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CONTROL IN SOCIAL ECONOMIC SYSTEMS

Optimal Insurance and Investment Strategies in Individual Risk Models with Background Risks

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Abstract—In the presented paper, a one-stage problem of simultaneous finding an optimal insurance and investment portfolio in the presence of background risks is studied. The goal functional is a functional of Markovitz type which depends on the first two moments of the final capital. It is shown that the optimal insurance necessarily belongs to a class of stop loss insurances, equations determining parameters of the stop loss insurance and investment portfolio are derived. A numerical example illustrating the theoretical results is given.

Keywords: optimal insurance, optimal investment, Markovitz functional, background risk

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1. INTRODUCTION

One of the first results of study of models with background risks [1] showed that without the assumption of background risks a discrepancy arises between theoretical and empirical data, when solving various problems of financial mathematics. Later, a number of papers appeared on the choice of an investment portfolio that maximizes the expected utility of the final capital in the presence of background risks. As an example, we can cite the article [2] which showed that "unfair" external risk (with a negative mean) increases risk aversion in the investor's utility function. A characteristic feature of such problems is the lack of a dependence between background risks and asset returns.

However, in [3], it is proved that there is a significant correlation between asset returns and certain background risks. Paper [4] investigated the impact of background risk on portfolio choice and the efficient frontier within the mean-variance criteria. In [5], the author reviewed existing results within the mean-variance and expected utility approaches in the presence of additive external risks.

Insurance as a tool for reducing losses from background risks has been used in continuous-time dynamic models. Decisions on portfolio formation from risky assets and consumption management under background risk were studied in [6]. Using the dynamic programming equations, the authors found that optimal insurance hedges investment risks and balances consumption growth and volatility. In [7], the problem of choosing insurance that reduces the loss from an external piece-wise constant background risk process was investigated; a characterization of the function dual to the Bellman function as a solution to the Hamilton–Bellman–Jacobi equation was found. In a similar setting, the problem of a firm's choice of insurance strategy against a jump-like process of productivity failures and against natural disasters was studied in [8] using the dynamic programming method, but without allowing for investment.

In this paper, we consider the one-stage problem of selecting optimal insurance coverage for a group of policyholders (clients)—the so-called individual risk model—and the simultaneous choice of an investment portfolio in a securities market with a risk-free asset. The maximized functional is a Markowitz-type functional, i.e., it depends only on the first two moments of the final insurer's capital, which consists of the capital received from the insurance transaction and the capital earned on investments. It is assumed that there is a background risk which affects the amount of clients' losses, and a background risk which affects the overall return on investment. The results obtained differ from earlier findings in several respects. First, instead of insuring only background risks, an optimal risk-sharing function for the group of clients is derived. Second, the problem of choosing both insurance and an investment portfolio is investigated. Third, the normal approximation of aggregate risk, which is widely used in financial mathematics, is not employed here, and the results are therefore "exact" in this sense.

Section 2 describes insurance and investment markets, first without background risks and, then, with background risks. An optimization problem is formulated with the variance as a risk measure. Section 3 is devoted to finding its solution, where an optimal insurance and optimal portfolio being found as solutions to separate problems due to the specifics of the objective functional. Section 4 solves a similar optimization problem, but using the standard deviation as a risk measure. Necessary optimality conditions are obtained. Sections 5 and 6 contain a numerical example and concluding remarks, respectively.

2. PROBLEM FORMULATION

2.1. Description of Investment and Insurance Markets

First, consider an investment market of $n+1$ assets (see, for example, [10]), where the return vector $\bar{R} = (R_0, \dots, R_n)$ and there is no background risk. Let $R_0 = m_0$ almost sure (a.s.), i.e., it is a risk-free asset. Note that the presence of such an asset allows one to avoid spending the entire initial capital $w > 0$ on risk investments and, for example, when $m_0 = 1$, one can avoid investing the amount wa_0 and put it "in one's pocket" (without taking inflation into account). Let us denote by $\bar{a} = (a_0, \dots, a_n) \in R^{n+1}$ the investment portfolio, where a_i is the share in percentage of the initial capital $x_0 > 0$ invested into the ith asset. A typical budget constraint is $\sum_{i=0}^{n} a_i = 1$. This means self-financing of the investor-insurer and his ability to "short sell", i.e. borrowing some assets at current prices in order to invest money in others with subsequent reimbursement at the prices of the next quotation. The investor's capital $X_{\bar{a}} = w\sum_{i=0}^{n} a_i R_i = w(\sum_{i=1}^{n} a_i(R_i - m_0) + m_0)$, where $w > 0$ is the initial capital. After substituting $a_0 = 1 - \sum_{i=1}^{n} a_i$, we have:
the expected value and variance

