(\frac{1}{3}U_1^0, \frac{1}{3} + \frac{1}{3}U_2^0, \frac{2}{3} + \frac{1}{3}U_3^0\right), \quad \sigma \to 0,$$

i.e., we obtain the uniform distribution on the cube $[0,1/3]\times[1/3,2/3]\times[2/3,1]$. The cases $\nu=2,3$ are considered by analogy.

Proof of Theorem 1. We use the representation

$$(X_1,X_2,X_3)=(a_{\nu1},a_{\nu2},a_{\nu3})+\lambda(\eta_{\nu1},\eta_{\nu2},\eta_{\nu3}),$$

where ν takes values 1, 2, and 3 with probabilities p_1, p_2, and p_3, respectively, independently of η_{ij}, $i,j=1,2,3$.

Let G_i, $i=1,2,3$, denote the distributions of $\eta_{i1}-\eta_{i2}$.

We have

$$\mathbf{P}(X_1<X_2)=\sum_{i=1}^{3}p_iG_i\left(\frac{a_{i2}-a_{i1}}{\lambda}\right)\to p_1+p_3=1-p_2>\frac{1}{2},\quad \lambda\to0,$$

since $a_{12}>a_{11}$, $a_{22}<a_{21}$, and $a_{32}>a_{31}$. Similarly,

$$\mathbf{P}(X_2<X_3)\to1-p_3,\quad \mathbf{P}(X_3<X_1)\to1-p_1,\quad \sigma\to0,$$

and

$$P_{X_1X_2X_3}\to\min\{1-p_1,1-p_2,1-p_3\}=1-\max\{p_1,p_2,p_3\}>\frac{1}{2},\quad \sigma\to0.$$

Proof of Theorem 2. This result is established by analogy with Proposition 2. We use the representation

$$(X_1,X_2,X_3)=(a_{\nu1},a_{\nu2},a_{\nu3})+\lambda(\eta_1,\eta_2,\eta_3),$$

where the random variables η_j, $j=1,2,3$, are continuous, independent, and identically distributed. Let G denote their distribution.

Under the condition $\nu=1$, we have

$$U_j=\sum_{i=1}^{3}p_iG\left(\frac{a_{1j}-a_{ij}}{\lambda}+\eta_j\right),\quad j=1,2,3,$$

and consequently,

$$(U_1,U_2,U_3)\xrightarrow{d}\left(p_1U_0^1,p_1+p_2U_2^0,p_1+p_2+p_3U_3^0\right),\quad \lambda\to0.$$

In other words, we obtain the uniform distribution on the parallelepiped $[0,p_1]\times[p_1,p_1+p_2]\times[p_1+p_2,1]$ in the limit. The cases $\nu=2,3$ are considered by analogy.

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Data Quality Management in Problem-Solving Using Research Infrastructures over Heterogeneous Data Sources

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Abstract—Problem-solving based on available research data, especially in the context of open science and research infrastructures, should ensure the possibility of their multiple reuse. Data quality metrics are important characteristics that affect not only the accuracy of research results but also the assessment of data suitability, the feasibility of solving specific research problems, the choice of methods for working with data, object matching, data compatibility, and other aspects of reuse. This requires an assessment of various data quality dimensions at different levels of aggregation, from entire datasets to individual values. This study presents an approach to integrated data quality management based on data specifications, as well as data and metadata quality requirements. Various data quality assessment dimensions, including accuracy, completeness, and provenance, are discussed. The developed approach is applied to problem-solving using multiple data sources in stellar astronomy.

Keywords: data quality, data reuse, formal specifications, non-functional requirements

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1. INTRODUCTION

Research inevitably encounters the need to assess the quality of the available data used in problem-solving [1]. At different stages of scientific research, there is a need to assess the quality of data. When searching and selecting suitable datasets, it is important to evaluate their applicability based on information about the quality of the data in these datasets. When creating samples that include only suitable data, as well as when data cleaning and improving to prepare them for research, it is necessary to evaluate the quality of data related to specific objects or characteristics. When using research methods, it is important to take into account the quality of the source data both to assess the quality of the results obtained and to assess the quality of the methods themselves. Thus, data quality metrics are an important and even necessary element in solving various research problems. The lack of information about the quality of the data or ignoring it can significantly reduce the quality of the results, even to their falseness, or make it impossible to conduct a study.

The volume of research data is steadily growing in many disciplines. Given the heterogeneity of research data sources and their diversity, problems arise with the compatibility of data obtained by different methods, created for different purposes, and with different quality requirements.

In some sources, information about the quality of the data presented may be contained in the description of datasets, accompanying documentation, or be known from external research. In other sources, the data is accompanied by quality assessments included in the structure of the

sets themselves. In some cases, there is no information about the quality of the data, but it can be assessed using statistical methods by analyzing the data itself or its samples. Data quality assessment methods can also vary significantly. Depending on the tasks being performed, different quality dimensions and different criteria for their evaluation may be important. In some cases, it is necessary to evaluate the quality of solutions in different ways based on information about the quality of the source data. Thus, heterogeneity is observed both in the metadata itself, concerning data quality, and in their application.

To support research and manage relevant data, research infrastructures are being created that accumulate data, provide services and metadata, ensuring their reuse. Due to their importance, approaches to data quality management in research infrastructures require thorough research. When working with large amounts of data and many heterogeneous sources, it is necessary to strive for automated quality management. Therefore, metadata regarding data quality must be clearly defined, accessible, and understandable to both humans and machines.

This article sets out the problem of developing the presentation and application of various data quality dimensions in research infrastructures. The following section provides an analysis of the state of research in this area. Section 3 provides a classification of approaches to data quality management. Metadata specifications and principles of their use in research infrastructures are described. Section 4 provides an example of data quality management for problem-solving in stellar astronomy using data from large multicolor photometric surveys of the sky. General conclusions are drawn about the development of research infrastructures to increase the effectiveness of research.

2. THE STATE OF RESEARCH IN DATA QUALITY MANAGEMENT

Quality problems are so sensitive in the national and global economy that approaches, methods, and standards have long been worked out to solve them. In computer science, data quality issues, as part of the general problem of the quality of real-world objects, play an equally important role. Databases and information systems are considered independent objects [1], which, like other objects in the real world, may have an acceptable or unacceptable state, which can be assessed by data quality metrics. At the same time, the data in them reflect the state of real-world objects, and it is necessary to evaluate both the correspondence of the state of information objects to the characteristics of real-world objects and the possibility of identifying real-world objects from the data. Ignoring data quality issues can lead to serious consequences not only in the information sphere but also negatively affect other areas of human activity.

Data on the qualitative and quantitative characteristics of real-world objects are used to assess their quality. Thus, the quality of objects (products, materials, and others) is always related to the quality of data about them. Moreover, in the modern research paradigm, which is based on the extraction of new knowledge from research data, collecting a large volume of data of varying quality about the studied objects from various sources requires assessment and consideration of their quality in research, which ultimately determines the development of science.

There may be requirements for data used in human activities. The quality of data (for example, characteristics such as accuracy and reliability) about real-world objects can determine the applicability and usage characteristics of both the data itself and the objects of research. Poor data quality, even if the objects under study themselves are of high quality, can make it difficult to solve problems and interfere with the proper use of both data and research objects.

For a long time, multi-criteria models that include sets of quality dimensions have been the prevailing approach in assessing data quality. Even in early studies, such as [2], the importance and relevance of various data quality dimensions were discussed. The main dimensions are accuracy, completeness, integrity, and relevance, as well as veracity, unambiguity, reliability, and volume of data. At the same time, the set of dimensions used depends on the objectives defined for a specific

research domain. Some studies emphasize the relationship between data quality and the quality of products and real-world objects, which confirms the effectiveness of such approaches to data quality assessment. From this point of view, the value of data is determined by its consumer, which draws attention to such quality dimensions as the significance and correctness of the data [3].

Over the course of research, an approach has been established in which the quality model is defined as a space of quality dimensions. In [4], the quality assessment methodology is defined by a model that includes a specific set of metrics, methods for evaluating them, and possibly methods for summarizing them. Depending on the tasks being performed, different methods for evaluating quality metrics can be used in each specific model, such as algorithms, rules, heuristics, or machine learning models that provide solutions to problems related to certain aspects of data quality.

For example, data completeness can be evaluated as the completeness of tuples (filling in all attributes), attributes (estimating the number of missing values among attribute values in tuples), or as coverage of all existing objects of a given type. The dimension of accuracy can be defined as the error of values, syntactic accuracy (the coincidence of names used for one object), or semantic accuracy (consistency of facts). The data volume can be determined by the amount of memory used or the expected number of tuples returned, and so on. In other words, the methods and implementations of quality assessment may depend both on the characteristics of datasets or samples and on specific limitations of the domain to which the data relate. The semantics of a particular metric, despite the same names, can vary significantly in different tasks. Accordingly, data quality requirements can be formulated in research problems both in terms of datasets or data warehouses and the applied problem domain.

Similar approaches are defined in current data quality standards. There are international and corresponding Russian national standards related to data quality management, in particular, ISO/TS 8000-1:2011 (GOST R 56214-2014 in Russia [5]) and a set of related standards. These standards comply with established principles of data quality modeling and introduce quality dimensions such as accuracy, completeness, and provenance (or source) of data without limiting specific implementations of quality assessment methods based on these criteria. Also, the standards define conceptual schemes for quality management and principles for developing criteria for evaluating these metrics, including principles of unambiguous syntactic and semantic coding. The quality of data within these standards indicates the extent to which the data meets the requirements of consumers and meets the established criteria. The principles of presenting requirements and improving data quality are established, allowing consumers to request data of appropriate quality and accurately determine whether the data obtained meets the established standards.

The ISO 19157:2013 [6] standard (GOST R 57773-2017 in Russia) is also interesting. It establishes the practice of quality management when working with spatial data sources with different resolutions and granularities. This standard allows you to select the most suitable data in terms of quality to solve problems at the required scale.

