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NONLINEAR SYSTEMS

Energy Quasi-Optimal Angular Acceleration of a Spacecraft Based on the Poinsot Concept

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Abstract—This paper is devoted to the kinematic problem of the optimal, in terms of energy consumption, program angular acceleration of a spacecraft under arbitrary boundary conditions imposed on its angular position and angular velocity. A quasi-optimal analytical solution of the problem is obtained within the classical Poinsot concept of the angular motion of a solid body as a generalized coning motion and Pontryagin's maximum principle. This solution is brought to an algorithm. Supporting analytical and numerical examples are provided to show either the proximity of the quasi-optimal and optimal solutions or their complete coincidence, depending on the boundary conditions.

Keywords: spacecraft, solid body, angular acceleration, Poinsot concept, optimal and quasi-optimal control, analytical solution, algorithm

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1. INTRODUCTION

In many spacecraft guidance and control problems, it is required to know the optimal, in some sense, program angular acceleration of a spacecraft under arbitrary (given) boundary conditions [1–5]. In the literature, as a rule, such acceleration is either numerically found or analytically constructed (e.g., [5]) based on polynomials (splines) by representing the spacecraft attitude quaternion via polynomials and expressing the angular velocity vector through this quaternion. However, no guarantees (rigorously proved theorems or considerations from theoretical mechanics) are given that these analytical solutions will approximate the optimal angular motion trajectory of the spacecraft well enough on the entire set of its angular motions under any boundary conditions.

In this paper, we study the kinematic problem of the optimal, in terms of energy consumption, program angular acceleration of a spacecraft under arbitrary boundary conditions for its angular position and angular velocity. It is difficult to obtain the optimal analytical solution of this problem under arbitrary boundary conditions: in the general case, the exact solution of the attitude problem of a solid body by its known angular velocity (the Darboux problem) remains unknown [6, 7]. Following the classical Poinsot concept of the angular motion of a solid body as a generalized coning motion, we reformulate the optimal angular acceleration problem of a spacecraft in this class of motions. The spacecraft trajectory is given by explicit expressions containing arbitrary quaternionic and scalar constants and two arbitrary scalar functions (the parameters of the generalized coning motion). These parameters and their derivatives are employed as variables to pose and solve an optimization problem in which the second derivatives of the parameters act as controls. Note that the generality of the original problem is almost not violated: the known exact solution of the classical optimal angular acceleration problem in the planar rotation case [4] or in the new special case of regular precession (considered below) and the similar solutions of the problem within the above concept completely coincide; in other cases, in numerical calculations of the solution of the original problem and the analytical solution proposed, the relative error between the values of an optimization criterion (the determinative characteristic of the problem) constitutes at most 1%, including spacecraft rotations to large angles. Therefore, the analytical solution proposed in this paper can be treated as quasi-optimal with respect to that of the classical optimal angular acceleration problem of a spacecraft. Explicit expressions for the attitude quaternion and the angular velocity vector of a spacecraft are derived. An explicit formula for the angular acceleration of a spacecraft is obtained by differentiating the expression for the angular velocity vector. Finally, an analytical algorithm for solving the problem is presented, which can be used aboard the spacecraft.

The analytical solution method proposed below has previously been successfully applied to dynamic optimal rotation problems for spacecraft of various configurations with different optimization criteria [8, 9], as well as as to the design of quasi-optimal (in terms of time) angular acceleration of a spacecraft under arbitrary boundary conditions [10].

2. CLASSICAL PROBLEM FORMULATION

The angular motion of a spacecraft as a solid body around its center of gravity is described by a system of differential equations in vector-matrix form:

$$2\begin{bmatrix} \dot\lambda_0 \\ \dot\lambda_1 \\ \dot\lambda_2 \\ \dot\lambda_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & -\lambda_2 & -\lambda_3 \\ \lambda_0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & \lambda_0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & -\lambda_0 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}, \tag{2.1}$$

$$\begin{bmatrix} \dot\omega_1 & \dot\omega_2 & \dot\omega_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \end{bmatrix}^{\mathrm{T}}, \tag{2.2}$$

which is equivalent to the short representation [4]

$$2\dot{\mathbf{\Lambda}} = \mathbf{\Lambda} \circ \boldsymbol{\omega}, \tag{2.3}$$

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon}, \tag{2.4}$$

where $\mathbf{\Lambda}(t) = [\lambda_0(t),\, \lambda_1(t),\, \lambda_2(t),\, \lambda_3(t)]^{\mathrm{T}}$ or $\mathbf{\Lambda}(t) = \lambda_0(t) + \lambda_1(t)i_1 + \lambda_2(t)i_2 + \lambda_3(t)i_3$ is the normalized quaternion that describes the angular position of the spacecraft ($||\mathbf{\Lambda}|| = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$); $\boldsymbol{\omega} = [\omega_1(t),\, \omega_2(t),\, \omega_3(t)]^{\mathrm{T}}$ or $\boldsymbol{\omega}(t) = \omega_1(t)\mathbf{i}_1 + \omega_2(t)\mathbf{i}_2 + \omega_3(t)\mathbf{i}_3$ is the angular velocity vector of the spacecraft, which can be treated as a quaternion with the zero scalar part $\boldsymbol{\omega}(t) = [0,\, \omega_1(t),\, \omega_2(t),\, \omega_3(t)]^{\mathrm{T}}$; $\boldsymbol{\varepsilon} = [\varepsilon_1(t),\, \varepsilon_2(t),\, \varepsilon_3(t)]^{\mathrm{T}}$ is the angular acceleration vector of the spacecraft; the imaginary units i_1, i_2, and i_3 of the Hamiltonian correspond to the orthonormal basis vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$, respectively, of the three-dimensional vector space; $\circ$ indicates the quaternion product; finally, $^{\mathrm{T}}$ means the vector transpose. The phase coordinates $\mathbf{\Lambda}$ and $\boldsymbol{\omega}$ are assumed to be continuous functions, and the angular acceleration $\boldsymbol{\varepsilon}$ (control) is assumed to be a piecewise continuous [11].

The boundary conditions imposed on the attitude and angular velocity of the spacecraft are arbitrary:

$$\mathbf{\Lambda}(0) = \mathbf{\Lambda}_0, \quad \mathbf{\Lambda}(T) = \mathbf{\Lambda}_T, \tag{2.5}$$

$$\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \quad \boldsymbol{\omega}(T) = \boldsymbol{\omega}_T. \tag{2.6}$$

It is required to find the optimal control $\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)$ that minimizes the energy consumption functional

$$J = \int_0^T ||\boldsymbol{\varepsilon}||\, dt, \tag{2.7}$$

where the time T is arbitrary and fixed.

The problem formulation (2.1)–(2.7) is kinematic since it neglects the dynamic configuration of the spacecraft. Note that the optimal angular acceleration of the spacecraft can be found as the derivative of its optimal angular velocity in the full dynamic optimal attitude problem of a spacecraft, where the Euler dynamic equations are used instead of the vector equation (2.4) and the control function is the vector of the external control moment applied to the spacecraft [8, 9].

3. DIMENSIONLESS VARIABLES

We reformulate the problem by replacing the original dimensional variables with dimensionless ones:

$$t^{\text{dimless}} = t^{\text{dim}}/T, \quad \boldsymbol{\omega}^{\text{dimless}} = \boldsymbol{\omega}^{\text{dim}}T, \quad \boldsymbol{\varepsilon}^{\text{dimless}} = \boldsymbol{\varepsilon}^{\text{dim}}T^2, \quad J^{\text{dimless}} = J^{\text{dim}}T^3.$$

In this case, all the expressions will not change, except for the optimization functional

$$J = \int\limits_0^1 ||\boldsymbol{\varepsilon}||\, dt. \tag{3.1}$$

Below we will solve problem (2.3)–(2.6), (3.1) in the dimensionless variables for $T = 1$, with the removed superscripts in the problem formulation.

4. REDUCTION TO A BOUNDARY VALUE OPTIMIZATION PROBLEM

Following Pontryagin's maximum principle [4, 11], we introduce a quaternion $\boldsymbol{\Psi}(t)$, conjugate to the spacecraft attitude quaternion $\boldsymbol{\Lambda}(t)$, and a vector $\boldsymbol{\varphi}(t)$, conjugate to the spacecraft angular velocity vector $\boldsymbol{\omega}(t)$. The Hamilton–Pontryagin function is given by

$$H = -\left(\boldsymbol{\varepsilon},\, \boldsymbol{\varepsilon}\right) + \left(\boldsymbol{\Psi},\, \boldsymbol{\Lambda} \circ \boldsymbol{\omega}\right)/2 + \left(\boldsymbol{\varphi},\, \boldsymbol{\varepsilon}\right), \tag{4.1}$$

where (. , .) means the inner product of vectors.