$$\mu(a) = w\left(\sum_{i=1}^{n} a_i \triangle m_i + m_0\right), \text{ where } \triangle m_i = m_i - m_0,\ m_i = E\,R_i,\ a \in R^n,$$

$$\sigma^2(a) = w^2 a C a^T,$$

where a^T denotes the transposed row vector a. Let's introduce natural assumptions:

$$0 < m_0 < \min_{i=1,\dots,n} m_i,$$

the risk-return covariance matrix C of dimension $n \times n$ is positive definite.

Let us now describe the insurance market. A group of l clients is homogeneous, i.e., their losses $X_1, \dots, X_l$ are independent and identically distributed random variables (i.v.) with a distribution function $F_1(x)$. The insurer chooses a Borel-measurable risk sharing function $I(x) : 0 \le I(x) \le x$, $x \in [0, \infty)$, then the share of the ith client's loss paid by the insurer is equal to $I(X_i)$. The insurer's initial capital is $w > 0$; the premium payed by the insured (see, for example, [11]) is $d = (1+\alpha)EI(X_1)$, where $\alpha > 0$ is a given loading coefficient.

2.2. Model with Background Risks

Consider the insurance and investment model, taking into account the background risks. The insurer invests his initial capital w[1] so that his final capital consists of insurance income $S_1 = l(1+\alpha)E\,I(X_1+Z)-\sum_1^l I(X_j+Z)$ and income from investments $S_2 = w[\sum_1^n a_i(R_i-m_0)+m_0]+wV$. Here: Z is the background insurance risk such that $X_j + Z \geq 0$ a.s. and $\sigma^2(I(X_1+Z)) < \infty$, V is the total investment background risk, $\sigma^2(V) < \infty$.

Remark 1. Here, the losses of clients are $X_j^Z \overset{def}{=} X_j+Z$, $j = 1,\dots,l$ are assumed to be identically distributed, but are not independent, since Z enters the expression for each loss. Moreover, X_j and Z are assumed to be dependent. Such a model is atypical for problems of "classical" actuarial mathematics [11], in which independence of losses is assumed. Background risks W and V, affecting, respectively, the losses of policyholders and the total return on investing risky assets, are assumed to be independent. This condition seems justified, since here the insurance and investment markets are of different nature and are not related.

The Markowitz functional under the study has the form

$$J[I,a] \equiv E\,(S_1+S_2) - \beta\sigma^2(S_1+S_2) \to \max_{I,a},\ 0 \leq I(x) \leq x,\ a \in R^n. \tag{1}$$

Here, $\beta > 0$ has the meaning of the investor-insurer's caution coefficient: the larger β, the more cautious the insurer. Substituting the expressions for S_1 and S_2, we obtain

$$\begin{gathered} J[I,a] = l\alpha E\,I(X_1^Z) + w(a\triangle m^T + m_0 + E\,V) \\ -\beta\left(l\sigma^2[I(X_1^Z)] + l(l-1)cov(I(X_1^Z), I(X_2^Z)) + w^2 aCa^T + 2aQ^T + \sigma^2(V)\right), \end{gathered} \tag{2}$$

where $Q^T = cov(R,V)$ is a column vector with components $cov(R_i,V)$, $i = 1,\dots,n$.

3. MAIN RESULTS

Let us return to problem (1) and consider an auxiliary problem

$$J[I,a] \to \max_I,\ 0 \leq I(x) \leq x, \tag{3}$$

for a fixed $a \in R^n$, the expression for $J[I,a]$ is given in (2). The following statement gives the necessary conditions for optimality in (3), establishing: the form of the optimal $I^*(x) = x \wedge k^*$ (stop loss insurance with the so-called insurer retention level k^*), where $x \wedge y \overset{def}{=} \min\{x,y\}$, and the equation for finding the parameter k^*.