Problems with the availability of suitable research data arise in many data-intensive domains. In this regard, the field of research infrastructures is actively developing. They combine research data, services, and tools, allowing them to be used repeatedly.

One of the ideological foundations of their development has become the guiding principles of FAIR-data [7], which propose a direction for ensuring the findability (F), accessibility (A), interoperability (I) and, as a result, the reusability (R) of research data. When developing research infrastructures and how they work, data quality issues are often reduced to approaches for evaluating data warehouses in terms of compliance with FAIR principles.

Major interdisciplinary initiatives of the research community aimed at providing and managing data, as well as analyzing it, address issues of data quality, mainly from the perspective of FAIR

principles, shifting the focus from data quality management to assessing the quality of the data management infrastructure for the possibility of improving data quality.

The RDA initiative has developed the FAIR Data Maturity Model [8], which defines a set of indicators, their priorities, and methods for assessing compliance with the principles of FAIR. It is used as a general approach to evaluate different methodologies. In particular, the FAIRsFAIR [9] project was created to promote the principles of FAIR for data generated by researchers. Popular explanations, examples, and solutions have been developed that promote data interoperability and reuse, as well as tools, standards, and practices for data management in various scientific disciplines. Based on the FAIR Data Maturity Model and FAIRsFAIR developments, the F-UJI [10] tool has been implemented to assess the degree of compliance with the principles of FAIR research datasets and provide recommendations. Thus, such developments create an alternative approach to assessing data quality. Evaluating compliance with FAIR principles provides a tangible assessment of the quality of data management, rather than the data itself, basically. On the other hand, this may devalue the principles of FAIR itself as a strategic direction in the development of data management. After all, these principles remain promising until the declared possibility of autonomous data management by a machine (machine-actionability) is sufficiently impossible, that is, the transition from processing predefined sets of metadata and instructions by a machine to ensuring the correct interpretation of data and metadata that the machine has not previously worked with.

The ELIXIR initiative specializes in biomedical and biological information resources. Here, data quality issues are also more often discussed precisely in the context of compliance with FAIR principles, in particular, in the recommendations of the FAIR CookBook [11]. However, no specific recommendations have been developed on data quality management.

The ELIXIR [12, 13] research infrastructure (in particular, the Data and Interoperability platforms within it) supports the storage of large amounts of data, advanced metadata management, data integrity control, duplicate and data detection mechanisms, and other tools used to manage data quality. At the same time, there are no specialized tools for data quality specification apart from assessing compliance with the FAIR principles.

Within the framework of the ESIP initiative dedicated to data management in geo-sciences, research related to data quality management was conducted. The Data Quality Working Group (DQWG) and Information Quality Cluster (IQC) have developed recommendations for the application of best practices and standards in this domain, principles of data life-cycle management to ensure their integrity, quality and reuse, and have contributed to the implementation of these recommendations among data providers, software developers and research groups [14–16]. These solutions are primarily related to the established practice of working with spatial data, such as standardizing attributes and metadata, using quality flags in data, and compliance with specialized standards.

Also, within the framework of ESIP, a matrix for assessing the maturity of research data has been developed, combining the basic principles of data curation during long-term storage and utilization. Part of these principles is data quality management [17].

International standards for data quality management also continue to evolve. The set of spatial data quality management standards is in the process of updating [18].

The most common products related to data quality management are data cleaning tools as part of integration systems. The tasks solved in such products are detecting errors and duplicates, typecasting and standardizing the representation of values, solving the problem of missing data, and others. However, these products can also provide data quality assessment tools and algorithms for calculating quality metrics. Such products include, in particular, Talend Data Quality [19], which is part of a relational database integration system. This tool uses SQL-based business rules to

represent data requirements and monitor data quality based on dimensions such as completeness, accuracy, and consistency of data, with the ability to detect specific problem areas. IBM InfoSphere Quality Stage [20], part of IBM's product line for supporting data management processes, provides continuous monitoring of quality events in data streams, error correction, data cleanup, and metadata enrichment.

Semantic Data Quality Management (SDQM) [21] is based on the analysis of ISO 9000 standards and research of data quality models with sets of dimensions, as well as semantic Web technologies [22], including RDF technologies [23]. Semantic approaches based on RDF, a language with extensible semantics for describing resources, allow accompanying resources identified in the global information space with metadata, the semantics of which is defined by dictionaries or ontologies in various namespaces, setting resource requirements in terms of these dictionaries, and finding relevant resources. Moreover, the RDF model has become the most commonly used for defining metadata schemas. To define the quality model, the data quality management dictionary [24] is introduced, the data provenance model for web resources [25] and other dictionaries are reused.

Big Data Quality Management Framework (BDQMF) [26] is designed to manage the quality of big data at all stages of its life-cycle. BDQMF covers the entire data quality management process, from requirements definition and preprocessing to data analysis and quality monitoring. The main approaches include creating a data quality profile with quality targets, evaluating and improving quality at each stage, validating and optimizing quality rules to improve data accuracy, and monitoring and visualizing data after processing.

From the point of view of FAIR principles, data quality issues should be considered in the context of ensuring data reuse (R) [27]. The main principle (R1) in this direction indicates the need to describe data richly with accurate and up-to-date information (attributes). Such information refers to the resources and metadata needed to evaluate the reusability of data, rather than just describing its semantics. This information may include details about the provenance of the data, the license terms of its use, and other non-functional properties, which nevertheless play an important role in the selection of data by researchers to solve their research problems [28]. Metadata based on data quality metrics should also be included in this category.

The data quality standards developed by the W3C consortium include the PROV [29] standard, which establishes a model for describing the provenance of data. This standard defines a lot of non-functional metadata that is captured during any data manipulation. They include information about the authorship, the method of obtaining, transformations of the data, information about how some data were used as the sources when creating the data in question, and much more. However, in practice, the available data is most often accompanied by minimal information, limited by information about the authorship, affiliation, and time of creation of datasets, and rarely uses all the rich features provided by this standard.

In addition, the W3C consortium has developed an approach to data quality specification DQV (Data Quality Vocabulary) [30] based on RDF technologies. The definitions in this model allow you to describe a vocabulary of quality metrics for datasets, including sets of dimensions and methods for evaluating quality metrics, such as estimates of the volume, completeness, accuracy, and other characteristics of the contents of data catalogs. Specifications in the DCAT (Data Catalog Vocabulary) [31] format are used to communicate with the described data catalogs.

The DCAT specification allows you to describe catalogs, datasets, more general concepts of resources, their representations in various formats and sources, individual records, as well as data services that provide access to data through software interfaces for querying and obtaining the necessary parts of data. Thus, data quality specifications can be associated with entire catalogs and datasets, as well as with individual fragments of them. DCAT recommendations are evolving. Version 2 adds some data identification capabilities and backward references to the DQV model.

The recent version 3 adds support for versions and series of datasets. The mutual references with the DQV model provide enhanced data quality specification capabilities in the development of DCAT.

In general, approaches to building data quality models can be considered well-established and flexible enough for further development within the framework of existing concepts. However, despite this, such approaches are rarely used to describe data in the form of well-defined models. Metadata related to data quality is most often present in documentation in an arbitrary form or included directly in datasets but is not isolated as separate quality metadata and is poorly structured. This is especially noticeable, for example, in the case of errors in measuring the characteristics of objects. Often, the values of characteristics and their errors are presented in catalogs as equivalent attributes of an object.

3. QUALITY METADATA AND A DATA QUALITY MANAGEMENT APPROACH

Data quality specifications in the DQV [30] format offer an RDF schema that allows you to specify a structure for describing data quality. This scheme, among other things, includes:

- Sets of quality dimensions (Dimension): characteristics that are used to evaluate various aspects of data quality;
- Metric categories (Category): classification of metrics according to certain criteria, which facilitates their organization and use;
- Values of quality metrics (QualityMeasuration): specific values that reflect the quality level according to the specified metrics;
- Methods for evaluating quality metrics (Metric): methods for calculating or generating quality metrics, they describe exactly how the quality of the data was assessed, as well as the types of data that are used to represent the values of quality metrics;
- Datasets and their Representations (Distribution): linking quality metadata to specific datasets or copies of them in specific formats;
- Data Services (DataService): linking quality metadata to services that provide access to data.

A quality metadata model for a quality specification can be built based on specifications similar to those offered in DQV. However, such a basic model requires significant expansion to take into account the specification of quality metadata at various levels of data aggregation, the definition of actions with data depending on their quality, the establishment of quality requirements for the data being searched or created, and other important features.

In order to form the necessary types of specifications included in the quality model, the quality metadata was classified according to various criteria. Linking the elements of this classification allows you to define the specifications of quality metrics, establish their relationship with the described data, determine actions with data based on their quality, and formulate data requirements that need to be taken into account.

1. Classification by quality dimensions:
   - volume;
   - completeness;
   - accuracy;
   - provenance;
   - relevance;
   - reliability;
   - and others.
2. Classification by type of quality metric values:
   - boolean flag or bit vector;
   - real value;

- a set of categorical values of quality grades;
- a set of categorical values distinguishing types of violations or compliance with quality requirements;
- the same types with empty values (unknown quality).

3. Classification according to the method of value specification:
- constant;
- formulas or rules;
- tabular.

4. Classification by metadata source:
- metadata of the dataset;
- metadata in the data schema;
- external metadata repositories;
- metadata in publications;
- evaluation based on an experiment;
- statistical evaluation of data;
- evaluation on a representative sample of data;
- volatile metadata (requiring reassessment).

5. Classification by data aggregation method:
- dataset;
- relation;
- data sample (slice, query conditions);
- related semantics of entities or processes;
- tuple;
- specific entity or process;
- attribute;
- related attribute semantics;
- attribute value.