The conjugate system of equations has the form

$$\begin{cases} 2\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \circ \boldsymbol{\omega} \\ \dot{\boldsymbol{\varphi}} = -\text{vect}\left(\widetilde{\boldsymbol{\Lambda}} \circ \boldsymbol{\Psi}\right)/2, \end{cases} \tag{4.2}$$

where vect(.) and $\tilde{}$ are the vector part and conjugation of the quaternion, respectively. The linear differential systems of equations for the variables $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}$ coincide, hence their solutions differ by a quaternion constant $\mathbf{C}$:

$$\boldsymbol{\Psi} = \mathbf{C} \circ \boldsymbol{\Lambda}. \tag{4.3}$$

Therefore, in view of the notation adopted in [4]:

$$\mathbf{p} = \text{vect}\ \left(\widetilde{\boldsymbol{\Lambda}} \circ \boldsymbol{\Psi}\right) = \widetilde{\boldsymbol{\Lambda}} \circ \mathbf{c}_v \circ \boldsymbol{\Lambda}, \tag{4.4}$$

where $\mathbf{c}_v$ is the vector part of the quaternion $\mathbf{C}$, the expressions (4.2) become

$$\begin{cases} \mathbf{p} = \widetilde{\boldsymbol{\Lambda}} \circ \mathbf{c}_v \circ \boldsymbol{\Lambda} \\ \dot{\boldsymbol{\varphi}} = -\mathbf{p}/2. \end{cases} \tag{4.5}$$

Thus, due to the self-conjugation of the linear differential system of equations (2.1) (or (2.3)), the dimension of the boundary value problem of the maximum principle is reduced by four [4]. The maximum condition (3.1) yields the following continuous structure of optimal control:

$$\boldsymbol{\varepsilon}^{\text{opt}} = \boldsymbol{\varphi}/2. \tag{4.6}$$

From (2.3), (2.4), (4.4), and (4.5) it follows that $\dot{\boldsymbol{\omega}} = \boldsymbol{\varphi}/2$, $\ddot{\boldsymbol{\omega}} = -\mathbf{p}/4$, and $\dddot{\boldsymbol{\omega}} = -\dot{\mathbf{p}}/4 = \ddot{\boldsymbol{\omega}} \times \boldsymbol{\omega}$, where the symbol $\times$ means the vector product. Written as a single equation, they become

$$\dddot{\boldsymbol{\omega}} = \ddot{\boldsymbol{\omega}} \times \boldsymbol{\omega}. \tag{4.7}$$

As a result, the optimal angular acceleration of the spacecraft is found by solving the boundary value problem (2.3), (4.7), (2.5), (2.6).

Using the expression (4.4), we write the Hamilton–Pontryagin function (4.1) as

$$H = -\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}\right) + \left(\mathbf{p}, \boldsymbol{\omega}\right)/2 + \left(\boldsymbol{\varphi}, \boldsymbol{\varepsilon}\right). \tag{4.8}$$

Based on formulas (2.3), (2.4), and (4.5), we also indicate the vector first integral [12] relating the phase and conjugate variables of the problem: $\mathbf{p}/4 + \boldsymbol{\varphi} \times \boldsymbol{\omega} = \mathbf{const}\ \forall\ t \in [0, T]$.

Despite that this paper considers the problem without control constraints (the constraint on the magnitude of the angular acceleration is indirectly realized here by minimizing the integral control-quadratic functional), the expressions (4.4)–(4.6) can be used to estimate the magnitude of the optimal angular acceleration vector. For a continuously differentiable vector function $\boldsymbol{\varepsilon}^{\mathrm{opt}}$ of the optimal control, we have

$$\mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(t) \mid = \left| \int\limits_0^t \dot{\boldsymbol{\varepsilon}}^{\mathrm{opt}}(\tau)\, d\tau + \boldsymbol{\varepsilon}^{\mathrm{opt}}(0) \right| \leq \int\limits_0^t \mid \dot{\boldsymbol{\varepsilon}}^{\mathrm{opt}}(\tau) \mid\ d\tau + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(0) \mid$$

$$= \int\limits_0^t \mid \mathbf{p}(\tau) \mid\ d\tau/4 + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(0) \mid = \mid \mathbf{c}_v \mid t/4 + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(0) \mid.$$

In other words, at any time instant $t \in [0, 1]$, the magnitude of the vector function $\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)$ must lie inside a sphere of variable radius

$$||\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)|| \leq \left(\mid \mathbf{c}_v \mid t/4 + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(0) \mid\right)^2\ \forall\ t \in [0, 1]. \tag{4.9}$$

Similarly, on the other side, we obtain

$$||\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)|| \leq \left(\mid \mathbf{c}_v \mid (1-t)/4 + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(1) \mid\right)^2\ \forall\ t \in [0, 1]. \tag{4.10}$$

Consequently, the magnitude of the optimal angular acceleration vector must belong to the intersection of the sets defined by inequalities (4.9), (4.10), i.e.,

$$\begin{gathered} ||\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)|| \leq \gamma^2(t)\ \forall\ t \in [0, 1], \\ \gamma^2(t) = \min_t \left\{ \Big(\mid \mathbf{c}_v \mid t/4 + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(0) \mid\Big)^2, \Big(\mid \mathbf{c}_v \mid (1-t)/4 + \mid \boldsymbol{\varepsilon}^{\mathrm{opt}}(1) \mid\Big)^2 \right\}, \end{gathered} \tag{4.11}$$

where the vector constant $\mathbf{c}_v$ is determined from the given constants of the problem statement.

Note that in the special case of the boundary conditions on the angular velocity (2.6) corresponding to a planar Euler rotation, when the exact solution of the problem is known, $\mid \mathbf{c}_v \mid$ in (4.11) is explicitly expressed through the given boundary conditions on the spacecraft angular position (2.5):

$$\mid \mathbf{c}_v \mid = \mid \mathrm{vect}(\boldsymbol{\Lambda}_T \circ \widetilde{\boldsymbol{\Lambda}}_0)/\sin(\arccos(\mathrm{scal}(\boldsymbol{\Lambda}_T \circ \widetilde{\boldsymbol{\Lambda}}_0))) \mid, \tag{4.12}$$

where scal(.) means the scalar part of the quaternion, and the magnitude of the vector function $\boldsymbol{\varepsilon}^{\mathrm{opt}}(t)$ can reach the boundary of the region (4.11), (4.12).

5. PARTIAL SOLUTION OF THE PROBLEM IN ELEMENTARY FUNCTIONS

In the class of regular coning motions, we obtain a new analytical partial solution of the optimal angular acceleration problem of a spacecraft, expressed in elementary functions. In this class of motions, the optimal angular velocity of the spacecraft has the form

$$\boldsymbol{\omega}(t) = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_1 \beta \sin \gamma t + \mathbf{i}_2 \beta \cos \gamma t + \mathbf{i}_3 \gamma) \circ \mathbf{Q}, \tag{5.1}$$

where $\mathbf{Q}$ (quaternion), β, and γ are arbitrary constants; in addition,

$$||\mathbf{K}|| = \mathrm{q}_0^2 + \mathrm{q}_1^2 + \mathrm{q}_2^2 + \mathrm{q}_3^2 = 1. \tag{5.2}$$

The quaternion $\mathbf{Q}$ defines the rotation of the vector $\mathbf{i}_1\beta\sin\gamma t + \mathbf{i}_2\beta\cos\gamma t + \mathbf{i}_3\gamma$ about some constant axis passing through a fixed point of the spacecraft. Let us show that the angular velocity (5.1) satisfies equation (4.7). Differentiating the expression (5.1) three times with respect to t yields $\dot{\boldsymbol{\omega}}(t) = \beta\gamma\widetilde{\mathbf{Q}} \circ (\mathbf{i}_1\cos\gamma t - \mathbf{i}_2\sin\gamma t) \circ \mathbf{Q}$, $\ddot{\boldsymbol{\omega}}(t) = -\beta\gamma^2\widetilde{\mathbf{Q}} \circ (\mathbf{i}_1\sin\gamma t + \mathbf{i}_2\cos\gamma t) \circ \mathbf{Q}$, and $\dddot{\boldsymbol{\omega}}(t) = \beta\gamma^3\widetilde{\mathbf{Q}} \circ (-\mathbf{i}_1\cos\gamma t + \mathbf{i}_2\sin\gamma t) \circ \mathbf{Q}$. By substituting (5.1) and the above derivatives into (4.7), we verify the desired equality with $\dddot{\boldsymbol{\omega}} = \ddot{\boldsymbol{\omega}} \times \boldsymbol{\omega} = (\ddot{\boldsymbol{\omega}} \circ \boldsymbol{\omega} - \boldsymbol{\omega} \circ \ddot{\boldsymbol{\omega}})/2$.