Statement 1. *Problem* (3) *has solution* , $I^*(x) = x \wedge k^*$, *where* $0 < k^* < \infty$ *is the only root of the optimality equation*

$$\xi(k) = 0, \quad \text{where} \quad \xi(k) \overset{def}{=} \alpha - 2l\beta\left(k - \int_0^k (1 - F_1^Z(x))dx\right). \tag{4}$$

Proof.

Lemma 1. *There is a solution to problem* (3).

The proof of the lemma is given in Appendix.

[1] Here it is possible to pay the total premium ld in parts up to the end of the insurance period, so the capital available for investment at the beginning of the period is equal to w.

Denote $I_\rho(x) = \rho I^*(x) + (1-\rho)I(x)$, where $\rho \in [0,1]$, the function I^* is a solution of (3) and I is an arbitrary risk sharing function. Find the derivative $\frac{d}{d\rho}J[I_\rho, a]\Big|_{\rho=1}$, taking into account that

$$\frac{d}{d\rho}E\,I_\rho(X_1^Z)\Big|_{\rho=1} = \int\limits_0^\infty (I^*(x) - I(x))dF_1^Z(x),$$

$$\frac{d}{d\rho}\sigma^2(I_\rho(X_1^Z))\Big|_{\rho=1} = 2\int\limits_0^\infty (I^*(x) - E\,I^*(X_1^Z))(I^*(x) - I(x))dF_1^Z(x)$$

and

$$\frac{d}{d\rho}cov\,(I_\rho(X_1^Z), I_\rho(X_2^Z))\Big|_{\rho=1} = 2\int\limits_0^\infty\int\limits_0^\infty I^*(y)(I^*(x) - I(x))dG(x,y) - 2\int\limits_0^\infty E\,I^*(X_1^Z)(I^*(x) - I(x))dF_1^Z(x),$$

where the distribution functions are: $F_1^Z(x) = P(X_1^Z \le x)$ and $G(x,y) = P(X_1^Z \le x, X_2^Z \le y)$, the latter is obviously symmetric in its arguments. Since the set of risk sharing functions is convex, the necessary optimality condition for I^* in (3) is the inequality

$$\frac{d}{d\rho}J[I_\rho, a]\Big|_{\rho=1} \ge 0$$

for any risk sharing function I. Taking into account expression (2), this condition can be reformulated as: I^* maximizes the integral

$$\max_I \int\limits_0^\infty\int\limits_0^\infty I(x)\phi(x,y)dG(x,y), \tag{5}$$

where $\phi(x,y) = \alpha - 2\beta[I^*(x) - E\,I^*(X_1^Z) + (l-1)(I^*(y) - E\,I^*(X_1^Z))]$. According to the Neyman–Pearson lemma (see, for example, [12]) the function I^* maximizes in (5) if and only if

$$I^*(x) = \begin{cases} x, & \phi(x,y) > 0, \\ 0, & \phi(x,y) < 0 \end{cases} \tag{6}$$

up to a set of zero measure G. Let us consider (6) not for all $(x,y) \in R_+^2$, but only on the subset $\{(x,y) : x = y,\ x \ge 0, y \ge 0\}$, and the integral in (5) is taken over the distribution function $G(x,x)$, so the integral is $\int_0^\infty I(x)\phi(x,x)dG_1(x)$, where the distribution function $G_1(x) = P(X_1^Z \le x, X_2^Z \le x)$. Then the necessary condition for fulfilling (6) is

$$I^*(x) = \begin{cases} x, & \psi(x) > 0, \\ 0, & \psi(x) < 0 \end{cases} \tag{7}$$

up to a set of zero measure G_1.

$$\text{Here the function } \psi(x) \stackrel{def}{=} \phi(x,x) = \alpha - 2l\beta[I^*(x) - E\,I^*(X_1^Z)]. \tag{8}$$