6. Classification of actions of data quality improvement:
- deleting data according to the level of aggregation (dataset, sample, tuple, entity, attribute, value);
- assigning quality metadata (metric, weight);
- selection of quality data;
- data enrichment from external sources;
- averaging methods based on data quality;
- averaging methods without taking into account data quality;
- ignoring data quality (including all data without regard to quality).

7. Classification of data quality requirements:
- restrictions on the composition and quality of the source data;
- requirements for improving the quality of the source data;
- declared required quality assessments of the results;
- characterization of the dependence of the quality of the results on the quality and characteristics of the source data;
- the need to evaluate the quality of the results or the inability to evaluate them.

In cases where the necessary metadata at the dataset level is missing or requires adjustments on subsets of data (for example, parameter slices), they can be evaluated statistically on all data or representative samples of them, and then existing metadata can be supplemented in the same format.

Quality metadata at the level of tuples or attribute values is usually included directly in the data presented in catalogs or datasets as additional relationship attributes.

Quality metadata related to a specific level of data aggregation is distributed to nested levels of aggregation. Thus, any data tuple or value used can be associated with quality metadata collected from different levels of data representation, including metadata of the dataset as a whole, metadata of attributes, values, and others. When defining requirements for quality metadata, restrictions can be used on the metadata of entire datasets, as well as on the metadata of specific data values or their provenance.

Quality weight assessment approaches based on quality metrics are often used to make decisions about further actions. The weight can be expressed as numerical values, as well as special labels, such as the absence of an assessment (nan) or indication of data removal (del). If there are no quality attribute values, the quality weight is set to nan, which does not affect the total quality weight of the tuple. If at least one weight value for a tuple is marked as del, it means that the tuple should be excluded from the analysis, and other weight values are not taken into account. The tuple quality weights defined at the sample or dataset level affect the overall quality weight of the tuple. The presence of multiple weights for a tuple is generalized by calculating their average value. The same principles apply to the quality weights of individual attribute values. Thus, the quality weights of tuples and attributes are also extended to the weights of each of the attribute values.

4. AN EXAMPLE OF PROBLEM-SOLVING USING QUALITY SPECIFICATIONS

As an example of the application of the data quality management model, a quality model was developed, and the problem of improving the quality of data in the domain of stellar astronomy was solved based on a variety of large multi-color photometric sky surveys. To study different types of stars, their spectra reconstructed from photometric data in different emission bands, and their observed parameters and estimated values of astrophysical parameters, the problem was set to cross-match and merge data on the same objects from different photometric catalogs. An example of the analysis is presented on the data of the catalogs SDSS [32], UKIDSS [33], GALEX [34, 35].

Catalog analysis shows a variety of factors that affect the quality of data and affect the quality of object matching:

- Catalogs have different sky coverage due to the location of the observatories that conducted the observations.
- Each catalog contains data obtained through various sets of photometric filters available in telescopes and providing observations in different bands of the emission spectrum of objects.
- Telescopes have different optical resolutions and, accordingly, have different positioning (calibration) accuracy and object discrimination. It is expected that the accuracy of distinguishing objects in observations in the Galactic plane may be lower since the number of objects in the field of view is greater.
- Telescopes have different limits of the minimum and maximum magnitudes of objects: brightness values below the minimum are recorded within the margin of error, and values above the maximum are saturated and do not reflect the actual values.
- Catalogs include various sets of observation quality flags, as well as flags for classifying objects, detecting artifacts, and other information about observations.
- Observations carried out in different epochs are reflected in the position of objects due to their movement and in the intensity of the brightness as a result of its variability.
- Data distortions can also be related to various observing conditions: the time of year and day, the object's position relative to the zenith, and weather conditions.
- The location of an object at the edges of the field of view affects the noise level compared to observations in the center of the field and may also cause glare associated with the equipment.

The problem of cross-matching catalogs and sky surveys remains relevant in astronomy and is usually solved for a pair of catalogs with different matching quality requirements. In accordance with the requirements of specific tasks, data is preprocessed, or observations with low quality are deleted. However, the procedure may vary depending on the specifics of each case.

Based on the analysis of the problem, the following tasks were set:

1. Catalog coverage assessment: It is necessary to determine whether the catalogs sufficiently cover sites in certain sky viewing directions. It is important that one field of view contains data from several surveys with different sets of filters, which will cover most of the pass bands in the emission spectrum of objects.

2. Cross-matching method: It is necessary to develop a method for solving the problem of cross-matching surveys of different qualities (resolutions), including matching multiple observations of objects both within the same catalog and between different catalogs.

3. Estimation of the matching radii: It is necessary to solve the problem of estimating the radii of matching objects between catalogs depending on the direction of observation relative to the plane of the Galaxy.

4. Data quality assessment: It is required to evaluate the data quality on observational objects based on the values of catalog metadata and flags present in catalog structures.

5. Matching list generation: It is necessary to develop an approach to the formation of object matching lists and merging data to obtain tuples describing the magnitudes of objects in different emission bands.

6. Presentation of the results of the merge: It is required to present the results of the merge in the form of a catalog and associated metadata.

4.1. Data Completeness Issues

Task 1, related to the problem of completeness of data in surveys, appears in two aspects.

First, this refers to the data coverage of most directions in the sky by the catalogs used. This coverage significantly affects the solution of problems such as estimating the positional accuracy of catalogs and the ability to estimate absorption in the interstellar medium in various directions in the sky, which also depends on the angle of observation relative to the Galactic plane. Data completeness values in the specified directions are evaluated statistically based on catalog data. In this context, it is not the percentage coverage of the sky that is important but the coverage of large areas in different directions. For a specific field of view, a selection of catalogs can be chosen to provide good coverage.

Secondly, the completeness of the data from the used photometric catalogs is estimated from the point of view of covering most bands of the observed spectrum of stellar emission. This characteristic is crucial for estimating the spectrum based on photometry data and depends on the set of filters used in telescopes. It is defined by a set of attributes in the catalog structure.

It is evident that the data from any single large photometric survey do not meet these two requirements. That is why it becomes necessary to integrate and match multiple catalogs so that the data obtained as a result of the catalog merging are suitable for stellar spectrum estimation, further classification, and parametrization.

4.2. Data Accuracy Issues

In task 2, the problem of errors in matching multiple observations of objects in the low-resolution catalog (GALEX) and then averaging the erroneous matching results to a single tuple is solved by choosing the sequence of catalog matching. First, all tuples of this catalog are matched with higher-quality catalogs. Then, if it is necessary to obtain an average tuple for multiple catalog

observations, the results of cross-matching with higher-quality catalogs are averaged. The cross-matching of objects from different catalogs by coordinates makes a joint accuracy calculated as a mean squared error that takes into account both uncertainties. Thus, a lower estimate of the joint accuracy is achieved and, accordingly, the probability of incorrect matching is reduced.

When solving task 3, the radius of object matching is usually taken as a constant: 1 angular second for SDSS and UKIDSS, 3 angular seconds for GALEX. These values are related to the positional accuracy and optical resolution of catalogs (the accuracy of distinguishing objects). Different values of coordinate accuracy can be found in the catalog-related publications, but the description of coordinate quality remains ambiguous. Moreover, it is desirable to estimate the radius of matching objects depending on the direction in the sky. Therefore, the usual approach turns out to be insufficient, and it becomes necessary to determine the optimal matching radius.

The problem of choosing the matching radius is solved by applying a statistical estimate that maximizes the number of unique coordinate matches for a given radius in a specific direction of observation relative to the Galactic plane [36]. Thus, the positional accuracy of catalogs is detected on data or representative samples, and metadata of accuracy specifications are generated for them. These metadata can be presented tabularly depending on the field in a certain direction in the sky and later used for each value of the coordinates value of the field objects as its accuracy.

4.3. Data Reliability Issues

Task 4 is related to evaluating the accuracy of data in different catalogs, which depends on a number of factors:

- Positional accuracy of objects: depends on the resolution of the equipment used to observe the sky.
- Location of objects in the field of view: objects located in the center of the field have higher accuracy than those located at the edge.
- Background overexposure: bright stars located close to the observed objects can cause interference.
- Artifacts: such as planets or satellites passing in the object field of view, as well as lens flares.
- Saturation of bright objects: bright stars at the upper limit of the magnitudes determined by the equipment can distort data on their brightness.
- Undetection of objects: objects at the lower limit of magnitudes may remain unnoticed.
- Accuracy of magnitude values: for each magnitude value, its accuracy in a certain emission band is indicated.

Some catalog attributes are flags that indicate the type of object, additional characteristics of its observations, and their quality. Data preparation takes into account the limitations imposed on catalog attributes that are not related to data quality, in accordance with the semantics of the task.

- The attributes Fr and Nr in GALEX indicate the angular size of the object at half-light. The restriction cuts off tuples with values of these attributes exceeding 0.003 (for stars).
- The class attribute in SDSS provides information about the type of object: star, galaxy, artifact, and others. The restriction cuts off tuples with values other than 6 (star).

Next, it is necessary to evaluate the quality of the tuples as a whole, both the observations and the values of the characteristics of the objects in them. The GALEX catalog uses the following flags for additional information about observations and data quality:

- Attributes Fexf and Nexf are vectors of bit values indicating various types of low-quality observations: objects with neighbors, mixed with other objects, truncated by the observation

boundary, and others. When these flags are not equal to 0, the magnitude value is assigned a quality weight of del.

- Attributes Fafl and Nafl are vectors of bit values indicating various types of artifacts: flares, spots, and others. If the values of these flags are not equal to 0, the magnitude value is also assigned a quality weight of del.
- Attributes nS/G and fS/G are real values that evaluate whether an object is a star or a galaxy. We can say that this is a classification of an object indicating the degree of its reliability. The tuple quality weight is set to 1 for parameter values exceeding 0.5; otherwise, the weight is 0.