Based on (2.3) and (5.1), the spacecraft trajectory can be explicitly written in the form of a regular precession:

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{Q}} \circ \mathbf{exp}\{\mathbf{i}_2\beta t/2\} \circ \mathbf{exp}\{\mathbf{i}_3\gamma t/2\} \circ \mathbf{Q}, \tag{5.3}$$

where $\mathbf{exp}\{.\}$ denotes the quaternion exponential [4].

The expressions (5.1)–(5.3) contain five arbitrary constants β, γ, and q_i, $i = 0, 1, 2$. (The constant q_3 is related to condition (5.2).) Let us satisfy the boundary conditions (2.5), (2.6). Due to the insufficient number of arbitrary constants in (5.1), we impose requirements on $|\boldsymbol{\omega}_0|$ and $\boldsymbol{\omega}_T$ in the course of solving the problem, while the unit vector $\boldsymbol{\omega}_0^e = \boldsymbol{\omega}_0/|\boldsymbol{\omega}_0|$ is arbitrary and given. At the point $t = 0$, by formula (5.1),

$$\boldsymbol{\omega}_0 = |\boldsymbol{\omega}_0|\,\boldsymbol{\omega}_0^e = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_2\beta + \mathbf{i}_3\gamma) \circ \mathbf{Q}, \tag{5.4}$$

$$\|\boldsymbol{\omega}_0\| = \left\|\widetilde{\mathbf{Q}} \circ (\mathbf{i}_2\beta + \mathbf{i}_3\gamma) \circ \mathbf{Q}\right\| = \left\|\widetilde{\mathbf{Q}}\right\| \|\mathbf{i}_2\beta + \mathbf{i}_3\gamma\| \; \|\mathbf{Q}\| = \beta^2 + \gamma^2; \tag{5.5}$$

at the right end of the spacecraft trajectory for $t = T = 1$, from (5.3) we have

$$\boldsymbol{\Lambda}_T = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{Q}} \circ \mathbf{exp}\{\mathbf{i}_2\beta/2\} \circ \mathbf{exp}\{\mathbf{i}_3\gamma/2\} \circ \mathbf{Q}, \tag{5.6}$$

with

$$\mathrm{scal}(\widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T) = \mathrm{scal}(\mathbf{exp}\{\mathbf{i}_2\beta/2\} \circ \mathbf{exp}\{\mathbf{i}_3\gamma/2\}), \tag{5.7}$$

where scal(.) means the scalar part of the quaternion. Based on (5.2) and (5.4)–(5.6), we find $|\boldsymbol{\omega}_0|$, β, γ, and $\mathbf{Q}$.

Let us represent (5.4) and (5.6) in the form

$$(\mathbf{i}_2\beta + \mathbf{i}_3\gamma) \circ \mathbf{Q} - \mathbf{Q} \circ \boldsymbol{\gamma}_0 = 0,$$
$$\mathbf{exp}\{\mathbf{i}_2\beta/2\} \circ \mathbf{exp}\{\mathbf{i}_3\gamma/2\} \circ \mathbf{Q} - \mathbf{Q} \circ \widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T = 0.$$

Equivalently, involving the $\boldsymbol{m}$- and $\boldsymbol{n}$-matrices isomorphic to quaternions [13], we have

$$\left(\begin{bmatrix} 0 & 0 & -\beta & -\gamma \\ 0 & 0 & -\gamma & \beta \\ \beta & \gamma & 0 & 0 \\ \gamma & -\beta & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -\omega_{01} & -\omega_{02} & -\omega_{03} \\ \omega_{01} & 0 & \omega_{03} & -\omega_{02} \\ \omega_{02} & -\omega_{03} & 0 & \omega_{01} \\ \omega_{03} & \omega_{02} & -\omega_{01} & 0 \end{bmatrix}\right) \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.8}$$

$$\left(\begin{bmatrix} \mu_0 & -\mu_1 & -\mu_2 & -\mu_3 \\ \mu_1 & \mu_0 & -\mu_3 & \mu_2 \\ \mu_2 & \mu_3 & \mu_0 & -\mu_1 \\ \mu_3 & -\mu_2 & \mu_1 & \mu_0 \end{bmatrix} - \begin{bmatrix} \nu_0 & -\nu_1 & -\nu_2 & -\nu_3 \\ \nu_1 & \nu_0 & \nu_3 & -\nu_2 \\ \nu_2 & -\nu_3 & \nu_0 & \nu_1 \\ \nu_3 & \nu_2 & -\nu_1 & \nu_0 \end{bmatrix}\right) \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{5.9}$$

where the coefficient matrix of the linear algebraic system (5.9) is determined by the components of the quaternions $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$:

$$\begin{cases} \boldsymbol{\mu} = \mathbf{exp}\{\mathbf{i}_2\beta/2\} \circ \mathbf{exp}\{\mathbf{i}_3\gamma/2\} \\ \mu_0 = \cos(\beta/2)\cos(\gamma/2), \quad \mu_1 = \sin(\beta/2)\sin(\gamma/2) \\ \mu_2 = \sin(\beta/2)\cos(\gamma/2), \quad \mu_3 = \cos(\beta/2)\sin(\gamma/2), \end{cases} \tag{5.10}$$

$$\boldsymbol{\nu} = \widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T. \tag{5.11}$$

Their norms are $\|\boldsymbol{\mu}\| = 1$ and $\|\boldsymbol{\nu}\| = 1$, and the coefficient matrices of systems (5.8), (5.9) have rank 2. By choosing two linearly independent equations in (5.8) and in (5.9), we obtain a homogeneous system of the form

$$\begin{bmatrix} 0 & \omega_{01} & \omega_{02} - \beta & \omega_{03} - \gamma \\ -\omega_{01} & 0 & -(\omega_{03} + \gamma) & \omega_{02} + \beta \\ \mu_1 - \nu_1 & 0 & -(\mu_3 + \nu_3) & \mu_2 + \nu_2 \\ 0 & \nu_1 - \mu_1 & \nu_2 - \mu_2 & \nu_3 - \mu_3 \end{bmatrix} \begin{pmatrix} K_0 \\ K_1 \\ K_2 \\ K_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.12}$$

A nontrivial solution of the system of equations (5.12) is due to the zero determinant of its coefficient matrix. Using this fact and formulas (5.4), (5.5), (5.10), and (5.11), we get

$$|\boldsymbol{\omega}_0| = (\mu_2\beta + \mu_3\gamma)/(\nu_1\omega_{01}^e + \nu_2\omega_{02}^e + \nu_3\omega_{03}^e). \tag{5.13}$$

Based on (5.5), (5.7), (5.10), and (5.13), we compile a system of two nonlinear equations to determine the constants β and γ :

$$\begin{cases} \left(\beta^2 + \gamma^2\right)\left(\nu_1\omega_{01}^e + \nu_2\omega_{02}^e + \nu_3\omega_{03}^e\right)^2 - \left(\mu_2\beta + \mu_3\gamma\right)^2 = 0 \\ \operatorname{scal}(\widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T) - \cos(\beta/2)\cos(\gamma/2) = 0. \end{cases} \tag{5.14}$$

The value of $|\boldsymbol{\omega}_0|$ is found from (5.3) using the solution of system (5.4).

The components of the quaternion $\mathbf{Q}$ are given by

$$\begin{matrix} q_3 = \pm\left[1 + (B_0/G)^2 + (B_1/G)^2 + (B_2/G)^2\right]^{-1/2}, & q_0 = B_0 q_3/G, \\ q_1 = B_1 q_3/G, & q_2 = B_2 q_3/G, \end{matrix} \tag{5.15}$$

where

$$\begin{cases} B_0 = -(\mu_2 + \nu_2)(\omega_{03} + \gamma) + (\mu_3 + \nu_3)(\omega_{02} + \beta) \\ B_1 = (\mu_1 - \nu_1)\omega_{01} + (\mu_3 + \nu_3)(\gamma - \omega_{03}) + (\mu_2 + \nu_2)(\beta - \omega_{02}) \\ B_2 = (\mu_1 - \nu_1)(\omega_{02} + \beta) + (\mu_2 + \nu_2)\omega_{01} \\ G = (\mu_1 - \nu_1)(\omega_{03} + \gamma) + (\mu_3 + \nu_3)\omega_{01}. \end{cases} \tag{5.16}$$

The boundary condition on the spacecraft angular velocity for $t = T = 1$ shall be

$$\boldsymbol{\omega}(T) = \boldsymbol{\omega}_T = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_1\beta\sin\gamma + \mathbf{i}_2\beta\cos\gamma + \mathbf{i}_3\gamma) \circ \mathbf{Q}. \tag{5.17}$$

Finally, if the boundary conditions on the spacecraft angular velocity $\boldsymbol{\omega}$ satisfy the requirements (5.13) and (5.17), the spacecraft angular motion trajectory will lie in the class of regular coning motions and will be determined by the explicit expressions (5.1) and (5.2). (For all $t \in [0, T]$, $\boldsymbol{\omega}(t)$ will belong to a coning surface determined in space by arbitrary given boundary conditions on the position, $\boldsymbol{\Lambda}_0$ and $\boldsymbol{\Lambda}_T$, and the arbitrary given direction of its initial value vector $\boldsymbol{\omega}_0^e$.)

