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Control in Linear Best-Response Games on Networks: A Review of Models and Methods

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Abstract—Numerous studies are devoted to systems of connected agents and network control. Historically, the greatest interest of control theory has been in averaging systems, particularly the consensus problem. However, network interaction can be characterized by more specific functions reflecting the dependence on the actions of network neighbors, which is particularly evident in models of strategic interaction on a network, the subject of game theory. This paper surveys game-theoretic models of network interaction from the class of linear best-response games. The models are formally described, and formulations of control problems in this class of games are considered. Special attention is paid to the connection with consensus models and the well-known related control problems that can be stated in terms of a strategic interaction of agents. Despite the general similarity with the models of linear systems from control theory, this class of games allows capturing qualitative aspects of strategic interaction of connected agents and highlighting the role of structural characteristics.

Keywords: peer effects, games on networks, linear systems, network control

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1. INTRODUCTION

The scientific interest in control of network interaction between different agents began to emerge in the mid-20th century, following the advent of the concept of networks. Networks were intended to describe systems of interacting agents; unlike a graph as a mathematical object strictly formalizing the nature of a link, networks do not specify the interaction mechanisms. In sociology [1, 2], economics [3, 4], and law [5, 6], the network nature of interaction implies the presence of formal and informal links between agents, which are not always possible to describe mathematically; but such links make the processes of decision-making, propagation/dissemination (e.g., of information, activity, diseases), etc., interdependent.

A peculiarity of many social systems is individual incentives of participants, which was reflected in the emergence, nearly during the same period, of game theory as a tool for describing conflicts. Since the end of the 20th century, the two approaches—network and game-theoretic—have begun to be used jointly. Games in which agents are connected by a network structure (or the network structure results from their interaction) use the results of graph theory and network analysis to determine the influence of network structure on strategic behavior.

Simultaneously, formal analysis and control methods of network systems were developed. In automatic control theory and its applications, the so-called averaging systems and consensus models [7–11] gained enormous popularity among researchers, particularly the classical DeGroot model [12] for the dynamics of opinion formation in a network structure. Mathematical methods for describing systems of strategically interacting agents have specific features, and the interaction result (outcome) often differs from consensus. Decisions made by agents under the influence of their interaction with other agents generate inefficiency, so it is topical to design and analyze various control methods.

Since the early 2000s, several control approaches have been proposed based on network characteristics of agents in socio-economic systems. Approaches to control the interaction of agents have become particularly popular in economics: models of competition and cooperation among firms, interbank interaction, and models of public goods consumption. Several game-theoretic models closely related to econometric models for identifying the influence of an agent's environment on their behavior have emerged; in such models, researchers describe the relationship between an agent's choice and the choices of neighbors. And the greatest interest has been attracted by games where the player's best response linearly depends on the actions of neighbors; formal statements of control problems in such models have arisen as well. This review is devoted to the models of the above class and corresponding control methods.

1.1. Terminology

In mathematical models of the relationship between agents' actions, the starting point is the concept of endogenous social effects [13]. Depending on the context, sociologists and social psychologists use various terms, e.g., peer influences, social norms, neighborhood effects, social interactions, herd behavior, conformism, etc. In other words, interacting with each other, agents (people, organizations, etc.) begin to learn, imitate, or copy the behavior of their environment.

In microeconomics and econometrics research, the influence of such interaction on individual behavior is usually called peer effects [16]. They are defined as the correlation of actions among members of the same reference group. To identify peer effects, the Linear-in-Means (LIM) model [13] was proposed; it studies the influence of social interactions on the behavior and outcomes of individuals in a group. The model is used to analyze how the behavior or characteristics of a group (in the original formulation, average values) affect the behavior of an individual member.

Historically, groups were understood as parts of a single population (e.g., residents of one community, classmates, colleagues, or companies competing in the same market), which was reflected in econometric models of peer effects by an explanatory variable characterizing the behavior of one of the disjoint groups. With the conceptualization of networks, it became possible to consider individual links between people (network effects)[1] to refine existing models and resolve a number of technical identification problems [16]. Identification issues of such models were discussed in [13, 16, 22, 23], as well as in the reviews [24, 25] devoted to econometric studies and [26, 27] devoted to the identification of peer effects in field and laboratory experiments.