The SDSS catalog uses one essential tuple quality flag:

- The attribute Q introduces categorical gradations and evaluates the quality of observation: 1 — low, 2 — acceptable, 3 — high. If the attribute value is 1, the quality weight of the tuple is set to 0; if the value is 2, then the weight is 0.5; if the value is 3, then the weight is 1.

The following quality flags are used in the UKIDSS catalog:

- The attribute cl is a mixed categorical and gradation metric that evaluates whether an object is a star, galaxy, or noise. Attribute values may indicate assignment to a particular class, indicating the degree of reliability. The quality weight value of the del tuple is assigned for all values except -2 and -1 (stars).
- The attribute p* is a real value that partially duplicates and clarifies the value of the cl attribute, estimating the probability that the object is a star. If the attribute value exceeds 0.7, the tuple quality weight value is 1; if the value is below 0.3, the value is 0; otherwise, the value is 0.5.
- Attributes pG and pN estimate a similar probability for galaxies and noise and are not considered in this task. Tuples classified as galaxy or noise using the cl attribute are deleted.

The exact way in which the weights of qualities, tuples, or values are formed based on the values of the flags was set by setting the problem in collaboration with experts in stellar astronomy. If there are no quality attribute values, the quality weight is assigned the value nan. Thus, based on the flags, the quality weights of the tuples and some attribute values in the tuples were determined. The quality weights are generalized to obtain general estimates of the quality of tuples or values in accordance with the principles described in the previous section. The generalized data quality weights are taken into account later when merging data.

4.4. Merging Data

When performing task 5, related to the formation of object matching lists, the principles of working with identified objects from several catalogs, proposed in [37], are applied. Match bundles are used that exclude transitive relationships of matched objects in multiple catalogs. Data is merged based on tuples included in match bundles and their generalized quality weights.

Depending on the tasks being solved on the identified objects, data merging can be performed according to different principles. For example, to solve the problem of the parametrization of objects, only high-quality data are important, and any low-quality observations are removed from consideration. However, if it is necessary to save low-quality objects to preserve the completeness of the data and improve its quality, such data can be included in the analysis.

Coordinates from the most accurate catalog are used for the identification of objects after merging and positioning them (in the example, this is SDSS). For identified multiple observations from the same catalog, collected in a single tuple, the coordinates can be averaged without taking into account the weights.

Flags mainly relate to the quality of observations of the object brightness in the sky. The fusion brightness data can be averaged from observational data in the same emission band.

Most catalogs contain attributes indicating the error (σ) of the magnitude measurement (m). To estimate the error of the merge result when selecting only reliable tuples (with a high quality weight: $r = 1$) a weighted mean can be used. In this case, the weight is inversely proportional to the square of the error, which minimizes the total variance. If more than reliable tuples are used in solving the problem, it is necessary to take into account the estimation of the quality weight (r) of the tuple or the value together with the error of the magnitude value. In this case, when calculating the weighted mean (2), the weight inverse to the square of the error is multiplied by the weight of the quality value (1).

$$w = \frac{r}{\sigma^2}, \tag{1}$$

$$m = \frac{\sum_i m_i w_i}{\sum_i w_i}. \tag{2}$$

The error of the result in this case can be estimated as follows (3):

$$\sigma = \sqrt{\frac{1}{\sum_i w_i}}. \tag{3}$$

4.5. Result Generation

The solution to problem 6 is to form the resulting relationship. It is created by combining matched tuples from different catalogs with a generalized evaluation of attributes for which multiple values have been found. The result should combine the following elements in a common tuple:

- coordinates from the most accurate catalog, which are also used to identify the object;
- weighted mean magnitude estimates for each emission band corresponding to the catalog bands;
- error estimates for each of the magnitude values.

This relationship can be represented as an independent catalog, which should be abundantly supplied with metadata. Such metadata should include a description of the provenance of the data and its quality, which will make it possible to assess the applicability of the data in problem-solving and ensure that it can be reused.

The description of the data provenance of the new catalog may include the following elements:

- at the catalog level: links to catalogs whose data were used to generate the result;
- at the attribute level: links to attributes (coordinates, magnitudes in the corresponding bands) from which attribute values were formed;
- a link to the description of the tuple matching method;
- a link to the description of the selection method (for coordinates) and the averaging of attribute values (for magnitudes).

The description of the catalog data quality may include:

- at the relation level: information about the completeness of the sky coverage and the completeness of the attributes (covered observation spectral bands), depending on the direction in the sky;
- information about evaluating the positional accuracy of the resulting data;
- information about the quality of fused tuples;
- information about the presence of weighted estimates of magnitude errors in the catalog attributes and their relationship to the attributes of the magnitude values themselves.

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5. CONCLUSION

The approaches to data quality specification in research infrastructures with multiple heterogeneous data sources of different qualities are investigated. Standards for describing data quality are based on multi-criteria models, which usually do not limit the composition and methods of evaluating quality metrics. The development of the principles of data quality specification is proposed to determine how to store and access quality metadata, the level of aggregation of the evaluated data and take into account non-functional data requirements when problem-solving and data processing tasks. These principles are illustrated when solving a problem in stellar astronomy. At different stages of its solution, various quality dimensions are evaluated, including completeness, accuracy, and reliability of data, different sources of quality metadata are used, such as information on the accuracy of physical quantities in catalogs, tuple quality flags, guaranteed quality from catalog documentation, and statistical estimates based on a sample of catalog data. Depending on the obtained estimates of quality dimensions, different data sources are used, data that is not suitable in quality is filtered out, the sequence of data processing in problem-solving algorithms is changed, and metadata of both data sources and intermediate and final results of solving the problem is generated and stored.

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Using Ensembles with Enhanced Divergence in Forecast Space in Recommender Systems

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Abstract—The paper considers the divergent decision forest method based on achieving a higher divergence in the forecast space compared to the standard random decision forest. It is based on including a new tree T_x in the ensemble at each step, which is constructed based on minimizing a special functional, which is the difference between the squared error of T_x and the squared divergence of the forecasts T_x and the current ensemble. The method is a development of similar previously developed methods that are intended for predicting numerical variables. The paper presents the results of applying the divergent decision forest method to solving classification problems that arise when creating recommender systems. The paper investigates the dependence of the forecast efficiency on the tree depth and one of the key parameters of the algorithm that regulates the contribution of two components to the minimized functional. Studies have shown that the accuracy of the proposed technology significantly exceeds the accuracy of the random decision forest and is close to the accuracy of the CatBoost method.

Keywords: ensemble method, machine learning, recommender systems

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1. INTRODUCTION

Methods based on the use of ensembles of regression or decision trees are one of the main branches of modern machine learning and are actively used in solving various applied problems [1]. Two popular directions of ensemble algorithms can be distinguished: random forests [2] and gradient boosting [3]. For both technologies, methods have been developed to solve both automatic classification problems and numerical variable prediction problems. When predicting a target numerical variable Y based on features $X_1, \ldots, X_n$, training is performed using the sample $S = \{s_1 = (y_1, x_1), \ldots, s_m = (y_m, x_m)\}$, where y_j and x_j are the values of the target variable Y and the vector of feature values $X_1, \ldots, X_n$ of the object s_j, respectively. Classification problems with two disjoint classes can also be viewed as binary numeric variable prediction problems with training from a similar type of sample. Let us formulate the main differences between gradient boosting-based methods and random forests. In the random forest method, individual trees are built independently. In this case, the tree at step k is aimed at predicting the target variable Y and is constructed using the sample S_k obtained by the bagging procedures [4] and the random subspace method [5]. The bagging procedure comes down to sample selection with replacement from S. The random subspace method assumes the use of a random subset of the original feature set $X_1, \ldots, X_n$ of a fixed size when training each new random forest tree. The output forecast of $\hat{A}_k$, consisting of k trees, is calculated as the average forecast over all trees included in the ensemble, that is,

$\hat{A}_k = \frac{1}{k}\sum_{j=1}^{k} A_j$. In the gradient boosting method, the optimal algorithm is sought as a linear combination of regression trees $\hat{A}_N = T_0 + \alpha_1 \times T_1 + \ldots + \alpha_N \times T_N$, where the original algorithm T_0 usually calculates a forecast identically equal to the average value of Y. At each step, a new term $\alpha_k \times T_k$ is added to the current linear combination $\hat{A}_{k-1} = A_0 + \alpha_1 \times T_1 + \ldots + \alpha_{k-1} \times T_{k-1}$. The choice of the latter is made based on minimizing the loss function for the linear combination $\hat{A}_k = T_0 + \alpha_1 \times T_1 + \ldots + \alpha_k \times T_k$. That is achieved by using the gradient descent procedure.

Let $L(\hat{A}_{k-1}, S)$ be some loss function depending on the predictions computed by $\hat{A}_{k-1}$ and the true values of Y of objects S. Let us associate the predictions computed for objects S with the variables $z_1, \ldots, z_m$. According to the gradient descent method, the expected minimum L will be achieved for the prediction at the kth step for object s_j, provided

$$\hat{A}_k(s_j) = \hat{A}_{k-1}(s_j) - \eta \frac{\partial L}{\partial z_j}\bigg|_{z_j = \hat{A}_{k-1}(s_j)}.$$

However, the forecast in the specified form cannot be calculated for objects for which the true value of y_j is unknown, since the loss function and its partial derivatives are defined only for known values of Y. Therefore, the authors of the method proposed to use instead of the values of g_j their forecasts calculated using a regression tree trained on a sample of $\{(g_1, x_1), \ldots (g_m, x_j)\}$. This tree becomes the tree T_k added to the linear combination. When training T_k, the bagging and the random subspace procedures are used.

Currently, there are several modifications of gradient boosting, including XGBoost [6], LightGBM [7], CatBoost [8]. In contrast to gradient boosting, conservative versions of the random forest method are widespread, i.e. the method remains virtually unchanged since its creation. At the same time, there is no convincing evidence that the ensemble generation scheme used in random forests is actually optimal when using simple averaging as an aggregation procedure.