The optimal angular acceleration from (2.4), (5.1) is

$$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \beta\gamma\widetilde{\mathbf{Q}} \circ (\mathbf{i}_1\cos\gamma t - \mathbf{i}_2\sin\gamma t) \circ \mathbf{Q}, \tag{5.18}$$

$$||\boldsymbol{\varepsilon}|| = \beta^2\gamma^2 = \text{const}. \tag{5.19}$$

The dimensionless optimization functional (3.1) takes the optimal value

$$J = \int_0^1 ||\boldsymbol{\varepsilon}||\, dt = \beta^2\gamma^2. \tag{5.20}$$

The conjugate variables $\boldsymbol{\varphi}$ and $\mathbf{p}$ are determined using formulas (5.18) and (4.5). The problem with the above restrictions has been completely solved.

Now we present an algorithm for constructing the minimum-energy angular acceleration of a spacecraft in the class of regular coning motions:

1) Use the given constant conditions $T = 1$, $\boldsymbol{\Lambda}_0$, $\boldsymbol{\Lambda}_T$ (2.5), and $\boldsymbol{\omega}_0^e = \boldsymbol{\omega}_0/\left|\boldsymbol{\omega}_0\right|$ and formulas (2.6), (5.10), (5.11), (5.15), and (5.16) to determine the unknowns β, γ, and $\left|\boldsymbol{\omega}_0\right|$.

2) Use $\boldsymbol{\Lambda}_0$, $\boldsymbol{\Lambda}_T$, β, γ, $T = 1$, and $\left|\boldsymbol{\omega}_0\right|$ and formulas (5.15) and (5.16) to find the quaternion $\mathbf{Q}$.

3) Use the expressions $\boldsymbol{\omega}_0^{\text{calc}} = \left|\boldsymbol{\omega}_0\right| \boldsymbol{\omega}_0^e$ and $\boldsymbol{\omega}_T^{\text{calc}} = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_1\beta\sin\gamma + \mathbf{i}_2\beta\cos\gamma + \mathbf{i}_3\Omega) \circ \mathbf{Q}$ to find the conditions $\boldsymbol{\omega}_0^{\text{calc}}$ and $\boldsymbol{\omega}_T^{\text{calc}}$ at the ends of the trajectory.

4) Compare the values of $\boldsymbol{\omega}_0^{\text{calc}}$ and $\boldsymbol{\omega}_T^{\text{calc}}$ with the given ones (2.6).

5) If equality holds in item 4), then the optimal solution of the problem is in the class of regular coning motions; use formulas (5.1), (5.3), and (5.18) and item 1) to calculate the angular velocity, trajectory, and angular acceleration vector of the spacecraft; determine the optimal value of the functional based on (5.10).

6) Calculate the conjugate variables $\boldsymbol{\varphi}$ and $\mathbf{p}$ by formulas (4.4), (4.5), (5.18).

This optimal solution of the problem in the class of regular coning motions and the previously known exact solution [4] in the class of planar Euler rotations (under the condition $\boldsymbol{\omega}_0\ \boldsymbol{\omega}_T\ ||\text{vect}(\widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T)$) for the special cases of boundary conditions on the spacecraft angular velocity will be used as analytical confirmations when constructing a quasi-optimal solution of the spacecraft angular acceleration problem in the class of generalized coning motions. Also, we will provide some suggestive considerations based on the numerical solution of the optimal angular acceleration problem of a spacecraft with arbitrary boundary conditions.

6. JUSTIFICATION OF THE APPROACH BASED ON NUMERICAL SOLUTIONS OF THE PROBLEM

Let us numerically solve the boundary value problem of the maximum principle (Section 4) for the original kinematic optimal acceleration problem (2.3)–(2.6), (3.1) of a spacecraft:

$$\begin{cases} 2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \boldsymbol{\omega} \\ \dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon} \\ \dot{\boldsymbol{\varphi}} = -\mathbf{p}/2 \\ \mathbf{p} = \widetilde{\boldsymbol{\Lambda}} \circ \mathbf{c}_v \circ \boldsymbol{\Lambda}, \quad \mathbf{c}_v = \mathbf{const}, \end{cases} \tag{6.1}$$

$$\boldsymbol{\Lambda}(0) = \boldsymbol{\Lambda}_0, \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \tag{6.2}$$

$$\boldsymbol{\Lambda}(T) = \boldsymbol{\Lambda}_T, \quad \boldsymbol{\omega}(T) = \boldsymbol{\omega}_T, \tag{6.3}$$

$$\boldsymbol{\varepsilon}^{\text{opt}} = \boldsymbol{\varphi}/2. \tag{6.4}$$

It is required to find $\boldsymbol{\varepsilon}^{\text{opt}}$, T^{opt}, $\boldsymbol{\Lambda}^{\text{opt}}$, $\boldsymbol{\omega}^{\text{opt}}$, and $\mathbf{c}_v$. The terminal conditions (6.3) are conveniently written in the phase space $\boldsymbol{\Lambda} \times \boldsymbol{\omega}$ of dimension 7 as

$$\text{vect}\ (\boldsymbol{\Lambda}(T) \circ \widetilde{\boldsymbol{\Lambda}}_T) = 0, \quad \boldsymbol{\omega}(T) = \boldsymbol{\omega}_T. \tag{6.5}$$

(For details, see [8, 9].)

A numerical solution method for such problems was described in [8, 9]. Also, for comparison, we provide kinematic characteristics based on the calculation results in the full dynamic optimal rotation problem of a spacecraft (in the sense of minimizing energy consumption) as a solid body of various dynamic configurations [9] in dimensionless variables ($T = 1$) under the same boundary conditions:

$$\mathbf{\Lambda}_0 = (0.7951,\ 0.2981,\ -0.3975,\ 0.3478),\ \ \boldsymbol{\omega}_0 = (0.2739,\ -0.2388,\ -0.3), \tag{6.6}$$

$$\mathbf{\Lambda}_T = (0.8443,\ 0.3984,\ -0.3260,\ 0.1485),\ \ \boldsymbol{\omega}_T = (0.0,\ 0.0,\ -0.59). \tag{6.7}$$

Spacecraft 1. An arbitrary spacecraft, $I_1 = 0.9869$, $I_2 = 1.1843$, and $I_3 = 0.7895$.

Spacecraft 2. An arbitrary spacecraft, $I_1 = 0.9506$, $I_2 = 1.3308$, and $I_3 = 0.5704$.

Spacecraft 3. The International Space Station (ISS) in its early version [14], $I_1 = 4\,853\,000$ kg $\times$ m^2, $I_2 = 23\,601\,000$ kg $\times$ m^2, and $I_3 = 26\,278\,000$ kg $\times$ m^2, or the dimensionless variables $I_1 = 0.2358$, $I_2 = 1.1466$, and $I_3 = 1.2766$.

Spacecraft 4. Space Shuttle, which has characteristics almost like a dynamically symmetric solid body: $I_1 = 3\,400\,648$ kg $\times$ m^2, $I_2 = 21\,041\,672$ kg $\times$ m^2 or $I_1 = 0.1967$, $I_2 = 1.2168$, and $I_3 \approx I_2$.

Spacecraft 5. An arbitrary spacecraft, $I_1 = 0.9116$, $I_2 = 1.3674$, and $I_3 = 0.5470$.

Table 1 shows the values of the attitude quaternion and angular velocity vector of the spacecraft obtained by solving the optimal angular acceleration problem (6.1)–(6.5) and the dynamic optimal rotation problem [9] for spacecraft 1–4 at the intermediate point $t = 0.5$ of the time interval. For comparison, the last line of Table 1 contains the data obtained by solving the quasi-optimal angular acceleration problem of a spacecraft considered in Sections 6 and 7 of the paper.

Table 1. Values of the attitude quaternion and angular velocity vector

Spacecraft	λ_0	λ_1	λ_2	λ_3	ω_1	ω_2	ω_3
1	0.8095	0.3628	−0.3766	0.2670	−0.0499	−0.0115	−0.4941
2	0.8093	0.3631	−0.3765	0.2674	−0.0496	−0.0116	−0.4949
3	0.8077	0.3654	−0.3773	0.2678	−0.0506	−0.0159	−0.4949
4	0.8086	0.3634	−0.3780	0.2668	−0.0519	−0.0137	−0.4948
Boundary value problem (6.1)–(6.5)	0.8096	0.3625	−0.3768	0.2668	0.0502	−0.0114	−0.4937
Modified problem	0.8099	0.3627	−0.3756	0.2673	−0.0488	−0.0098	−0.4938

For the spacecraft under consideration, Table 2 presents the components of the angular acceleration vectors $\boldsymbol{\varepsilon} = [\varepsilon_1(t),\ \varepsilon_2(t),\ \varepsilon_3(t)]^{\mathrm{T}}$ at the points $t = 0$, $t = 0.5$, and $t = T = 1$ of the optimal control process.