The value $\psi(0) = \alpha + 2l\beta E\,I^*(X_1^Z) > 0$. As x increases from 0, the function $I^*(x) = x$ according to (7), therefore $\psi(x)$ decreases (see (8)) up to the point $x = k^*$ where the abscissa axis is touched. As x increases from k^*, the function $\psi(x)$ can neither decrease nor increase. Indeed, when $\psi(x)$ decreases from zero, we obtain a contradiction with (7), since $I^*(x) = 0$ for such x. Suppose that $\psi(x)$ increases from zero, then (see (7)) $I^*(x) = x$ and $\psi(x)$ must decrease. Thus, $I^*(x) = x \wedge k^*$, $k^* \in (0,\infty)$. Obtain an equation for finding the level k^* by substituting the function $I_k(x) = x \wedge k$ into expression (8) and $E\,X_1^Z \wedge k = \int_0^k (1 - F_1^Z(x))dx$ instead of $E\,I^*(X_1^Z)$. As a result, we get the optimality equation (4) in Statement 1. Note that the found optimal stop loss insurance is distinct from the deductible obtained in [13] for a continuous-time background risk insurance model with a random horizon.

Remark 2. Let $0 < \mathrm{IS} \le \infty$ denote the maximum possible value of X_1^Z or, more precisely, $\mathrm{IS} = \sup \mathit{supp}\, F_1^Z$—the upper bound of the distribution support F_1^Z. The case when the retention level $k^* \ge IS$ means the full compensation for the client's loss.

To conclude the analysis of (1), let us study the problem

$$\max_a J[I,a],\ a \in R^n \tag{9}$$

for a fixed function $I(x)$. The objective function has the form

$$E\,S_1 + w(a\triangle m^T + m_0 + E\,V) - \beta[\sigma^2(S_1) + w^2(aCa^T + 2aQ^T + \sigma^2(V))].$$

Calculating its gradient and equating it to zero, we get $J'_a[I,a] = w\triangle m - 2w^2\beta(aC + Q) = 0$. From here

$$a^* = \left(\frac{\triangle m}{2w\beta} - Q\right)C^{-1}. \tag{10}$$

Due to the positive definiteness of the matrix C, the function $J[I,a]$ is strictly concave in a. Therefore, the found vector a^* is the unique solution to (9). Thus, Statement 2 below is proved.

Statement 2. *The only solution to problem* (9) *is portfolio a^* defined in* (10).

Note that the income from insurance S_1 depends only on I, and the income from investment S_2 depends only on a. Thus, we obtain from Statements 1 and 2: the solution to the problem of maximizing the Markowitz functional with background risks is: stop loss insurance $I^*(x) = x \wedge k^*$, where k^* is defined in (4), and a portfolio $\bar{a}^* = (a_0^*, a^*) \in R^{n+1}$, in which $a_0^* = 1 - \sum_{i=1}^n a_i^*$.

4. OPTIMAL INSURANCE AND INVESTMENT IN A MODEL WITH STANDARD DEVIATION OF THE CAPITAL

The standard deviation $\sigma(S_1 + S_2)$, rather than the variance $\sigma^2(S_1 + S_2)$, is used as the risk measure in the Markowitz functional (see, for example, [10, 11] for its use in financial models). The expressions for $S_1 = S_2^I$ and $S_2 = S_2^a$ are given in Section 2.2. The studied problem is

$$J[I,a] \equiv E\,(S_1 + S_2) - \beta\sigma(S_1 + S_2) \to \max_{I,a},\ 0 \le I(x) \le x,\ a \in R^n. \tag{11}$$

The advantage of this approach is that both terms in (11) have the same dimension, for example, roubles. The detailed expression for the objective functional is (see (2))

$$\begin{gathered} J[I,a] = l\alpha E\,I(X_1^Z) + w(a\triangle m^T + m_0 + E\,V) \\ -\beta\sqrt{l\sigma^2[I(X_1^Z)] + 2l(l-1)cov(I(X_1^Z), I(X_2^Z)) + w^2(aCa^T + 2aQ^T + \sigma^2(V))}. \end{gathered} \tag{12}$$

Problems with a similar formulation, but without background risk, were studied in [14].
Let's study an auxiliary problem

$$J[I,a] \to \max_I,\ 0 \le I(x) \le x. \tag{13}$$

Statement 3. *Problem* (13) *has a solution $I_a^*(x) = x \wedge k_a^*$, where $0 < k_a^*$ is the minimal root of the equation*

$$\xi_a(k) = 0,\ \textit{where}\ \xi_a(k) = \alpha - l\beta[kE\,I^k(X_1^Z)]/(\sigma^2(S_1^{I^k} + \sigma^2(S_2^a))^{1/2}, \tag{14}$$

with the sharing rule $I^k(x) = x \wedge k$.