2. CONSTRUCTION OF FORESTS WITH ENHANCED DIVERGENCE

In the [9–11] a new approach was proposed aimed at constructing an ensemble based on the condition of minimizing forecast losses, calculated as the average forecasts for the ensemble. In other words, the problem of selecting trees for the ensemble in such a way that the losses were as minimal as possible when using $\hat{A}_k$ as a forecast was set. In this case, the usual root-mean-square error was used as a loss functional:

$$L(S, A) = \frac{1}{m}\sum_{j=1}^{m}\left(y_j - \hat{A}_k(x_j)\right)^2.$$

It was shown that the mean squared error for the algorithm A_k is calculated by the formula

$$L(S, \hat{A}_k) = \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{k}(y_j - T_i(x_j))^2 - \frac{1}{km}\sum_{j=1}^{m}\sum_{i=1}^{k}\left(\hat{A}_k(x_j) - T_i(x_j)\right)^2. \quad (1)$$

One can pose the problem of finding such a set of trees $T_1, \ldots, T_k$ that achieves a minimum of losses (1). This problem is extremely labor-intensive. However, a simple heuristic approach can be used, when at each step a tree is added to the ensemble that simultaneously approximates the dependence and is maximally distant from the current ensemble in terms of the calculated forecasts. To implement this approach, in [9–11] it was proposed to add a tree T_k at step k, in which the following functional would be as minimal as possible

$$Q(T_k, \hat{A}_k, \mu) = \frac{1}{m}\sum_{j=1}^{m}(y_j - T_k(x_j))^2 - \mu\frac{1}{m}\sum_{j=1}^{m}\left(\hat{A}_k(x_j) - T_k(x_j)\right)^2, \quad (2)$$

where μ is a parameter from the interval $(0,1)$. The functional $Q(T_k, \hat{A}_k, \mu)$ can be rewritten as

$$Q(T_k, \hat{A}_k, \mu) = \frac{1}{m}\sum_{j=1}^{m}(y_j - T_k(x_j))^2 - \theta\frac{1}{m}\sum_{j=1}^{m}\left(\hat{A}_k(x_j) - T_k(x_j)\right)^2,\ \theta = \frac{k^2\mu}{(k+1)^2}. \tag{3}$$

Note that the minimum of the functional (2) is achieved if each point x_j is associated with a forecast t_j^o equal to the value of the auxiliary variable t_j which minimizes the function

$$(y_j - t_j)^2 - \theta\left(\hat{A}_{k-1}(x_j) - t_j\right)^2.$$

It is easy to show that the necessary conditions for an extremum are satisfied in the case when

$$t_j^o = \frac{y_j - \theta\hat{A}_{k-1}(x_j)}{1-\theta}. \tag{4}$$

It is obvious that $\theta \to 1$ for $k \to \infty$ and $\mu = 1$, which leads to instability of the estimates by formula (4). Thus, the additional factor $\mu \in (0,1)$ is introduced. To find the optimal ensemble, an approach was proposed in [10, 11], in which at step k a new tree T_k is added to the ensemble, predicting the values t_j^o calculated by formula (4) depending on the vector descriptions. The tree T_k is trained using the sample $\{(t_1^o, x_1), \dots (t_m^o, x_m)\}$. The use of the hyperparameter μ, which affects the efficiency of the constructed ensembles, is a heuristic technique. There is no theoretically justified method for calculating the optimal value of μ. It is assumed that the optimal value of μ can be selected as a result of experiments, which is one of the goals of this study.

3. APPLICATION FOR AUTOMATIC CLASSIFICATION

The approach described above is based on the use of quadratic loss functions and is intended for solving problems of predicting numerical variables. However, the method can also be used to solve automatic classification problems. When solving the latter, cross-entropy is usually used to estimate the accuracy of the approximation of a dependence, i.e., a value proportional to the minus logarithm of the likelihood function calculated based on the Bernoulli distribution. In the case of a binary classification problem, the target value Y takes values from the set $\{0,1\}$, where the equality $Y = 1$ indicates membership in the target class K. Suppose that some tree T calculates the probabilities of belonging to class K of objects from $S = \{s_1 = (y_1, x_1), \dots, s_m = (y_m, x_m)\}$. Let p_j be the probability of object s_j belonging to the target class, calculated by tree T from its description x_j. The cross-entropy loss for tree T on sample S is estimated by the formula

$$L(T,S) = -\ln\prod_{j=1}^{m} p_j^{y_j}(1-p_j)^{1-y_j} = -\sum_{j=1}^{m}[y_j\ln p_j + (1-y_j)\ln(1-p_j)]. \tag{5}$$

The losses according to the formula (5) are an analogue of the quadratic losses for the tree T at the sample S and correspond to the left term in the formula (2). For the estimation involving the deviation of the new tree T_k from the ensemble A_k, the following approach can be proposed. Let p_j^k be the probability of the object s_j belonging to the target class, calculated by the tree T_k from its description x_j; $\hat{p}_j^k$ be the probability of the object s_j belonging to the target class, calculated by the ensemble A_k. The deviation of the new tree from the ensemble is estimated by the formula

$$D(T_k, A_k, S) = -\ln\prod_{j}^{m}(p_j^k)^{\hat{p}_j^k}(1-p_j^k)^{1-\hat{p}_j^k} = \sum_{j=1}^{m}[-\hat{p}_j^k\ln p_j^k - (1-\hat{p}_j^k)\ln(1-p_j^k)].$$

To find the optimal values of probabilities p_j^k, the same approach can be used as in the case of quadratic losses. At the first stage, the values $p_1^o, \dots, p_m^o$ are sought for which the minimum of the

following functional is achieved

$$Q(T_k, A_k, S) = L(T, S) - \mu D(T_k, A_k, S).$$

Next, a new tree T_k is trained using the sample

$$\{(p_1^o, x_1), \ldots, (p_m^o, x_m)\}.$$

The ensemble A_k at step k is unknown. Therefore, instead of $Q(T_k, A_k, S)$, a similar functional can be used

$$Q(T_k, A_{k-1}, S) = L(T, S) - \mu D(T_k, A_{k-1}, S). \tag{6}$$

It is easy to show that the minimum of the functional (6) is achieved at

$$p_j^o = \frac{y_j - \mu \hat{A}_{k-1}(x_j)}{1 - \mu}. \tag{7}$$

Unfortunately, using (7) leads to a violation of the condition $p_j^o \in [0, 1]$. This condition can be preserved by moving to the problem of conditional optimization. However, the latter leads to a significant complication of the algorithm. A simpler solution is based on the use of formula (7) with subsequent training of a new tree from the sample

$$\{(p_1^o, x_1), \ldots, (p_m^o, x_j)\}. \tag{8}$$

To separate the classes, the estimate of the value p_j is compared with a threshold, for the search of which ROC analysis tools are used. We will further call this method a divergent decision forest (DDF).

4. APPLICATION TO RECOMMENDER SYSTEMS

The method proposed in the previous sections depends on a set of hyperparameters, which include both the multiplier μ and the hyperparameters used in the construction of individual trees. The successful application of most machine learning methods depends on the selected values of the hyperparameters. At the same time, accurate theoretical estimates for the choice of hyperparameters are usually absent. Therefore, their optimal values have to be sought through experiments with data. The goal of this work was an experimental search for optimal values of hyperparameters for a divergent decision forest. The studies were carried out in the context of solving problems of predicting user preferences that arise when creating recommender systems. The choice of this class of problems is associated with both the widespread use of recommender systems in various sectors of the economy, and with the intensive use of machine learning methods in this area [12]. Three problems of assessing the preferences of Internet users when choosing a computer game and two problems of assessing the preferences of Internet users when choosing a sticker in social networks were considered. The assessment was based on information about the interaction of users with the corresponding catalogs. The HR5 (hitrate at 5) indicator was used for assessment, that is the proportion of users for whom at least one relevant object was among the first 5 recommendations. The following notations are used: k is the number of users, p is the number of objects, n is the number of features and M is the number of rows in the training sample. The characteristics of the considered problems are presented in the Table 1.

The aim of the work was to study the dependence of the algorithm accuracy in terms of the HR5 indicator on the depth of the trees and the parameter μ. The results for the game0 and sticker1 tasks are presented in the Table 2 and 3, respectively. The value $\mu = 0$ corresponds to the usual random decision forest (RF) model.

Table 1. Characteristics of the problems considered

problem	k	p	n	M
game0	414	187	41	4132
game1	1706	190	41	16 327
game2	3888	398	55	29 065
sticker1	2685	118	40	27 032
sticker2	5942	197	41	44 613

Table 2. Relationship between HR(5) and tree depth and parameter μ for the game0 problem

	$\mu = 0.0$	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.9$
$depth = 3$	0.584	0.594	0.599	0.613	0.640
$depth = 5$	0.596	0.604	**0.623**	**0.630**	**0.657**
$depth = 7$	0.591	0.606	0.621	0.621	0.623
$depth = 9$	0.592	**0.618**	0.606	0.606	0.570
$depth = 11$	**0.606**	0.601	0.606	0.611	0.493

Table 3. Relationship between HR(5) and tree depth and parameter μ for the sticker1 problem

	$\mu = 0.0$	$\mu = 0.25$	$\mu = 0.5$	$\mu = 0.75$	$\mu = 0.9$
$depth = 3$	0.447	0.449	0.448	0.458	0.514
$depth = 5$	0.482	0.488	0.498	**0.516**	**0.527**
$depth = 7$	0.475	0.491	**0.501**	0.508	0.513
$depth = 9$	0.479	0.492	0.500	0.507	0.477
$depth = 11$	**0.482**	**0.494**	0.499	0.494	0.467

Table 4. The best HR(5) value for each of the models

	RF	DDF ($\mu \neq 0$)	catboost
games0	0.606	0.657	**0.664**
games1	0.642	**0.654**	0.637
games2	0.513	0.550	**0.562**
stickers1	0.482	**0.527**	0.513
stickers2	0.337	0.372	**0.385**
mean	0.516	0.552	0.5522

The tables show that when the tree depth does not exceed 7 the HR(5) increases with the growth of μ. At the same time, when the tree depth is greater than 7, the HR(5) indicator decreases.