Table 2. Values of the angular acceleration

Spacecraft	$\varepsilon_1(0)$	$\varepsilon_2(0)$	$\varepsilon_3(0)$	$\varepsilon_1(0.5)$	$\varepsilon_2(0.5)$	$\varepsilon_3(0.5)$	$\varepsilon_1(1)$	$\varepsilon_2(1)$	$\varepsilon_3(1)$
2	−0.9649	0.7262	−0.4736	−0.3051	0.2053	−0.2902	0.5357	−0.1154	−0.0965
3	−0.9399	0.7235	−0.4449	−0.3207	0.2039	−0.2942	0.5681	−0.1076	−0.0940
4	−0.9262	0.6536	−0.4422	−0.3182	0.2586	−0.2978	0.5591	−0.2446	−0.0886
5	−0.7914	0.6537	−0.4345	−0.3968	0.2549	−0.3002	0.7341	−0.2381	−0.0875
Boundary value problem (6.1)–(6.5)	−0.9854	0.7259	−0.4892	−0.2917	0.2087	−0.2878	0.5077	−0.1272	−0.0985
Modified problem	−0.9647	0.7634	−0.4932	−0.3103	0.1687	−0.2847	0.5350	−0.0220	0.1024

We also describe the kinematic characteristics of the optimal motion when the initial and terminal states of the spacecraft are determined by the relations (6.6) and

$$\mathbf{\Lambda}_T = (0.79368,\ 0.49375,\ -0.26823,\ 0.23309),\ \ \boldsymbol{\omega}_T = (0.2,\ 0.3,\ -0.2), \tag{6.8}$$

respectively.

Table 3, with reference to problem (6.1)–(6.5) and the solution of the dynamic optimal rotation problem [9] for spacecraft 3–5, shows the components of the attitude quaternion and the angular velocity vector at $t = 0.5$; Table 4, the components of the angular acceleration vectors at $t = 0$, $t = 0.5$, and $t = T = 1$.

Table 3. Orientation quaternion and angular velocity vector

Spacecraft	λ_0	λ_1	λ_2	λ_3	ω_1	ω_2	ω_3
3	0.79080	0.41371	−0.35618	0.27679	0.31768	−0.01578	−0.52304
4	0.79103	0.41317	−0.35664	0.27635	0.31932	−0.01491	−0.52314
5	0.78999	0.41442	−0.35860	0.27492	0.32414	−0.01453	−0.52402
Boundary value problem (6.1)–(6.5)	0.78995	0.41434	−0.35808	0.28583	0.32168	−0.01522	−0.52451

Table 4. Angular acceleration

Spacecraft	$\varepsilon_1(0)$	$\varepsilon_2(0)$	$\varepsilon_3(0)$	$\varepsilon_1(0.5)$	$\varepsilon_2(0.5)$	$\varepsilon_3(0.5)$	$\varepsilon_1(1)$	$\varepsilon_2(1)$	$\varepsilon_3(1)$
3	0.36496	0.53821	−1.00244	−0.07017	0.44837	0.10585	−0.54722	0.88317	1.18497
4	0.28551	0.51819	−1.00680	−0.04259	0.45769	0.10771	−0.55803	0.86381	1.18517
5	0.19271	0.34271	−1.05149	−0.05270	0.54857	0.12304	−0.41047	0.70409	1.16712
Boundary value problem (6.1)–(6.5)	0.26547	0.38677	−1.00250	−0.06733	0.52385	0.10458	−0.43888	0.74958	1.18404

The same calculations were carried out for other boundary conditions. According to Tables 1–4 and other calculations, in the dynamic optimal rotation problem, the kinematic characteristics of the optimal motion depend significantly on the initial and terminal states, less significantly on its configuration, and are close enough to the results of the kinematic optimal angular acceleration problem of the spacecraft. Therefore, the kinematic optimal angular acceleration problem (2.3)–(2.7) has a general nature for spacecraft of arbitrary configurations. In this case, expressions for the attitude quaternion and angular velocity vector in the kinematic problem can be derived analytically in explicit form based on the solution of the so-called modified optimal reorientation problem of the spacecraft in the class of generalized coning motions. And the control angular acceleration of the spacecraft can be determined by differentiating its angular velocity vector. We consider this in more detail.

7. THE MODIFIED OPTIMAL ANGULAR ACCELERATION PROBLEM OF A SPACECRAFT

The general solution of the fundamental attitude problem of a solid body by its known angular velocity (2.1), (2.3), called the Darboux problem, is unknown. Therefore, we obtain a solution in the class of generalized coning motions. For this purpose, let the angular velocity vector $\boldsymbol{\omega}(t)$ be defined as

$$\boldsymbol{\omega}(t) = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_1\dot{\mathrm{B}}(t)\sin\Gamma(t) + \mathbf{i}_2\dot{\mathrm{B}}(t)\cos\Gamma(t) + \mathbf{i}_3\dot{\Gamma}(t)) \circ \mathbf{Q}, \tag{7.1}$$

where the functions $\mathrm{B}(t)$ and $\Gamma(t)$ (the parameters of the generalized coning motion) are arbitrary. In this case, equation (2.3) has an exact solution [8, 9] satisfying the initial condition (2.5):

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{Q}} \circ \mathbf{exp}\{-\mathbf{i}_3\Gamma(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(\mathrm{B}(t) - \mathrm{B}(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3\Gamma(t)/2\} \circ \mathbf{Q}. \tag{7.2}$$

Clearly, formulas (7.1) and (7.2) include expressions for the angular velocity and spacecraft trajectory in the known exact solutions of the optimal angular acceleration problem when the angular velocity vector keeps a constant direction on the entire time interval of spacecraft motion (a planar

Euler rotation of a spacecraft [4]) or makes a regular precession (see Section 5 of this paper). Moreover, formulas (7.1) and (7.2) with arbitrary functions $\mathrm{B}(t)$ and $\Gamma(t)$ are a direct generalization of (5.1) and (5.3) with the linear functions βt and γt, respectively. The Darboux problem can generally be reduced, using bijective changes of variables [8, 9], to an equation of the form (2.3), where the angular velocity takes the form $\boldsymbol{\omega}^*(t) = -\boldsymbol{\omega}(t)$ and belongs to the class of similar motions. (The quaternionic differential equation (2.3) with the angular velocity $\boldsymbol{\omega}^*(t)$ still has no explicit solution.) Thus, the form of the angular velocity vector (7.1) corresponds to the classical Poinsot concept (kinematic interpretation) of the angular motion of a solid body as a generalized coning motion [7].

Let the second derivatives of the functions B and Γ be treated as controls. With the designations

$$\dot{\mathrm{B}} = \mathrm{B}_1, \quad \dot{\Gamma} = \Gamma_1, \tag{7.3}$$

we compile the controlled system:

$$\dot{\mathrm{B}} = \mathrm{B}_1, \quad \dot{\Gamma} = \Gamma_1, \quad \dot{\mathrm{B}}_1 = u_1, \quad \dot{\Gamma}_1 = u_2, \tag{7.4}$$

where B, B_1, Γ, and Γ_1 are the phase coordinates of this problem, and u_1, u_2 are the controls. The quaternion $\mathbf{Q}$ is defined as

$$\mathbf{Q} = \mathbf{Q}_2 \circ \mathbf{Q}_1, \quad \mathbf{Q}_1 = \mathbf{exp}\{\mathbf{i}_1\alpha_1/2\}, \quad \mathbf{Q}_2 = \mathbf{exp}\{\mathbf{i}_2\alpha_2/2\}, \tag{7.5}$$

where α_1 and α_2 are arbitrary constants. Note that the quaternions $\mathbf{Q}_1$ and $\mathbf{Q}_2$ rotate the angular velocity vector $\boldsymbol{\omega}$ (7.1) about the axes $\mathbf{i}_1$ and $\mathbf{i}_2$, respectively. Due to the additive constant included in the function $\Gamma(t)$, the rotation about the axis $\mathbf{i}_3$ is already incorporated in the expression (7.1). The conjugation of the quaternion $\widetilde{\mathbf{Q}}$ has the form

$$\widetilde{\mathbf{Q}} = \widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2, \quad \widetilde{\mathbf{Q}}_1 = \mathbf{exp}\{-\mathbf{i}_1\alpha_1/2\}, \quad \widetilde{\mathbf{Q}}_2 = \mathbf{exp}\{-\mathbf{i}_2\alpha_2/2\}. \tag{7.6}$$