Linear best-response games can be treated as generative models of peer/network effects: whereas econometric models focus on identifying these effects and causal relationships from real data, network games provide a theoretical basis for explaining the formation of peer effects by modeling the strategic behavior of agents. In game-theoretic models, the main attention is paid to the role of the interaction structure: how the structure affects equilibrium in the game; what is the result of local interaction effects of nodes, in terms of aggregated characteristics; which nodes and links are key. The interest in linear models is mostly motivated by the crucial role of the network in the analysis of mutual influence of nodes, equilibrium, and control problems.

[1] In their work on the theory of industrial organization [17], the authors proposed applying the term "network effects" to markets with increasing returns to scale (e.g., see [18]), and the term "network externalities" to markets where agents' actions generate inefficient equilibria. While some of the models below do capture firm interactions in markets [19, 20], the terms are not always appropriate according to [17]. With this aspect in mind, the term "network effects" will be used here both as a synonym for peer effects and in a broader sense, referring to situations in which agent's actions depend on the actions of those connected to this agent in the network, as in [21].

1.2. Network Interventions

In recent decades, many results have been obtained concerning the structure of real networks, their formation mechanisms, and the way networks affect the behavior of their participants. Examples of using network structure to elaborate economic policy have appeared [28]; and tools for estimating treatment effects in the cases of interconnected individuals have been proposed [29].

In the research of strategic interaction networks, authors examine how information about the network should be considered for solving control problems in such models.

Network effects are examples of externalities, i.e., situations where a human's well-being depends on the actions of other people without mutually agreed compensation [30]. In this vein, linear best-response games describe the influence of linear externalities in the actions of players connected to the environment. From the perspective of game-theoretic interaction models, this means that equilibrium is inefficient in the presence of externalities; for details, see subsection 2.1.

In economics, externalities have always been regarded as a source of inefficiency, and the classical way to use or counteract externalities is direct intervention in the economic process (economic interventionism). Similarly, the concept of network interventions has emerged, i.e., the process of using social interaction network data to change the outcomes of network participants.[2] The term "network interventions" gained particular popularity in the literature after the same-name paper [31], where it was formulated rather in the context of sociological studies. There exist various network intervention strategies, and assessing the applicability and effectiveness of these methods is of interest. As will be shown below, equilibrium in the game-theoretic models under consideration is always inefficient, which motivates researchers to develop control methods.

In statistical causal inference methods, the estimation of the average treatment effect traditionally assumes independent treatment effects across individuals [32]. Under network interaction, agents can influence each other, violating this assumption. In practical situations, the interaction outcomes of actors depend on the behavior of their network neighbors, and the presence of links influences the average treatment effect. This is usually described by the term "network interference": under network interaction effects, the average effect for treated individuals and the overall effect diverge, and ignoring these factors underestimates the final treatment effect [29].

1.3. Applications

There are many quantitative assessment examples for the influence of group behavior on individual outcomes. Throughout this paper, the variable x_i denotes the decision made by agent i. In most cases, x_i is understood as the agent's effort or action in a process of interest to researchers. Standard examples in the literature include the following:

- educational processes, probably the most popular application area [14, 15, 33, 34]. The variable x_i can be the student's average grade or the results of sociological surveys. While econometric models analyze how student's performance depends on the environment, the game-theoretic setting formalizes the strategic nature of efforts: for example, students choose the time spent on education or participation in additional activities, taking into account the choices of their classmates and friends;
- social norms and deviant behavior. There are many situations in which unethical behavior spreads in social networks. For example, some corporate cultures encourage weak ethical norms [35], and the presence of unethical peers in an academic context increases students' temptation to cheat [36, 37]. In this case, game-theoretic interaction models allow investigating the role of social norms in the choice of individual efforts and their position in the network structure. Here, the variable x_i is an indicator function characterizing the performance of a certain action;

[2] Network interventions can be treated as an analog of the term "network control" used for technical systems.

- crime networks [38]. This is historically the first example[3] underlying the linear quadratic game model [39] (see subsection 2.2) and the key player problem (see subsection 3.2); here, x_i is the crime effort profile. Econometric models assess how links between agents manifest deviant behavior [40]. Network games show how the effects of strategic complementarity (or substitutability, see subsection 2.1) in the efforts of interacting agents propagate through the network;
- competition and cooperation of firms [19]. Here, x_i is the output (normalized sales volume). The authors investigate the impact of collaboration in R&D work. In general, the issue of interest is the role of interfirm relations in economic development: the effect of exogenous expansion of business networks on firm productivity and the underlying mechanisms [41].