Table 4 presents a comparison of the proposed technology with standard random decision forests, as well as with the CatBoost method. Standard decision forests correspond to $\mu = 0$. The table shows that the accuracy in terms of the HR5 metric for the proposed technology significantly exceeds the accuracy of the random decision forest and is close to the accuracy of the CatBoost method.

5. CONCLUSION

A divergent decision forest method has been developed for solving binary classification problems. The method is based on the development of approaches previously proposed for solving regression problems [9–11]. Using the example of problems related to recommender systems, a study of the

divergent decision forest (DDF) method was conducted. The method is based on achieving higher divergence in the prediction space in comparison with the standard random decision forest.

A significant dependence of the efficiency on the tree depth and on the coefficient μ, which regulates the relative contributions of the components responsible for the approximation of the target variable and the divergence of ensembles, is shown. At the same time, with a small tree depth, the accuracy of the algorithm increases with an increase in μ. Such an effect is not observed with a large tree depth. However, in general, DDF outperforms the standard random decision forest corresponding to $\mu = 0$. Experiments have shown that the efficiency of DDF is close to the well-known boosting model CatBoost. The DDF method is based on sequential generation of tree sets. Therefore, its Performance is close to the performance of conventional random forests and gradient boosting variants. One of the problems of the method is the choice of the optimal value of the hyperparameter μ. This problem can be solved using known tools for choosing hyperparameters. However, experiments indicate that higher efficiency is achieved with high values of the hyperparameter, that is, with μ equal to 0.8 or 0.9. Unlike the methods of decision and regression trees, which have high transparency and interpretability, methods using large ensembles of trees, unfortunately, lose these properties. The latter also applies to the DDF method. This drawback can be compensated by various known methods for achieving interpretability. Data mining methods and statistical analysis can also be used to increase transparency.

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Minimax Approach for Using the Qualitative Preferences in the Multicriteria Evaluation

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Abstract—Decision support systems in analytical systems based on the use of big data involve the formation of integral assessments of population objects using all parameters or some subset of them. The article discusses the problem of obtaining a multi-objective (multi-parameter) assessment of objects and an approach that involves the use of importance weights in the presence of high-quality and, possibly, incomplete information about the relative importance of certain criteria. The fundamental principle of various quantitative assessments of the mutual preference of private particular criteria for various objects in the population is considered while maintaining the system of preferences of the entire set of objects. The approach used assumes that the decision maker formulates qualitative information about the relative preference of certain criteria in the form of a not necessarily complete preference graph. For each object, weighting coefficients are calculated automatically according to the principle of a guaranteed result by solving an optimization problem using generalized logical criteria of maximum risk and maximum caution. For special cases of preference systems, analytical relationships and algorithms for calculating weight coefficients are given. This technique ensures the use of additional qualitative information about the preferences of certain criteria, obtaining numerical values of significance weighting coefficients and solving the problem of multicriteria assessment based on the principle of a guaranteed result.

Keywords: multicriteria optimization, multicriteria estimation, interactive procedure, weighting coefficients, qualitative preference information

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1. INTRODUCTION

Problems of multicriteria assessment are widespread in various fields of activity, and, consequently, in various subject areas. In this case, the assessment task is understood as the situation of the need to obtain an integral assessment of the objects in the population for the subsequent final choice of the decision maker. The objective complexity of such problems lies in the impossibility of achieving the best values for all criteria at the same time, so we are talking about choosing a compromise solution, that is, a solution that cannot be improved by any of the criteria without worsening the other criteria. One of the most common approaches to solving such problems is to generate a numerical value estimate for each feasible solution and therefore solve a single-criteria optimization problem or, in the case of a discrete set of feasible solutions, a selection problem [1].

There are approaches to the problem of multi-criteria choice that take into account qualitative information about the preferences of decision makers [2]. The main disadvantage of these methods is the requirement for the decision maker to introduce a sufficiently large number of additional parameters (exact numerical values), the absence (at least partial) of which makes the applicability of these methods impossible. The described technique involves the use of exactly the amount of information that the decision maker is ready to provide (in a specific case, the absence of such information at all) [4–6].

One of the most common methods for solving multicriteria (multiobjective) problems is the use of a generalized optimality criterion, which includes importance weights that reflect the decision maker's idea of the relative importance of certain optimality criteria [2]. The importance of criteria is understood here in the sense of the axiomatic theory of importance, which allows us to assume that if there is additional information of the form the ith criterion is more preferable than the jth criterion ($Q_i \succ Q_j$), then for weighted coefficients, the ratio $w_i > w_j$ should be true. In this case, of course, the decision maker requires exact numerical values of the weighting coefficients.

The weights may be assigned by the decision maker in various ways [3]. All known methods of individual examination require from the decision maker or expert complete and accurate answers (most often with numerical estimates) to the questions that the chosen method asks. The described approach allows the use of incomplete and/or qualitative information about the preferences of particular criteria.

2. MULTICRITERIA ESTIMATION PROBLEM STATEMENT

In general, the problem of multi-criteria choice can be formulated as follows.

The decision maker (DM) has formed a set of a number of feasible alternatives, from which DM needs to make a single and final choice. The choice is made as selection of a certain subset of alternatives (in a particular case, one alternative) such that the decision maker considers them the best and equivalent to each other in terms of the best quality.

Also in practice, the problem of multicriteria (also multiobjective) evaluation of options is considered and solved, as a result of which the decision maker seeks to order the options by preference for further analysis. Obviously, this task (estimation) is a generalization of the previous one (choice) and does not contradict practice.

Traditionally, the following components of the decision-making model are considered:

— initial set of options (solutions, alternatives) from which the choice is made;
— the principle of optimality, on the basis of which the best option (or best options) is selected;
— decision maker (DM), who determines the process of finding a solution, as well as experts and consultants who assist him in this.

Without limiting the generality of the mathematical formulation, we will assume that the set of feasible solutions is a discrete finite set of alternatives, each of which is denoted by a number:

$$D = \{1, \ldots, m\}. \tag{1}$$

The choice task is to choose the variant $x^* \in D = \{1, 2, \ldots, m\}$, which is the best (optimal) from the decision maker's point of view.

Since there is no optimal solution in most cases of a multi-criteria choice problem, the term "rational decision" has become established in the literature, which means the presence of rational reasons understandable to other people that led to the choice of this solution from a set of acceptable ones.

Many approaches to solving problems of multi-criteria choice involve the use of an estimation problem. The task of evaluation is to determine for each alternative a numerical value that characterizes the quality (utility, efficiency, etc.) of this option in terms of the subject area and the relative importance of particular criteria in accordance with the individual opinion of the decision maker. At the same time, for comparability of alternatives, this generalized characteristic should have a direction and scale.

In the future, we will assume that the measurement scales of particular optimality criteria are defined and the numerical value of the assessment by option can be obtained for each option and each criterion.

We will also assume that all partial criteria are reduced to a dimensionless form and to a single measurement scale $[\alpha, \beta]$, while $0 \leqslant \alpha < \beta$.

The general idea of evaluation is to construct a scalar function $F(Q(x))$, which has the property of ordering the original options according to their preference. The procedure for constructing such a function is called folding or combining the vector optimality criterion (as an option, folding or combining particular criteria):

$$\min_{x \in D} Q_1(x), \quad \min_{x \in D} Q_2(x), \quad \cdots, \quad \min_{x \in D} Q_n(x), \tag{2}$$

subject to

$$D = \{1, 2, \ldots, m\} \tag{3}$$

reduces to a one-criterion (scalar) problem

$$\min_{x \in D} \{F(Q_1(x), \ldots, Q_n(x))\}. \tag{4}$$

The function F is called the generalized criterion for the optimality of the problem (2). To take into account the relative importance of particular criteria, weights of the relative importance $w = (w_1, \ldots, w_n)$ are used. Various functions can be used as a generalized criterion, particulally generalized logical criteria of optimality:

"maximal caution" criterion:

$$F_{L_{\min}}(w, Q_1(x), \ldots, Q_n(x)) = \min_{1 \leqslant i \leqslant n} (w_i Q_i(x)), \tag{5}$$

"maximal risk" criterion:

$$F_{L_{\max}}(w, Q_1(x), \ldots, Q_n(x)) = \max_{1 \leqslant i \leqslant n} (w_i Q_i(x)). \tag{6}$$

Where the weights $w_j,\ j = 1, \ldots, n$ reflect DM's view of the importance of private optimality criteria. The importance of criteria is understood here in the sense of the axiomatic theory of importance, which allows us to assume that if additional information of the form ith criterion is no less important than the jth criterion ($Q_i \succeq Q_j$), then the following relation is true for the weight coefficients and:

$$Q_i \succeq Q_j \Leftrightarrow w_i \geqslant w_j. \tag{7}$$

Again, to solve the original selection problem, the exact assignment of weighting coefficients of importance in numerical form is required. In this case, it is usually assumed that the weighting coefficients of importance belong to the region of admissible values of the following form (hereinafter we will refer to it as the region of admissible values of the weighting coefficients):

$$D_w^0 = \left\{ w \in R^n |\ w_i \geqslant 0, \quad i = 1, 2, \ldots, n; \ \sum_{i=1}^{n} w_i = 1 \right\}. \tag{8}$$

It is known that the choice of weight coefficients of the importance of particular criteria in strict accordance with can lead to a certain problem: if for some particular criterion Q_i the value of the corresponding weight coefficient of importance is zero (which does not contradict the construction of the region D_w), then this criterion will essentially be struck out of consideration and it makes no difference what value it will have. Mathematically, this will lead to a weak efficiency of the solutions obtained (proof in [Podinovsky, Nogin]).