In view of (7.5) and (7.6), the functions $\boldsymbol{\omega}$ (7.1) and $\boldsymbol{\Lambda}$ (7.2) will satisfy the boundary conditions (2.5) and (2.6) if

$$\widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2 \circ (\mathbf{i}_1\mathrm{B}_1(0)\sin\Gamma(0) + \mathbf{i}_2\mathrm{B}_1(0)\cos\Gamma(0) + \mathbf{i}_3\Gamma_1(0)) \circ \mathbf{Q}_2 \circ \mathbf{Q}_1 = \boldsymbol{\omega}_0, \tag{7.7}$$

$$\widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2 \circ (\mathbf{i}_1\mathrm{B}_1(T)\sin\Gamma(T) + \mathbf{i}_2\mathrm{B}_1(T)\cos\Gamma(T) + \mathbf{i}_3\Gamma_1(T)) \circ \mathbf{Q}_2 \circ \mathbf{Q}_1 = \boldsymbol{\omega}_T, \tag{7.8}$$

$$\boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2 \circ \mathbf{exp}\{-\mathbf{i}_3\Gamma(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(\mathrm{B}(T) - \mathrm{B}(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3\Gamma(T)/2\} \circ \mathbf{Q}_2 \circ \mathbf{Q}_1 = \boldsymbol{\Lambda}_T. \tag{7.9}$$

For the controlled system (7.4), the optimization problem is posed as follows: find optimal controls $u_1(t)$ and $u_2(t)$ transferring system (7.4) from the state

$$\mathrm{B} = \mathrm{B}(0), \quad \mathrm{B}_1 = \mathrm{B}_1(0), \quad \Gamma = \Gamma(0), \quad \Gamma_1 = \Gamma_1(0) \tag{7.10}$$

to the state

$$\mathrm{B} = \mathrm{B}(T), \quad \Gamma_1 = \Gamma_1(T), \quad \Gamma = \Gamma(T), \quad \Gamma_1 = \Gamma_1(T) \tag{7.11}$$

that satisfy the relations (7.7)–(7.9), where the parameters α_1 and α_2 are to be determined, and implement the optimality criterion

$$J = \int_0^T (u_1^2 + u_2^2)\,dt. \tag{7.12}$$

We write the expressions (7.7)–(7.9) as

$$(\mathbf{i}_1\mathrm{B}_1(0)\sin\Gamma(0) + \mathbf{i}_2\mathrm{B}_1(0)\cos\Gamma(0) + \mathbf{i}_3\Gamma_1(0)) = \mathbf{Q}_2 \circ \mathbf{Q}_1 \circ \boldsymbol{\omega}_0 \circ \widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2, \tag{7.13}$$

$$(\mathbf{i}_1\mathrm{B}_1(T)\sin\Gamma(T) + \mathbf{i}_2\mathrm{B}_1(T)\cos\Gamma(T) + \mathbf{i}_3\Gamma_1(T)) = \mathbf{Q}_2 \circ \mathbf{Q}_1 \circ \boldsymbol{\omega}_T \circ \widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2, \tag{7.14}$$

$$\begin{gathered}\mathbf{exp}\{-\mathbf{i}_3\Gamma(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(\mathrm{B}(T) - \mathrm{B}(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3\Gamma(T)/2\} \\ = \mathbf{Q}_2 \circ \mathbf{Q}_1 \circ \widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T \circ \widetilde{\mathbf{Q}}_1 \circ \widetilde{\mathbf{Q}}_2.\end{gathered} \tag{7.15}$$

The control angular acceleration of a spacecraft is determined from equation (2.4) by differentiation:

$$\dot{\boldsymbol{\omega}} = \boldsymbol{\varepsilon}. \tag{7.16}$$

This problem will be called the modified optimal angular acceleration problem of a spacecraft; its exact solution is acceptable as an approximate or quasi-optimal solution of the classical optimal problem (2.3)–(2.6), (3.1).

8. SOLUTION OF THE MODIFIED OPTIMAL ANGULAR ACCELERATION PROBLEM OF SPACECRAFT

The Hamilton–Pontryagin function of problem (7.4)–(7.12) is given by

$$H = -(u_1^2 + u_2^2) + \psi_1 \mathrm{B}_1 + \psi_2 \Gamma_1 + \psi_3 u_1 + \psi_4 u_2, \tag{8.1}$$

where the conjugate variables satisfy the differential system

$$\dot{\psi}_1 = 0, \;\; \dot{\psi}_2 = 0, \;\; \dot{\psi}_3 = -\psi_1, \;\; \dot{\psi}_4 = -\psi_2. \tag{8.2}$$

The general solution of system (8.2) has the form

$$\psi_1 = c_1, \;\; \psi_2 = c_2, \;\; \psi_3 = -c_1 t + c_3, \;\; \psi_4 = -c_2 t + c_4, \tag{8.3}$$

where $c_1, \ldots, c_4$ are arbitrary constants.

The maximum condition for the Hamilton–Pontryagin function (8.2) yields the following expressions for the optimal controls:

$$u_1 = \psi_3/2 = (-c_1 t + c_3)/2, \;\; u_2 = \psi_4/2 = (-c_2 t + c_4)/2. \tag{8.4}$$

By substituting formulas (8.4) into equations (7.4), we find their general solution containing eight unknown constants $c_1, \ldots, c_8$:

$$\begin{aligned} &\mathrm{B} = -c_1 t^3/12 + c_3 t^2/4 + c_5 t + c_6, && \Gamma = -c_2 t^3/12 + c_4 t^2/4 + c_7 t + c_8, \\ &\mathrm{B}_1 = -c_1 t^2/4 + c_3 t/2 + c_5, && \Gamma_1 = -c_2 t^2/4 + c_4 t/2 + c_7. \end{aligned} \tag{8.5}$$

Based on (8.4), let $c_6 = 0$ in the expression for the function B (8.5). The nine unknown constants of the problem, $c_1, \ldots, c_5$, c_7, c_8, α_1, and α_2 are determined from the nine equations of system (7.13)–(7.15). (Due to the requirement $||\boldsymbol{\Lambda}|| = 1$, three scalar equations are independent in the quaternionic representation (7.15).) Note that these equations can be solved by the modified Newton method, where the initial approximations will be the values of the constants $c_1, \ldots, c_5$, c_7, c_8, α_1, and α_2 from the exact solution of the problem in the special case (see Section 5 of the paper).

By using (7.1), (7.2), and (8.5), we obtain explicit dependencies determining the laws of change of the angular velocity and trajectory of the spacecraft in the optimal angular acceleration problem in the class of generalized coning motions. In view of (7.1), (8.4), and (8.5), formula (7.16) gives an analytical expression for the angular acceleration vector $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \dot{\boldsymbol{\omega}} = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_1(u_1 \sin\Gamma + \mathrm{B}_1\Gamma_1 \cos\Gamma) + \mathbf{i}_2(u_1 \cos\Gamma - \mathrm{B}_1\Gamma_1 \sin\Gamma) + \mathbf{i}_3 u_2) \circ \mathbf{Q}. \tag{8.6}$$

Due to (7.3), (7.5), and (7.6), the components of the vectors $\boldsymbol{\omega}$ (7.1) and $\boldsymbol{\varepsilon}$ (8.6) can be explicitly written as

$$\begin{aligned} \omega_1 &= \mathrm{B}_1 \sin\Gamma \cos\alpha_2 - \Gamma_1 \sin\alpha_2, \\ \omega_2 &= \mathrm{B}_1(\sin\Gamma \sin\alpha_1 \sin\alpha_2 - \cos\Gamma \cos\alpha_1) + \Gamma_1 \sin\alpha_1 \cos\alpha_2, \\ \omega_3 &= \mathrm{B}_1(\sin\Gamma \cos\alpha_1 \sin\alpha_2 - \cos\Gamma \sin\alpha_1) + \Gamma_1 \cos\alpha_1 \cos\alpha_2, \\ \varepsilon_1 &= u_1(\sin\Gamma + \mathrm{B}_1\Gamma_1 \cos\Gamma) \cos\alpha_2 - u_2 \sin\alpha_2, \\ \varepsilon_2 &= u_1(\sin\Gamma \sin\alpha_1 \sin\alpha_2 + \cos\Gamma \cos\alpha_1) \\ &\quad + \mathrm{B}_1\Gamma_1(\cos\Gamma \sin\alpha_1 \sin\alpha_2 - \sin g \cos\alpha_1) + +u_2 \sin\alpha_1 \cos\alpha_2, \\ \varepsilon_3 &= u_1(\sin\Gamma \cos\alpha_1 \sin\alpha_2 - \cos\Gamma \sin\alpha_1) \\ &\quad + f_1\Gamma_1(\cos\Gamma \cos\alpha_1 \sin\alpha_2 + \sin\Gamma \sin\alpha_1) + u_2 \cos\alpha_1 \cos\alpha_2. \end{aligned}$$

The norm of the control angular acceleration vector of a spacecraft, appearing in the functional (2.7) (or (3.1)) of the original problem, is expressed in terms of the elements of the modified problem as follows:

$$||\boldsymbol{\varepsilon}|| = u_1^2 + \mathrm{B}_1^2\Gamma_1^2 + u_2^2. \tag{8.7}$$

Thus, the modified optimal angular acceleration problem of a spacecraft has been completely solved.