Proof. The existence proof for (13) repeats the arguments in the proof of Lemma 1.
As before, we denote by $I_\rho(x) = \rho I^*(x) + (1-\rho)I(x)$, where $\rho \in [0,1]$ and $I^*(x) = I_a^*(x)$ maximizes

$J[I,a]$. The necessary condition for the optimality of I^* in (13)

$$\frac{d}{d\rho}J[I_\rho,a]\bigg|_{\rho=1} \geq 0 \text{ for any risk sharing function } I.$$

This inequality can be rewritten taking into account the found expressions for the derivatives of $E\,I_\rho(X_1^Z)$, $\sigma^2(I_\rho(X_1^Z))$ and $cov\,(I_\rho(X_1^Z), I_\rho(X_2^Z))$ at the point $\rho=1$ (see the proof of Statement 1): $I^*(x) = I_a^*(x)$ is the solution to the problem of maximizing the integral

$$\max_I \int_0^\infty \int_0^\infty I(x)\phi_a(x,y)dG(x,y),$$

where $\phi_a(x,y) = \alpha - \beta[I^*(x) - E\,I^*(X_1^Z) + (l-1)(I^*(y) - E\,I^*(X_1^Z)]/(\sigma^2(S_1^{I^*} + \sigma^2(S_2^a))^{1/2}$. According to the Neyman–Pearson lemma, the function $I^*(x)$ gives maximum if and only if

$$I^*(x) = \begin{cases} x, & \phi_a(x,y) > 0, \\ 0, & \phi_a(x,y) < 0 \end{cases} \tag{15}$$

up to a set of zero measure G. As in the proof of Statement 1, we consider (15) not for all $(x,y) \in R_+^2$, but only on the subset $\{(x,y) : x = y,\, x \geq 0, y \geq 0\}$ Then, the necessary condition for fulfilling (15) is

$$I^*(x) = \begin{cases} x, & \psi_a(x) > 0, \\ 0, & \psi_a(x) < 0 \end{cases} \tag{16}$$

up to a set of zero measure G_1. Here the function

$$\psi_a(x) \stackrel{def}{=} \phi_a(x,x) = \alpha - l\beta[I^*(x) - E\,I^*(X_1^Z)]/(\sigma^2(S_1^{I^*} + \sigma^2(S_2^a))^{1/2}. \tag{17}$$

Note that the denominator in (17) is always positive, since $\sigma^2(S_2^a) > 0$. The value $\psi_a(0) = \alpha + 2l\beta E\,I^*(X_1^Z) > 0$. As x increases from 0, the function $I^*(x) = x$ according to (16), therefore $\psi_a(x)$ decreases (see (17)) up to the point $x = k_a^*$ where the abscissa axis is touched. As x increases from k_a^*, the function $\psi_a(x)$ neither decreases nor increases. Indeed, when $\psi_a(x)$ decreases from zero, we obtain a contradiction with (16), since $I^*(x) = 0$ for such x. Suppose that $\psi_a(x)$ increases from zero, then (see (16)) $I^*(x) = x$ and $\psi_a(x)$ must decrease, not increase. Thus, $I^*(x) = x \wedge k_a^*$, where $k_a^* > 0$.

For finding the retention level k_a^*, substitute the function $I^k(x) = x \wedge k$ instead of $I^*(x)$ into the expressions for $E\,I^*(X_1^Z)$ and $\sigma^2(S_1^{I^*}$ in (17). As a result, we obtain the optimality equation (14) in Statement 3. □

Note that the case $k_a^* = \infty$, i.e. (see (14)), the function $\xi(k) > 0$ on $[0,\infty)$ cannot be excluded. This means that the insurer covers the entire client's loss.