The main criticism of econometric models for identifying network effects is associated with the problem of endogeneity [13, 23], namely, the correlation between the characteristics and actions of agents, as well as between the environment and the model error. Various methods have been proposed to combat endogeneity, such as two-stage least squares with instrumental variables, nonlinear regression models, etc. (The interested reader can find the details in the surveys [24, 25].)

The purpose of this paper is to review control models and methods proposed for the case of the linear dependence of players' best-response functions on the actions of neighbors within the game-theoretic interaction framework on a network. This paper is organized as follows. In the Introduction, we briefly describe the history of network models of strategic interaction and control problems, highlighting the significant role of identification models for network effects (peer effects). Section 2 is devoted to linear best-response games, including their classification and analysis tools, as well as examples of such games and control problems solved in comparative statics. In Section 3, the Principal's problem—change the equilibrium outcome of the game—is presented; a classification of corresponding control methods, examples of control problems, and a description of solutions to several control problems for strategically interacting agents are provided. In the Conclusions, we indicate some examples from topical directions for further research.

2. LINEAR BEST-RESPONSE GAMES

2.1. Player's Payoff and Linear Best Response

Consider a noncooperative one-stage game on a network with complete information. A set of agents $N = \{1, \ldots, n\}$ strategically interact on the network defined by a symmetric adjacency matrix $G = \{g_{ij}\} \in \{0, 1\}^{n\times n}$, $g_{ii} = 0$: they try to benefit as much as possible, taking into account the actions of other players. Each participant i chooses some nonnegative action x_i and receives a payoff v_i from the decision. The payoff (utility) function

$$v_i(x) = f(x_i, z(x_{-i}, G)) \tag{1}$$

depends on the player's action x_i, i.e., the benefits and costs from the chosen action, and network effects (the actions of other players $x_{-i} = \{x_j\}_{j\neq i}$); the latter dependence is represented as a function of the player's environment $z(x_{-i}, G)$ considering the network structure (see the details below). In addition to these two components, the model may include the so-called pure externalities, i.e., the dependence of the agent's payoff on other exogenous parameters (not associated with players' actions).

Agents maximize their individual payoffs; for this purpose, each agent i needs to best respond to the actions of other agents x_{-i} :

$$BR_i(x_{-i}, G) = \operatorname*{Argmax}_{x_i \geqslant 0} v_i(x_i, z(x_{-i}, G)). \tag{2}$$

[3] https://lens.monash.edu/2019/08/06/1375976/network-science-identifying-key-players-in-collective-dynamics

The function $BR_i(x_{-i})$ is called the best-response/reply function of players: being independent of the actions of player i, this function specifies the latter's optimal response to the actions of other participants. An outcome $x^* = \{x_i^*\}_{i \in N}$ is called a Nash equilibrium in the class of pure strategies of the game if, for all i,

$$x_i = BR_i(x_{-i}, G). \tag{3}$$

Each agent is subjected to direct and indirect network effects. A direct effect on agent i is exerted by the network neighbors (the vertices adjacent to vertex i). Indirect effects arise due to the network nature of interaction: an agent is indirectly influenced by the neighbors of neighbors (second-level neighbors), those of neighbors (third-level neighbors), etc. Similar reasoning applies to the influence of network links on agents' actions: a direct effect of an edge of graph G is received by the vertices connected by this edge, and an indirect effect is received by other vertices connected to the original pair of vertices.

A key aspect of the game is the particular dependence of player i on the actions of network neighbors: in linear best-response games, the set of all best responses of player i is a singleton, and the best-response function linearly depends on the player's environment $z_i(x_{-i}, G)$; and equilibrium in such a game satisfies a system of linear algebraic equations, which are the best responses of players under $x \geqslant 0$. Currently, the following classification of player environment functions is common:

- local aggregate functions:

$$z(x_{-i}, G) = \beta \sum_{j=1}^{n} g_{ij} x_j;$$

  each player's payoff depends on the sum of the actions of network neighbors;

- local average functions:

$$z(x_{-i}, G) = \beta \sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j,$$

  where d_i is the degree of node i; each player's payoff depends on the average action of network neighbors.