If the boundary conditions on the angular velocity of a spacecraft are specified as described in Section 5, the solutions of the classical and modified optimal angular acceleration problems will coincide. The same is true when $\boldsymbol{\omega}_0\ \boldsymbol{\omega}_T\ ||\mathrm{vect}(\widetilde{\boldsymbol{\Lambda}}_0 \circ \boldsymbol{\Lambda}_T)$ (a planar Euler rotation of a spacecraft [4]); in this case, $\mathrm{B}_1^2\Gamma_1^2 = 0$ in (8.7), and the criterion (7.12) corresponds to the optimization criterion (2.7) (or (3.1)) of the classical problem. For arbitrary boundary conditions, assuming that $\int_0^1 \mathrm{B}_1^2\Gamma_1^2\,dt$ is small compared to $\int_0^1 ||\boldsymbol{\varepsilon}||\,dt$ in (8.7), the last term can be omitted. Then the modified optimal control problem of a spacecraft in the variables B, Γ, B_1, Γ_1, u_1, and u_2 with the functional (7.12) and the expressions (7.1), (7.2), (7.13)–(7.15), and (8.4)–(8.6) corresponds to the classical optimal angular acceleration problem in the class of generalized coning motions. Based on the considerations of Sections 5 and 6 of this paper, the modified problem can be treated as a quasi-optimal angular acceleration problem of a spacecraft under arbitrary boundary conditions.

Similar to the classical problem (see formulas (4.11) and (4.12)), at any time instant, the magnitude $||\boldsymbol{\varepsilon}||$ in the modified problem lies inside a variable bounded domain determined by the constants of the problem conditions. Due to (8.4), (8.7), and the above assumptions, we obtain

$$||\boldsymbol{\varepsilon}(t)|| \le \left((c_1^2 + c_2^2)t^2 + 2 \mid c_1c_3 + c_2c_4 \mid t + c_3^2c_4^2\right)/4\ \forall\ t \in [0, 1]. \tag{8.8}$$

Here is a quasi-optimal algorithm for finding the minimum-energy angular acceleration of a spacecraft.

1. In view of (8.3)–(8.5), use the boundary conditions $\boldsymbol{\Lambda}_0$, $\boldsymbol{\Lambda}_T$ (2.5), $\boldsymbol{\omega}_0$, $\boldsymbol{\omega}_T$ (2.6), $T = 1$, formulas (7.5), (7.6), and the nine scalar equations from (7.13)–(7.15) to find the nine arbitrary constants $c_1, \ldots, c_5$, c_7, c_8, α_1, and α_2; then determine the functions $\mathrm{B}(t)$, $\mathrm{B}_1(t)$, $\Gamma(t)$, and $\Gamma_1(t)$.
2. Calculate the quaternion $\mathbf{Q}$ by formula (7.5).
3. Find the law of change of the angular velocity of the spacecraft by (7.1):

$$\boldsymbol{\omega}(t) = \widetilde{\mathbf{Q}} \circ \left(\mathbf{i}_1\dot{\mathrm{B}}(t)\sin\Gamma(t) + \mathbf{i}_2\dot{\mathrm{B}}(t)\cos\Gamma(t) + \mathbf{i}_3\dot{\Gamma}(t)\right) \circ \mathbf{Q}.$$

4. Find the law of change of the angular motion trajectory of the spacecraft by (7.2):

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}_0 \circ \widetilde{\mathbf{Q}} \circ \mathbf{exp}\{-\mathbf{i}_3\Gamma(0)/2\} \circ \mathbf{exp}\{\mathbf{i}_2(\mathrm{B}(t) - \mathrm{B}(0))/2\} \circ \mathbf{exp}\{\mathbf{i}_3\Gamma(t)/2\} \circ \mathbf{Q};$$

5. Construct the angular acceleration vector of the spacecraft by formula (8.6):

$$\boldsymbol{\varepsilon} = \widetilde{\mathbf{Q}} \circ (\mathbf{i}_1(u_1\sin\Gamma + \mathrm{B}_1\Gamma_1\cos\Gamma) + \mathbf{i}_2(u_1\cos\Gamma - \mathrm{B}_1\Gamma_1\sin\Gamma) + \mathbf{i}_3u_2) \circ \mathbf{Q}.$$

9. NUMERICAL EXAMPLES

In this section, we compare the numerical solution results of the original (classical) optimal angular acceleration problem of a spacecraft and the quasi-optimal solution of this problem using the analytical algorithm from Section 8. The values of α_1, α_2, $c_1, \ldots, c_5$, c_7, c_8 in the quasi-optimal solution of the spacecraft rotation problem with the boundary conditions (6.6), (6.7) are given by

$$\begin{aligned} &\alpha_1 = -0.0421, && \alpha_2 = -0.2226, && c_1 = 3.2902, && c_2 = -1.4885, && c_3 = 2.2113, \\ &c_4 = -1.45, && c_5 = -0.4156, && c_6 = 0, && c_7 = -0.2221, && c_8 = -0.9216. \end{aligned}$$

As it turned out, the solutions of the two problems are close enough. In this example, the values of the functional (3.1) of the original and modified problems are 0.4782 and 0.4797, respectively;

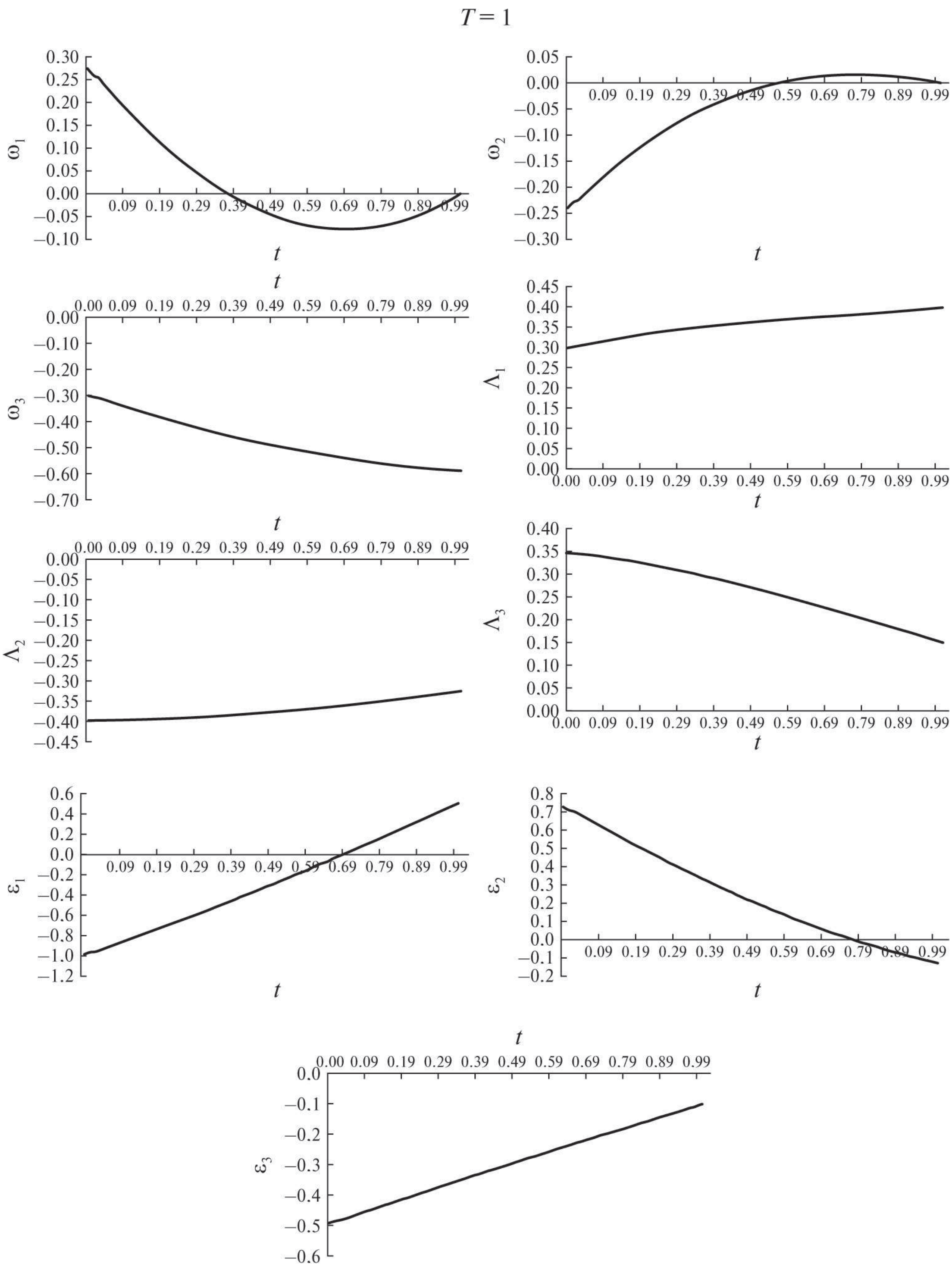

Fig. 1. The solution of the classical problem under arbitrary boundary conditions.

thus, the relative error between the values of (3.1) in the classical and modified problems is 0.3%. Figures 1 and 2 show the graphs of the projections of the spacecraft angular velocity vector $\omega_i(t)$, the components of the spacecraft attitude quaternion $\Lambda_i(t)$, and the projections of the spacecraft angular acceleration vector $\varepsilon_i(t)$, $i = 1, 2, 3$, depending on the time variable t; the graphs are quite similar for the two problems.