Let us now study the optimal investment problem for a fixed risk sharing function I,

$$\max_a J[I,a],\; a \in R^n. \tag{18}$$

Statement 4. *Let a_I^* be a solution to problem* (18). *Then this vector is the root of the equation*

$$\triangle m\left(\sigma^2(S_1^I) + w^2[aCa^T + 2aQ^T + \sigma^2(V)]\right)^{1/2} - w\beta(aC+Q) = 0. \tag{19}$$

Proof. The objective function is

$$E\,S_1^I + w(a\triangle m^T + m_0 + E\,V) - \beta\sqrt{\sigma^2(S_1^I) + w^2(aCa^T + 2aQ^T + \sigma^2(V))}.$$

Find its gradient in a and equate it to zero,

$$J_a'[I,a] = w\triangle m - w^2\beta(aC+Q)/\left(\sigma^2(S_1^I) + w^2[aCa^T + 2aQ^T + \sigma^2(V)]\right)^{1/2} = 0.$$

This equation is obviously equivalent to (19). If a_I^* is a solution to (18), then, taking into account the absence of restrictions on a, we obtain that a_I^* satisfies (19). □

Taking into account the results obtained in Statements 3 and 4, we have the following optimality conditions in the original problem $J[I, a] \to \max$, where the maximum is taken over I, a.

Statement 5. *Let* (I^{**}, a^{**}) *be a solution to problem* (1)*. Then* $I^{**}(x) = x \wedge k^{**}$*, where* (k^{**}, a^{**}) *is a solution to the system of n+1 nonlinear equations:* (19)*, where instead of* $I(x)$ *the function* $I^k(x) = x \wedge k$ *is substituted, and* (14)*.*

5. EXAMPLE

Let us give a numerical example of solving problem (1) with variance as a risk measure, which does not pretend to be of practical significance. Let the insureds number be $l = 10$, the number of risky assets $n = 2$, the vector of expected returns $E\bar{R} = (m_0, m_1, m_2) = (1, 1.1, 1.15)$, the vector of covariances of returns on risky assets and background risk $Q = (-0.3, -0.25)$ and the covariance matrix

$$C = \begin{pmatrix} 0.3 & 0.2 \\ 0.2 & 0.4 \end{pmatrix}.$$

The initial capital $w = 1$, the distribution function of client's loss with the background risk is equal to $F_1^Z(x) = 1 - p + p(1 - \exp(-2x))$, $x \geq 0$, where the probability of an insurance event

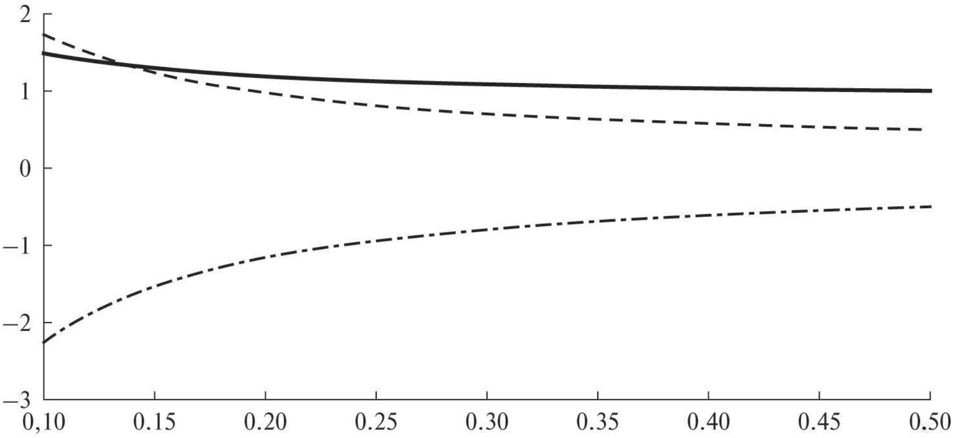

Fig. 1. Graphs of the dependence of the optimal portfolio components on β: a_1^*—solid line, a_2^*—dotted line, investment in the risk-free asset $a_0^* = 1 - a_1^* - a_2^*$—dash-dotted line.