In addition, it is known that the sum of the importance weights can be any positive number (for example, equal to 100, if it is convenient to work in percentage terminology).

Therefore, we finally come to the following formulation of the range of feasible values of weight coefficients of importance:

$$D_w^1 = \left\{ w \in R^n |\ w_i \geqslant w_0 > 0,\ \ i = 1, 2, \ldots, n;\ \ \sum_{i=1}^{n} w_i = R \right\}. \tag{9}$$

To solve the original selection problem, the decision maker needs to assign the exact values of the importance weights in numerical form from the range of their feasible values.

Approaches to the definition and methods for assigning weighting coefficients are widely described in the literature. Among them are the following [5]:

— ordering criteria by importance;
— determining the ratio of weight coefficients, while the decision maker specifies the ratio $w_i w_j$ in numerical form;
— construction of tables based on pairwise comparison of criteria by importance;
— method for determining weights using a set of sequential comparisons (Churchman-Akoff method);
— methods that use information about the quality of the optimal values of particular criteria;
— game-theoretic methods for assigning weight coefficients and others.

These methods have a number of significant drawbacks in their practical application.

(1) The statement (7) implies a "transition" from qualitative information ($Q_i \succeq Q_j$) to quantitative information ($w_i \geqslant w_j$). Obviously, this is a big uncertainty, which will eventually lead to different solutions.
(2) All of the listed methods for assigning numerical values of weight coefficients of importance are sequential algorithms that require accurate quantitative information from the decision maker at each step of this algorithm. As soon as the decision maker refuses to answer the next question, the entire algorithm terminates and the numerical values of w are not calculated.

3. ANALYSIS AND USE OF QUALITATIVE INFORMATION ON THE RELATIVE IMPORTANCE OF PARTICULAR OPTIMALITY CRITERIA – MINIMAX APPROACH

In a number of works [4–6], the main idea and concept of this approach are formulated, which are as follows.

(1) It is assumed that when using generalized optimality criteria with weighting coefficients of importance, their values can be different for each feasible solution of the domain D.
(2) The decision maker can formulate preferences on the set of particular criteria in the form of a set of relations, not necessarily for all possible pairs of particular criteria.
(3) Importance weight values are calculated automatically based on the guaranteed result principle. That is, for each feasible solution, such a set $w \in D_w$ is determined that provides the "worst" value of the generalized criterion for all possible values of the weight coefficients.

Let us consider in more detail a particular, but often occurring case, in which the decision maker establishes mutual preference for some L pairs of particular criteria (not necessarily for all C_n^2 possible pairs) preference of the ith particular criterion over the jth one on the entire set D of feasible solutions:

$$e_l = \{Q_i \succeq Q_j\},\ l = 1, 2, \ldots, L \leqslant \frac{n(n-1)}{2}. \tag{10}$$

The information (10) is qualitative, as it follows from it that the criterion i is more important than the criterion j, but it is impossible to say how much or how many times. Then, taking into account the relation (10), the range of acceptable values of the weight coefficients of importance D_w, defined as (8), narrows according to the relations introduced by the decision maker:

$$D_w^2 = \left\{ w \in R^n \;\middle|\; \begin{array}{l} w_i \geqslant w_0 > 0,\ i = 1, 2, \ldots, n; \\ \sum\limits_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant w_j \rangle_{e_l},\ l = 1, 2, \ldots, L \end{array} \right\}. \tag{11}$$

In some cases, the decision maker has the ability to clarify information about the relationship of weight coefficients w_i and w_j associated with a binary relation $\{Q_i \succeq Q_j\}$ using the parameter $g_l \geqslant 1$:

$$D_w^3 = \left\{ w \in R^n \;\middle|\; \begin{array}{l} w_i \geqslant w_0 > 0,\ i = 1, 2, \ldots, n; \\ \sum\limits_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant g_l w_j \rangle_{e_l},\ l = 1, 2, \ldots, L \end{array} \right\}. \tag{12}$$

If the ares of feasible values of the weight coefficients (11) and (12) are not empty, then the qualitative information is consistent.

In several works [4, 6], an approach was proposed and developed in which the weight coefficients of the importance of particular criteria are uncontrollable factors, and their values can be different for different alternatives.

Sometimes, in the decision making process, it is reasonable to take into account the dependence of the weight coefficients on the value of the particular criteria at each point of the domain of feasible solutions. Assuming that the decision maker can not accurately determine the numerical values of the weight coefficients w, they can be considered as uncontrolled factors and, applying the principle of guaranteed result, proceed to the following decision-making model:

$$x^* = \arg \min_{x \in D} \max_{w \in D_w} F(q(x), w) \tag{13}$$

subject to

$$D = \{1, 2, \ldots, N_D\}, \tag{14}$$

$$D_w^3 = \left\{ w \in R^n \;\middle|\; \begin{array}{l} w_i \geqslant w_0 > 0,\ i = 1, 2, \ldots, n; \\ \sum\limits_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant g_l w_j \rangle_{e_l},\ l = 1, 2, \ldots, L \end{array} \right\}. \tag{15}$$

4. QUALITY INFORMATION GRAPH

Qualitative information about the expert's preferences can be represented as a directed graph $G(V, E)$, where V is the set of vertices corresponding to particular criteria, E is the set of edges connecting the ith vertex with the jth vertex if and only if the relation $(Q_i \succeq Q_j)$ is completed.

In what follows, we will assume that information messages (7) satisfy the transitivity condition, that is, there are no closed cycles in the graph $G(V, E)$.

Let us divide the set I of graph vertices into tiers as follows:

— the first tier ($s = 1$) includes all vertices that do not include any arc;

— to the second tier ($s = 2$) those and only those vertices that include arcs from the vertices of the first layer;

— to the third tier ($s = 3$) those and only those vertices that include arcs from the vertices of the first and second layers, etc.

In this case, the vertices of the last tier ($s = S \leqslant N$) will not have a single outgoing arc.

In the special case of the absence of qualitative information (7), the graph $G(V, E)$ will be a collection of N isolated vertices. And in the particular case of linear ordering of particular criteria, the graph $G(V, E)$ will have the number of layers equal to the number of vertices ($S = n$).

As an example, consider a preference graph (Fig. 1) containing six vertices (criteria) and additional qualitative information introduced by the decision maker: $e_1 = \{Q_6 \succeq Q_4\}$, $e_2 = \{Q_6 \succeq Q_3\}$, $e_3 = \{Q_4 \succeq Q_1\}$, $e_4 = \{Q_3 \succeq Q_1\}$, $e_5 = \{Q_3 \succeq Q_2\}$, $e_6 = \{Q_1 \succeq Q_5\}$.

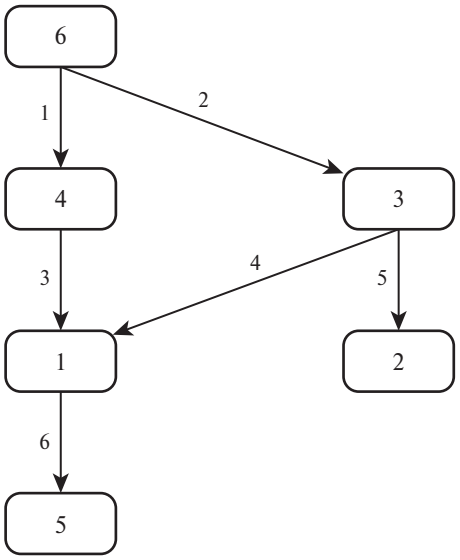

Fig. 1. Example of the preference graph.

For each vertex of the graph $v_i \in \{1, 2, \ldots, n\}$ corresponding to the particular criterion Q_i, we introduce the following notation: V_i is the set of vertices of the graph $G(V, E)$ from which there is a path to vertex i, including itself; n_i is the cardinality of the set V_i.

Let p be an arbitrary path in the graph $G(V, E)$: $p = \{l_1, \ldots, l_n(p)\}$, where $l_i \in E$ is an arc included in the path p; $n(p)$ is the number of arcs in the path p. Denote by P_i^k the set of all paths from vertex i to vertex k. Then we introduce the quantities $\bar{g}_k^i$ and $\bar{\bar{g}}_i$ as follows:

$$\bar{g}_k^i = \begin{cases} \max\limits_{p \in P_i^k} \prod\limits_{l \in p} g_l, & P_i^k \neq \emptyset \\ 1, & P_i^k = \emptyset \end{cases} \tag{16}$$

for each $i, k \in E$;

$$\bar{\bar{g}}_i = \max_{k \in E, k \neq i} \bar{g}_k^i. \tag{17}$$

Obviously, $\bar{\bar{g}}_i$ is a generalized estimate of the relative importance of a particular criterion i under the conditions of qualitative information (10). An example of calculating these quantities for the

graph shown in Fig. 1 is below. For all arcs of the graph, let us assign refinement coefficients g_l, $l = 1, \ldots, 6$ (Table 1).

Table 1. The example of refinement coefficients

Number of arc l	Coefficient g_l
1	1.2rule0pt3.4mm
2	1.3
3	1.0
4	1.5
5	1.3
6	2.0

Calculation of values (16) and (17) for all vertices of the graph is given in Table 2.