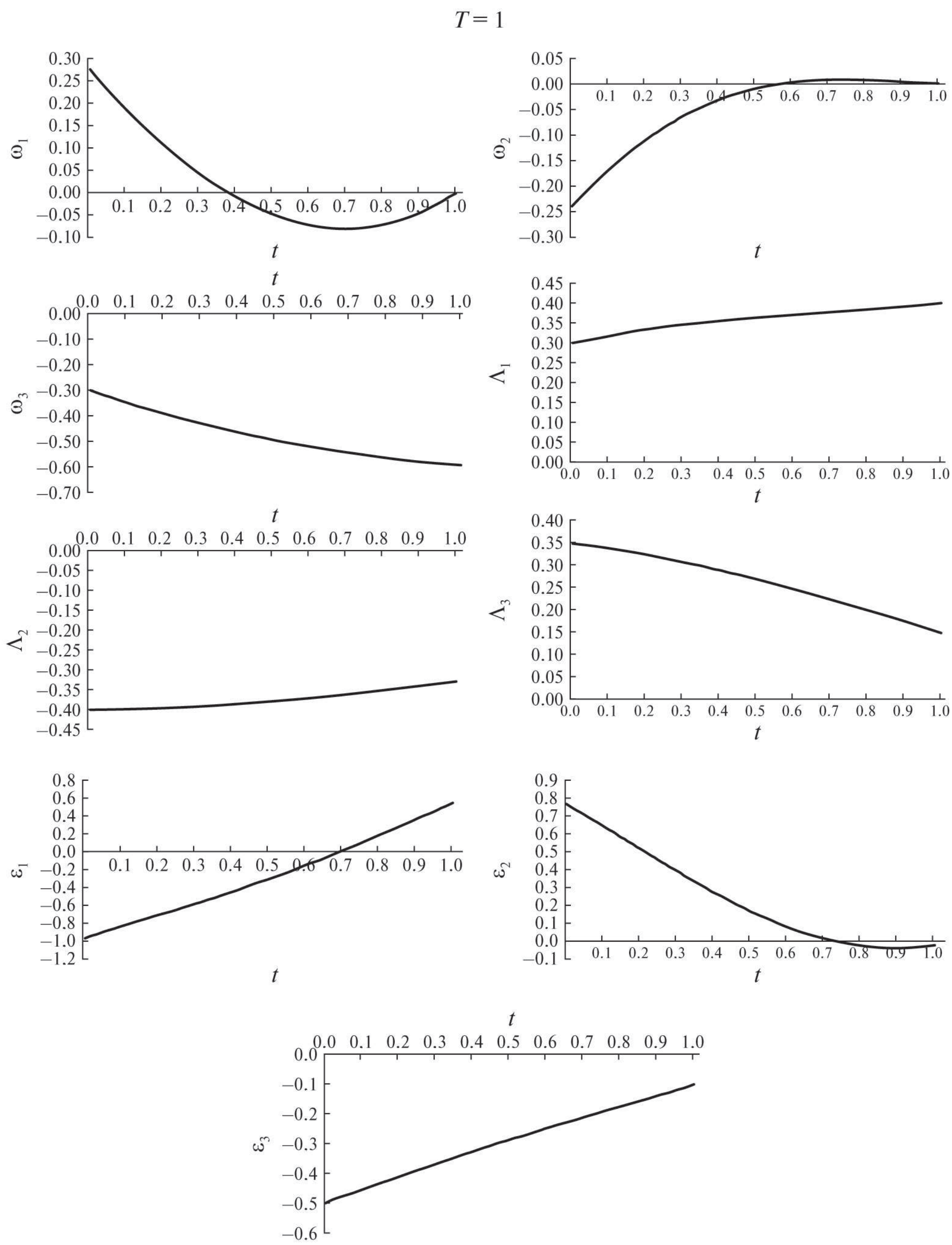

Fig. 2. The solution of the modified problem under arbitrary boundary conditions.

The energy-optimal angular acceleration problem of a spacecraft was also solved numerically in the following cases: the initial state is determined by (6.6), and the terminal position is set by rotating the spacecraft from the initial position to a certain angle about the Euler axis with the unit vector

$$(0.04500,\ -0.07519,\ -0.99615)\,. \tag{9.1}$$

Table 5 provides the components of the quaternions of the spacecraft terminal position for rotations to various Euler angles φ (deg) about the vector (9.1).

Table 5. Formation of terminal orientation quaternions

φ°	$\lambda_0(T)$	$\lambda_1(T)$	$\lambda_2(T)$	$\lambda_3(T)$
30.0	0.84643	0.40650	−0.31853	0.12982
60.0	0.84013	0.48716	−0.21783	−0.09703
90.0	0.77657	0.53461	−0.10229	−0.31727
120.0	0.66009	0.54564	0.02023	−0.51589
150.0	0.49863	0.51948	0.14136	−0.67935
180.0	0.30318	0.45792	0.25286	−0.79652

Next, Table 6 shows the values of the functional (3.1) in the classical and modified optimal angular acceleration problems when transferring the spacecraft from the initial (6.6) to terminal state $\boldsymbol{\Lambda}_T$ according to Table 5, with $\boldsymbol{\omega}_T = (0.0,\ 0.0,\ -0.59)$, also indicating the absolute and relative differences between them, $\Delta J = J^{\text{modif}} - J^{\text{classic}}$ and $\left(\Delta J/J^{\text{classic}}\right) \times 100\%$.

Table 6. Values of the functionals and their differences

Functionals and differences of their values	$\varphi = 30.0^\circ$	$\varphi = 60.0^\circ$	$\varphi = 90.0^\circ$	$\varphi = 150.0^\circ$	$\varphi = 180.0^\circ$
J^{classic}	0.52385	4.63277	15.31437	56.39081	86.78094
J^{modif}	0.52510	4.63724	15.32882	56.52086	87.51533
ΔJ	0.00125	0.00447	0.01445	0.13005	0.73439
%	0.24	0.10	0.09	0.23	0.85

Finally, Table 7 provides the same data for the cases where the terminal angular velocity of the spacecraft is given by the vector $\boldsymbol{\omega}_T = (0{,}0,\ 0{,}0,\ 0{,}0)$, i.e., when transferring the spacecraft to a state of rest.

Table 7. Values of the functionals and their differences

Functionals and differences of their values	$\varphi = 90.0^\circ$	$\varphi = 120.0^\circ$	$\varphi = 150.0^\circ$	$\varphi = 180.0^\circ$
J^{classic}	24.25074	45.17597	72.66431	106.71186
J^{modif}	24.28745	45.19513	72.77169	107.40843
ΔJ	0.03672	0.01916	0.10738	0.69657
%	0.15	0.04	0.15	0.65

The calculations for various boundary conditions show the closeness of the solutions of the classical and modified angular acceleration problems of a spacecraft. Therefore, the solution of the modified problem can be treated as a quasi-optimal solution of the classical optimal angular acceleration problem of a spacecraft.

10. CONCLUSIONS

The analytical quasi-optimal algorithm for solving the kinematic optimal, in terms of energy consumption, program angular acceleration problem of a spacecraft under arbitrary boundary conditions is theoretically justified and yields the optimal angular acceleration of a spacecraft with acceptable accuracy. This algorithm does not require a numerical solution of the boundary value problem of the maximum principle or any complex numerical solution; it provides ready-made analytical laws of quasi-optimal program control and program trajectory change that can be installed aboard a spacecraft. In this regard, the above results can be applied to nanoclass spacecraft with limited computing power. Recall that the Kotelnikov–Study transference principle [13] allows extending quaternionic formulas describing the control of angular motion to biquaternionic ones describing the control of the general spatial motion of a solid body. Based on the above results, this principle can be used to obtain an analytical quasi-optimal, in terms of energy consumption, program control algorithm for the spatial motion (maneuvering) of a spacecraft.

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