The parameter $\beta \in \mathbb{R}$ has a multiplicative effect and characterizes the strength of network interaction (social multiplier [42]). In the absence of the network component ($g_{ij} = 1$ for all i, j), both classes reduce to additive aggregation games [43, 44], where each agent is affected by the same set of other agents' strategies; in games on networks, this set depends on the agent and the network.

The distinction between the above classes is how agent i accounts for (or observes) the activity of network neighbors. Unlike local aggregation, where a change in the action of agent j directly affects agent i, in the case of averaging, agent i focuses on the average characteristic of the environment regardless of the (high or low) contribution of agent j. In other words, local aggregation models are intended to describe the so-called spillover effects (the propagation of contributions of some participants to others), while local averaging models reflect behavioral aspects related to the role of the environment in the agent's choice. As shown in [45, 46], local aggregation and averaging models[4] are different both in terms of payoff function properties and in comparative statics and solutions of control problems. In particular, the role of the parameter β also differs: in local aggregation games, increasing the strength of network interaction monotonically increases the players' marginal payoff, which is not the case in the local averaging model (see subsection 2.2).

[4] In this connection, resource network models [47] deserve attention, where both mechanisms—aggregation and averaging—are, in essence, employed simultaneously, albeit not within a game-theoretic framework.

Another classification basis, proposed in [48] (including for non-network models), is the sign of the environment function. Depending on this sign, game-theoretic models receive different interpretations:

- models with strategic complementarity [49] and, accordingly, positive network effect, where the contribution of neighbor j of player i increases the marginal payoff of player i (examples can be found in [39, 45]);
- models with strategic substitutability and, accordingly, negative network effect, where the effort of neighbor j of player i reduces the marginal payoff of player i (examples can be found in [50, 51]).

Formally, the matter concerns the sign of the second cross partial derivative of the player's payoff with respect to the individual action and the action of the neighbors: if $g_{ij} = 1$, then $\frac{\partial^2 v_i}{\partial x_i x_j} > 0$ for games with strategic complementarity and $\frac{\partial^2 v_i}{\partial x_i x_j} < 0$ for games with strategic substitutability. In other words, positive (negative) local network externalities manifest in the actions of agents: for games with strategic complementarity (substitutability), an increase in the efforts of other players (x_{-i}) leads to higher actions of player i bringing relatively greater (smaller, respectively) benefit compared to lower efforts of this player. Examples of models and players' best responses are presented in Table 1.

Table 1. Models of linear best-response games

	Strategic complementarity	Strategic substitutability
Local aggregation	Linear quadratic game $BR_i = b_i + \beta \sum_j g_{ij} x_j$	Local public goods game $BR_i = b_i - \beta \sum_j g_{ij} x_j$
Local averaging	Social norms in networks $BR_i = (1-\beta) b_i + \beta \sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j$	Anticonformism $BR_i = (1-\beta) b_i - \beta \sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j$

Within any economic model, an ideal state of the system (maximum social welfare) is commonly identified. In the case of control problems, this refers to the so-called first-best outcome/contract (social optimum), by analogy with Principal–agent problems in contract theory [52]. In works devoted to control problems in linear best-response games, social welfare is almost always understood as the total payoff of all agents:

$$W = \sum_{i=1}^{n} v_i$$

(see subsection 3.1 for details); and the social optimum (i.e., the solution of the problem $\max_x W$ and the corresponding equilibrium outcomes x^O and payoffs W^O) is studied in comparative statics, assuming that agents seek to maximize not their individual payoff functions but the welfare of all agents in aggregate. As will be demonstrated below (on examples of particular games), the Nash equilibrium is never the social optimum: the network effect either underestimates or overestimates the actions of agents. Therefore, it is topical to develop control methods in such games.

2.2. Mathematical Models of Network Effects: Coordination, Competition, Cooperation, and Awareness

Coordination game

The classical model of social influence and opinion dynamics in a network [53, 54] is a special case of the consensus problem, which has important applications in engineering control problems [8]. This model can also be treated as a microeconomic model of strategic decision-making by agents [55].