Table 2. The example of vertex characteristic calculation

Vertex i	Values of $\bar{g}_i^k (P_i^k \neq \emptyset)$	$\bar{\bar{g}}_i$
1	$\overline{g}_1^5 = 2.0$	2.0
2	–	1.0
3	$\bar{g}_3^5 = 2.0 \times 1.5 = 3.0; \bar{g}_3^2 = 1.3$	3.0
4	$\bar{g}_4^5 = 1.5 \times 1.0 = 1.5$	1.5
5	–	1.0
6	$\bar{g}_6^5 = \max\{1.2 \times 1.0 \times 2.0; 1.3 \times 1.5 \times 2.0\} = 3.9; \bar{g}_6^2 = 2.0 \times 1.3 = 2.6$	3.9

Algorithm 1

Require: $s = S; \bar{g}_i^k = 0, i, k \in \{1, \ldots, n\}$.

1: For all vertices i from tier s let $\bar{\bar{g}}_i = 1$.

2: $s \leftarrow s - 1$. If $s = 0$ (that is, all tiers are passed), then go to step 4, otherwise go to step 3.

3: For each vertex k belonging to layer s, we perform the following actions. Let J_k be the set of vertices in the layer $(s + 1)$ that have an arc from vertex k. Then, for each vertex $j \in J_k$, we put:

$$\bar{g}_i^k = g_l, \text{where} \quad l \quad \text{is number of vertex } (k, j) \quad \text{in the graph} \quad G(V, E);$$

$$\bar{g}_k^i = \max_{j \in J_k} \left\{ \bar{g}_k^j; \bar{g}_j^i \right\}, i \neq j; \ \bar{\bar{g}}_k = \max_{j \in J_k} \left\{ \bar{g}_k^j; \bar{\bar{g}}_k \right\}$$

and go to step 2.

4: For all $\bar{g}_i^k$, for which $\bar{\bar{g}}_k = 0, k, i \in \{1, \ldots, n\}$ assign $\bar{g}_i^k = 1$.

5. WEIGHTING COEFFICIENTS CALCULATION FOR MAXIMAL CAUTION CRITERIA

5.1. Common Case

Consider the solution to the problem of calculating weight coefficients of importance $w \in D_w^4$ using the generalized logical criterion "maximum caution":

$$x^* = \arg \min_{x \in D} \max_{w \in D_w} \max_{1 \leqslant i \leqslant n} F(q(x), w), \tag{18}$$

$$D_w^1 = \left\{ w \in R^n \;\middle|\; \begin{array}{l} w_i \geqslant w_0 \geqslant 0, \ i = 1, 2, \ldots, n; \\ \displaystyle\sum_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant g_l w_j \rangle_{e_l}, \ l = 1, 2, \ldots, L \end{array} \right\}. \tag{19}$$

For fixed values of the variable parameters $x \in D$, the values of the vector criterion $q(x)$ are definite.

Applying a linear model for determining mixed strategies in a zero sum matrix game [14], in the general case, the solution to problem (18)–(19) can be obtained by solving the following linear optimization problem:

$$\max v \tag{20}$$

subject to

$$D = \left\{ w \in R^n \left| \begin{array}{l} w_i \geqslant w_0 \geqslant 0,\ i = 1, 2, \ldots, n; \\ \sum\limits_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant g_l w_j \rangle_{e_l},\ l = 1, \ldots, L \\ w_i q_i(x) \geqslant v,\ i = 1, \ldots, n \end{array} \right. \right\}. \tag{21}$$

Consider an example of calculating weight coefficients. Let us assume that the values of partial optimality criteria for one feasible solution, normalized to the interval $[0, 1]$, are given in Table 3. In this table calculated weighting coefficients are given as well.

Table 3. The example of criteria values of the feasible solution and calculated weighting coefficients

Criteria	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
Criteria values q_i	0.1	1.0	0.5	0.3	0.9	0.3
Weighting coefficients w_i	0.1766	0.1767	0.2650	0.1767	0.0196	0.3445

The above solution to the problem of calculating the weight coefficients of importance of particular criteria was obtained using the LPSolv software package [15]. The source code in LP format to solve the problem is given below.

Listing 1. Source LP code

```
/* Objective function */
max: v;

/* Variable bounds */
w1 + w2 + w3 + w4 + w5 + w6 = 1;

w1 >= 0.01;
w2 >= 0.01;
w3 >= 0.01;
w4 >= 0.01;
w5 >= 0.01;
w6 >= 0.01;

w6 >= 1.2 * w4;
w6 >= 1.3 * w3;
w4 >= 1.0 * w1;
w3 >= 1.5 * w1;
w3 >= 1.3 * w2;
w3 >= 2.0 * w5;
```

0.1 w1 >= v;
1.0 w2 >= v;
0.5 w3 >= v;
0.3 w4 >= v;
0.9 w5 >= v;
0.3 w6 >= v;

The above solution was obtained by solving a linear optimization problem using the simplex method. For special cases of qualitative information about preferences and, therefore, the range of acceptable values of importance weight coefficients, the calculation of coefficient values can be carried out by faster algorithms.

5.2. Calculation in Particular Cases

Consider the solution to the problem of calculating weight coefficients of importance $w \in D_w^2$ (11) using the generalized logical criterion "maximum caution":

$$x^* = \arg\min_{x\in D}\max_{w\in D_w} F(q(x), w), \tag{22}$$

$$D_w^1 = \left\{ w \in R^n \;\middle|\; \begin{array}{l} w_i \geqslant w_0 \geqslant 0,\ i = 1, 2, \ldots, n; \\ \sum\limits_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant w_j \rangle_{e_l},\ l = 1, 2, \ldots, L \end{array} \right\}. \tag{23}$$

To calculate the values of the weight coefficients of importance w in the case of the condition of their non-negativity, that is, for the case $w_0 = 0$, the Algorithm 2 can be applied.

Algorithm 2

1: The graph $G(V, E)$ is divided to tiers.
2: Assigning an initial value $w'_j \leftarrow 0$ to each vertex j of the graph $G(V, E)$, $j = 1, \ldots, n$.
3: Consider the lowest layer as the first layer: $s \leftarrow S$.
4: $w'_j \leftarrow \max\left\{w'_j, R/q_j(x)\right\}$ to all vertices j of tier s.
5: Adjustment of w'_j values for vertices of higher layers:

$$w'_j = \max\left\{w'_j, \max_{k\in V_j} w'_k\right\},$$

V_j is the set of vertices, which have path to vertex j.
6: Let $s = s - 1$. If $s = 0$ i.e. all tiers have been passed, go to step 7. Otherwise repeat from the step 4.
7: Calculate final values of weighting coefficients:

$$w_j^* = \frac{R \times w'_j}{\sum\limits_{i=1}^{n} w'_i}$$

Consider a generalization of the extrema problem (22)–(23) for the case $w_0 \geqslant 0$. To solve this problem, the Algorithm 3 can be applied.

The proof of the correctness of the above algorithms is contained in [5].

Algorithm 3

Require: $V' = V = \{1, \ldots, n\}; R' = R; E' = E$.

1: Apply Algorithm 2 to graph $G(V, E)$ with $R = R'$ and get values w'_j.

2: If for all $j \in V'$ it is true $w'_j \geqslant w_0$ then assign $w^*_j = w'_j, j \in V'$ and the task has been solved. Otherwise go to the step 4.

3: For all vertex $m \in V'$, for which $w'_m < w_0$, make the following:
- assign $w'_m = w_0; R' = R' - w_0$;
- exclude the vertex m from consideration, for this assign $V' = V' \setminus m$ and from the set E' we exclude the arcs incident to the vertex m, if any.

4: Repeat from step 1.

6. WEIGHTING COEFFICIENTS CALCULATION FOR MAXIMAL RISK CRITERIA

Consider the solution to the problem of calculating weight coefficients of importance $w \in D^4_w$ (6) using the generalized logical criterion "maximum risk":

$$x^* = \arg \max_{w \in D_w} \max_{1 \leqslant i \leqslant n} F(q(x), w) \tag{24}$$

subject to

$$D^1_w = \left\{ w \in R^n \;\middle|\; \begin{array}{l} w_i \geqslant w_0 \geqslant 0, \; i = 1, 2, \ldots, n; \\ \displaystyle\sum_{i=1}^{n} w_i = R; \\ \langle w_i \geqslant g_l w_j \rangle_{e_l}, \; l = 1, 2, \ldots, L \end{array} \right\}. \tag{25}$$

For fixed values of the variable parameters $x \in D$, the values of the vector criterion $q(x)$ are definite as well.

The optimal solution of the optimization task (24)–(25) is the w^*, which has components:

$$w^*_i = \begin{cases} \dfrac{(R' \bar{g}^r_i)}{\displaystyle\sum_{k \in V_r} \bar{g}^r_k} + \bar{\bar{g}}_i w_0, & i \in V_r; \\ \bar{\bar{g}}_i w_0, & i \notin V_r, \end{cases} \tag{26}$$

here

$$R' = R - w_0 \times \sum_{i=1}^{n} \bar{\bar{g}}_i \geqslant 0, \tag{27}$$

values $\bar{g}^r_i$ and $\bar{\bar{g}}_i$ are determined by (16) and (17), respectively; V_i is the set of vertices of the graph $G(V, E)$ from which there is a path to vertex i, including itself; n_i is the cardinality of the set V_i, value r is obtained from the relations:

$$y_r = \max_{1 \leqslant k \leqslant n} y_k, \tag{28}$$

$$y_k = \max_{i \in V_k} \left(q_i(x) \left(\frac{(R' \bar{g}^r_i)}{\displaystyle\sum_{j \in V_r} \bar{g}^r_j} + \bar{\bar{g}}_i w_0 \right) \right). \tag{29}$$

The proof of formulas (24)–(29) is given in [10].

7. CONCLUSION

A technique for calculating the weight coefficients of the importance of particular criteria in solving problems of multi-criteria choice and evaluation is considered. The fundamental principle of different values of importance weight coefficients with a constant system of qualitative preferences allows the decision maker to formulate preferences in the form of a directed graph, which, in the case of completeness of the graph, is a strict order relation. This technique can be used both in building interactive decision-making systems and for reducing a multi-criteria problem into a single-criteria one for use in various optimization methods.

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