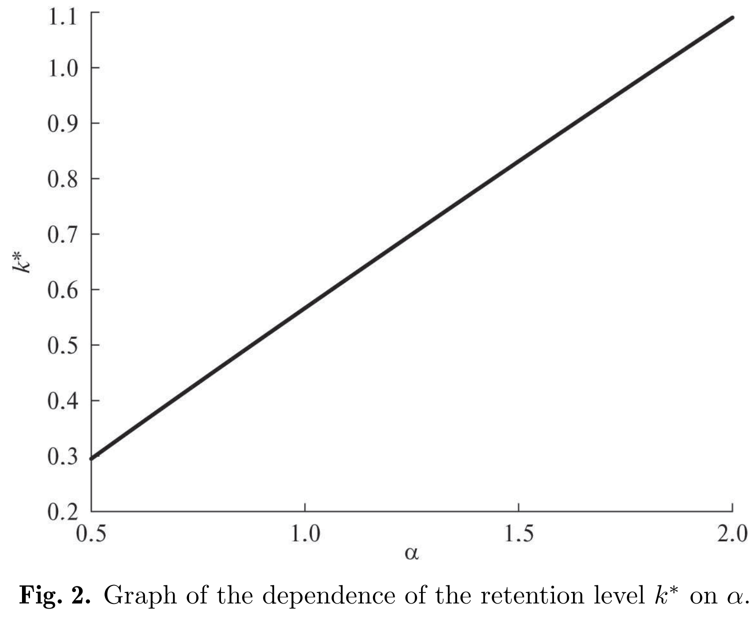

Fig. 2. Graph of the dependence of the retention level k^* on α.

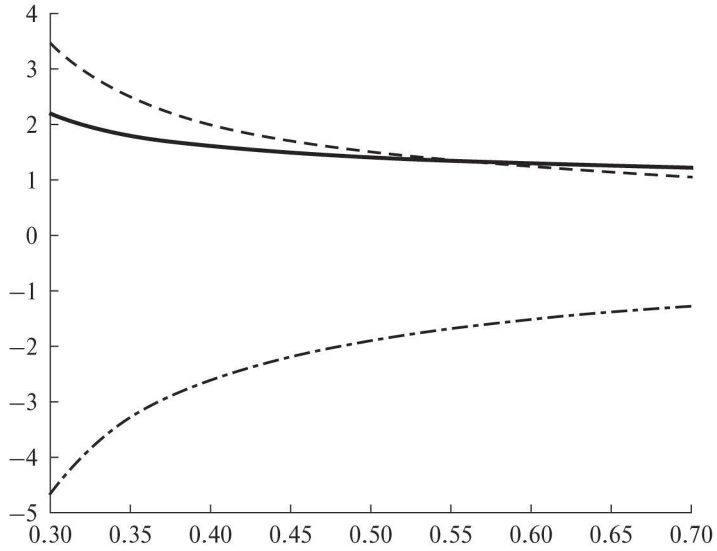

Fig. 3. Optimal portfolio components vs β: a_1^*—solid line, a_2^*—dotted line, $a_0^* = 1 - a_1^* - a_2^*$—dash-dotted line.

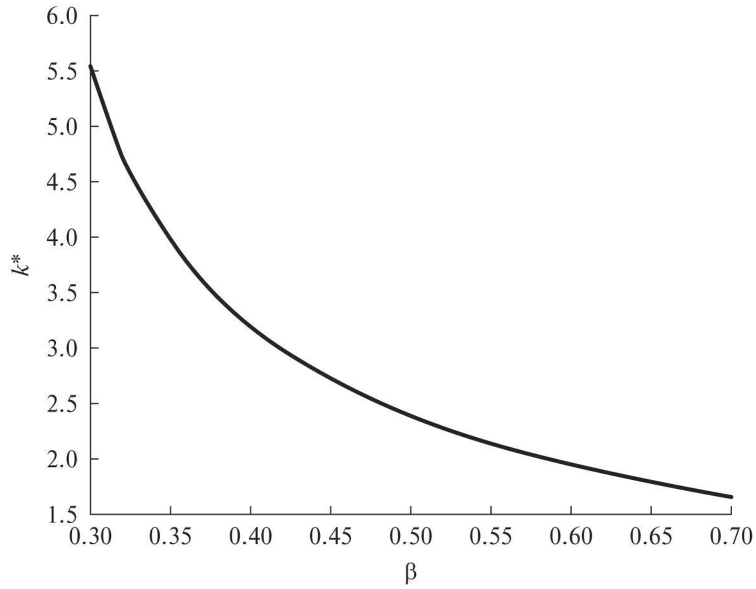

Fig. 4. Optimal level k^* vs β.