That is, let n agents seek to maximize their individual payoff characterized by the function

$$v_i = -\sum_{j=1}^{n} \frac{g_{ij}}{d_i}(x_i - x_j)^2. \tag{4}$$

In other words, it is disadvantageous for an agent to make a choice (e.g., technology or language) differing from those of neighbors. Due to the concavity of the function $v_i(x)$, the first-order optimality conditions are sufficient to calculate the system of best responses: the optimal response of player i to the actions of network neighbors,

$$BR(x_{-i}) = \sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j, \tag{5}$$

coincides with the opinion dynamics model. For instance, DeGroot's model can be interpreted as a game where participants exhibit behavior similar to that of network neighbors by best responding to their actions.

In such a game, any action profile where all agents take the same action constitutes a Nash equilibrium. Moreover, agents possess no individual characteristics distinguishing them from each other. Therefore, a model reflecting the interaction of agents with individual skills under the average behavior of their environment has become popular.

Social norms in networks

From a game-theoretic interaction standpoint, another modification of DeGroot's model is of interest, originally proposed in [56]. (The Friedkin–Johnsen model describes the best-response dynamics of players; the dynamics were analyzed, e.g., in [57].) In this modification, each node i additionally maintains a constant internal opinion. There are several popular versions of the model in a game-theoretic setting, particularly the one with incomplete information where agents try to determine some state of the world [58]. In this case, the interpretation of the model is close to the well-known beauty contest game [59].

Below, we consider the local averaging model, also known as the social-norm model [46]. Each player receives a payoff[5] of the form

$$v_i = b_i x_i - \frac{1}{2}\left(\frac{\beta}{1-\beta}\right)\left(x_i - \sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j\right)^2 - \frac{1}{2}x_i^2, \tag{6}$$

where $b_i > 0$ reflects the individual productivity of agent i, and $\beta > 0$ is the taste for conformism. Agents undertake costly actions $(-x_i^2/2)$, receive the marginal payoff $b_i x_i$ independent of the neighbors' efforts, and compare their individual contribution x_i with the average contribution of their network neighbors, seeking to minimize the difference between their individual action and the average action of their group. In view of the significant role of the term $\sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j = \overline{x}_i$, the authors called it the *social norm* of agent i in other publications in the context of sociological studies.

[5] Other payoff functions can be considered, e.g.,

$$v_i = b_i x_i - \frac{\theta}{2}\sum_j \frac{g_{ij}}{d_i}(x_i - x_j)^2 - \frac{x_i^2}{2}$$

or

$$v_i = b_i x_i - \theta x_i \sum_j \frac{g_{ij}}{d_i} x_j - \frac{1+\theta}{2} x_i^2,$$

where $\theta = \frac{\beta}{1-\beta}$. According to an observation of the authors [46], these payoff functions share the same best-response function; however, even if the equilibrium efforts x^* are identical, welfare analysis and its comparative statics may differ since the equilibrium utilities, and hence welfare, are not the same.

The first-order optimality conditions $\partial v_i/\partial x_i = 0$ yield the players' best responses

$$x_i = (1-\beta)b_i + \beta\overline{x}_i, \tag{7}$$

which can be equivalently written in the matrix form

$$x = (1-\beta)b + \beta\hat{G}x, \tag{8}$$

where $\hat{G} = \{\hat{g}_{ij} = g_{ij}/d_i\}^{n\times n}$. For $\beta < 1$, there exists a unique Nash equilibrium in pure strategies:

$$x^* = (1-\beta)(I-\beta\hat{G})^{-1}b. \tag{9}$$

First, note the effect of individual productivity. Although the influence of b_i on v_i^* is always positive, the mutual influence of agents' productivity can be multidirectional:

$$\frac{\partial v_i^*}{\partial b_j} \gtreqless 0 \iff x_i^* \gtreqless \overline{x}_i^*. \tag{10}$$

In other words, the equilibrium utility of agent i increases (decreases) in response to a small change in the productivity b_j of agent j iff the equilibrium response of player i is greater (smaller, respectively) than the equilibrium social norm of this player. Of main interest is the dependence of the game outcome on the parameter β. As shown by the authors, the dependence is generally nonmonotonic, and they considered two extreme cases as follows:

- pure individualism ($\beta = 0$), when each player's equilibrium response equals the individual productivity:
$$\sum_{j=1}^{n} x_j^* = \sum_{j=1}^{n} b_j;$$
- total conformism ($\beta \to 1$), when all agents choose the same effort equal to the weighted average of individual productivity:
$$\lim_{\beta\to 1}\sum_{j=1}^{n} x_j^* = n\sum_{j=1}^{n}\overline{d}_j b_j,$$

where $\overline{d}$ is the vector of normalized vertex degrees, $\overline{d_i} = d_i/\sum_{j=1}^{n} d_j$. When will the total contribution $\sum_j x_j^*$ be higher, under pure individualism ($\beta = 0$) or total conformism ($\beta \to 1$)? The answer depends on the correlation between the productivity b and the vertex degree distribution in the graph G :

$$\lim_{\beta\to 1}\sum_{j=1}^{n} x_j^* \gtreqless \sum_{j=1}^{n} x_j^*(\beta = 0) \iff Corr(\overline{d}, b) \gtreqless 0. \tag{11}$$

If b and $\overline{d}$ are positively (negatively) correlated, i.e., the agents with higher productivity occupy more (less) central positions in the network, total conformism will increase (decrease, respectively) the aggregate efforts.

The social optimum in the model, i.e., the solution of the problem

$$\max_x \quad b^T x - \frac{1}{2}x^T H(\beta)x, \tag{12}$$

where $H(\beta) := I + \frac{\beta}{1-\beta}\left(I-\widehat{G}\right)^T\left(I-\widehat{G}\right)$, is achieved by solving the system of all best responses

$$x_i = (1-\beta)b_i + \beta\overline{x}_i + \beta\sum_{j=1}^{n}\frac{g_{ij}}{d_i}\left(x_j - \overline{x}_j\right). \tag{13}$$

The distinction between the optimum and the Nash equilibrium in the game is the last term in this system of the best responses of players (the influence of the agent's individual action on the social norm of neighbors). In the Nash equilibrium, when deciding on their individual efforts, agents neglect this factor, creating a (positive or negative) externality: if i and j are neighbors, then

$$\frac{\partial v_i}{\partial x_j} \gtreqless 0 \iff x_i \gtreqless \overline{x}_i. \tag{14}$$

In other words, by exerting the effort x_j, agent j imposes a positive (negative) externality on the neighbor i iff the effort of i is above (below, respectively) the social norm of agent i. The optimum is achieved when the players' equilibrium responses match their individual productivity:

$$x^O = b. \tag{15}$$

In this case, a class of regular networks is distinguished where the aggregate efforts are always optimal: $x^* = x^O = b$. The reason is that in regular networks, the positive and negative externalities imposed by agents on their neighbors balance each other, so the aggregate effect becomes optimal.

Individual productivity is also crucial in the problem of adding and removing links between agents: the Principal takes into account only the productivity of agents, while information about the network structure does not matter:

(1) key-link adding: in any network, adding a link between two agents with high (low) productivity increases (decreases, respectively) their efforts and, moreover, increases (decreases, respectively) the efforts of all other agents in the network;
(2) key-link removing: regardless of the network structure, the Principal should remove the link between the two most productive agents.

The solution of the incentive problem will be discussed below when analyzing control in the linear quadratic game (subsection 3.2).

Linear quadratic game

The linear quadratic game on a network, proposed in [39], is a most well-known model of network games. The payoff function of player i,

$$v_i = x_i \left(b_i + \beta \sum_{j=1}^{n} g_{ij} x_j \right) - \frac{x_i^2}{2} \tag{16}$$

includes benefit from the individual action, $x_i b_i$, and the actions of neighbors, $\beta \sum_j g_{ij} x_j x_i$, as well as the quadratic cost of the decision, $x_i^2/2$. The parameter $b_i > 0$ is the standalone marginal return of player i. The parameter β reflects the dependence on neighbors' actions: for $\beta > 0$, the actions of players are strategic complements; for $\beta < 0$, strategic substitutes.

In the linear quadratic game, the first-order optimality conditions

$$\frac{\partial v_i}{\partial x_i} = b_i + \beta \sum_{j=1}^{n} g_{ij} x_j - x_i = 0 \tag{17}$$

lead to the following best-response functions:

$$BR_i(x_{-i}) = b_i + \beta \sum_{j=1}^{n} g_{ij} x_j. \tag{18}$$

In matrix form, the system of the best responses can be written as

$$(I - \beta G)x = b,$$

where I denotes an identity matrix of appropriate dimensions. If the matrix $(I - \beta G)$ is invertible (for details, see subsection 2.3), then there exists a unique Nash equilibrium: the equilibrium responses of players (in matrix form) are given by

$$x^* = (I - \beta G)^{-1} b. \tag{19}$$

The idea of the authors [39] was to create a bridge between the concept of Nash equilibrium and measures of centrality for network nodes. Consider a network with an adjacency matrix G and a scalar value $\beta \geq 0$. Then the vector of centralities [60] of the parameter β,

$$\sum_{k=0}^{+\infty} \beta^k G^k 1 = (I - \beta G)^{-1} 1$$

reflects the total number of paths in G starting at vertex i. The parameter β is a decay factor reducing the relative weight of the longest paths. Thus, the Nash equilibrium in the linear quadratic game on a graph exactly coincides with the vector of Katz–Bonacich centralities.