$p = P(X_1^Z > 0) = 0.2$. The caution coefficient β and the load coefficient α are variable parameters. We find the parameter k^* in the risk sharing function $I^*(x) = x \wedge k^*$ and the portfolio of investments into risky assets $a^* = (a_1^*, a_2^*)$ by applying Statements 1 and 2. The results of the numerical calculations are shown in Figs. 1 and 2.

From Fig. 1 it is evident that with the growth of the caution coefficient β investments in risky assets $a_1^*(\beta)$ and $a_2^*(\beta)$ decrease, and the function $a_0^*(\beta)$ increases, remaining negative. The latter means that the investor borrows money from a risk-free asset, but this amount decreases with the increase in β.

As quite expected, Fig. 2 shows that the level k^* in stop loss insurance increases with an increase in the loading coefficient α—as the insurance price grows, the insurer increases the share of the client's risk coverage. The function $k^*(\alpha)$ is close to linear one, which is explained by a smallness of the nonlinear part in equation (4) which determines k^*.

It is depicted on Fig. 3 that the investor borrows money from the risk-less asset and this sum decreases as well as investment into risky assets when β grows. Compared with Fig. 1, here the borrowed and invested sums are greater, which is explained by a relatively slow growth of standard deviation vs variance of the final capital.

According to Fig. 4, if the caution coefficient β increases then, naturally, the retention level k^* decreases, i.e. the insurer lessens his/her share of risk $X_1^Z \wedge k^*$.

6. CONCLUSION

This article studies a one-stage problem for an insurer-investor to select optimal insurance, i.e. risk sharing functions, for a group of clients and an investment portfolio in the presence of additive background risks. Two Markowitz-type functionals are considered, using variance and standard deviation as risk measures, respectively. In the first case, the explicit expressions for the insurance risk sharing function and the portfolio were found; in the second case, the necessary optimality conditions were found in the form of the system of nonlinear equations. Numerical examples are provided to illustrate the theoretical results. The following research directions on this topic seem promising: using not only the insurer's initial capital but also the insurance premiums received from clients as the initial investment capital; employing other objective functionals, such as the expected utility of final capital; and studying a multi-stage optimization problem with additive and/or multiplicative background risks.

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APPENDIX

A.1. PROOF OF LEMMA 1

Let $\{I_m\}$ be the maximizing sequence of risk sharing in problem (3), i.e.

$$J[I_m, a] \to J^* = \sup_I J[I, a].$$

Applying Helly's theorem to the sequence of random vectors $\{(I_m(X_1^Z, I_m(X_2^Z)\}$ (see the expression in (2) for $J[I, a]$), we obtain that there exists a subsequence $\{(I_s(X_1^Z), I_s(X_2^Z)\}$ that weakly converges to some limit (ξ, η). To prove that ξ and η are proper random variables, i.e. $P(\xi < \infty) = 1$ and $P(\eta < \infty) = 1$, it suffices to note that, by virtue of the definition $I_m(X_i^Z) \leq X_i^Z$ a.s., $i = 1, 2$. Since X_1^Z and X_2^Z are identically distributed and measurable with respect to the Borel sigma-algebra, the limit can be represented as $(\xi, \eta) = (I^*(X_1^Z), I^*(X_2^Z))$ for some measurable function $I^*(x)$, where $0 \leq I^*(x) \leq x$.

The equality $J^* = J[I^*, a]$ now follows from the weak convergence of $\{(I_s(X_1^Z, I_s(X_2^Z)\}$ to $\{(I^*(X_1^Z, I^*(X_2^Z)\}$, the continuity of the objective functional in (2) with respect to $E\,I(X_1^Z)$, $E\,I^2(X_1^Z)$, and $cov(I(X_1^Z), I(X_2^Z))$.

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