A more general model was considered in [61]: an arbitrary weighted network, the set of agent's admissible actions $x_i \in [0, L]$, and the players' best response

$$BR_i(x_{-i}) = \min\left(b_i + \beta \sum_{j=1}^{n} g_{ij} x_j, L\right). \tag{20}$$

For $L = \infty$, the original model is obtained.

The comparative statics of the model were investigated in [62]: the social optimum (the solution of $\max_x \sum v_i$) leads to the following system of players' best responses:

$$b_i - x_i + 2\beta \sum_{j=1}^{n} g_{ij} x_j = 0. \tag{21}$$

It can be equivalently written as

$$x_i^O = BR_i(x_{-i}) + \beta \sum_{j=1}^{n} g_{ij} x_j, \tag{22}$$

i.e., the equilibrium efforts are too small since each agent ignores the positive influence (positive externality arising from complementarity) of the individual action on the choices of neighbors. As a result, the equilibrium in the game turns out to be inefficient. Due to this effect, agents' actions are always underestimated, and the Principal in the incentive problem will choose unidirectional interventions: all agents will either receive additional subsidies or, conversely, be subjected to fines/taxes. (For a detailed discussion of this issue, see subsection 3.2.)

Obviously, the network structure monotonically affects the social optimum as well. In the case of symmetric matrices, the following result was established in [63]: for two matrices Σ, Σ' such that $\Sigma = -I + \beta G$ and $\Sigma' > \Sigma$, if $\sigma'_{ij} > \sigma_{ij}$ for all i, j, then

$$x^*(\Sigma') > x^*(\Sigma). \tag{23}$$

In the network formation model proposed in [64], agents choose other agents to create a link by maximizing their individual payoff according to formula (16). Due to the indicated properties of the function, agents with the highest centrality measure will receive the greatest benefit, and others will seek to create links with them.

Local public goods game

Games on networks with strategic substitutability describe situations where the action of one agent reduces the incentives for neighbors to increase their efforts. Such games model competitive scenarios, e.g., competition for limited resources, markets, or benefits, where players are connected through an interaction network, and their strategies are substitutable. This effect is opposite to strategic complementarity, where one agent's actions stimulate neighbors to increase efforts.

In the local public goods game on a network, agents decide on their individual contributions to the production of some non-excludable good (i.e., the agents cannot be excluded from its consumption), which benefits both themselves and their network neighbors. In many studies, this model was interpreted as a model of sectoral market theory [65], particularly an oligopoly [66]; and some versions were further developed in the field of business games and behavioral experiments [67]. This model is closely related to oligopoly theory because both consider strategic interaction among players with the strategic substitutability of actions.

Let firm i produce some quantity of goods x_i, and let its profit be $v_i(x) = x_i p(\sum_j x_j) - cx_i$, where $p(\sum_j x_j)$ denotes the inverse demand function, and c is the costs. In the case of linear inverse demand, the profit of firm i can be written as

$$v_i = x_i\left(b - \left(x_i + \beta \sum_{j=1}^{n} g_{ij}x_j\right)\right) - cx_i, \tag{24}$$

where $g_{ij} = g_{ji} = 1$ means that the products of firm i and firm j are substitutable, β is the degree of substitutability, and b is the market size. More general network models were studied in [19, 68]. Similar models were considered in [69, 70], where they were supplemented by the parameter r_i (type) of agent i (reflecting the efficiency or qualification of the latter's activity). The payoff function of players has the form

$$v_i = x_i\left(b - \sum_{j=1}^{n} x_j\right) - \frac{x_i^2}{2r_i}, \tag{25}$$

and the best response on an arbitrary graph G can be represented as

$$\left(\frac{1}{r}I + \beta G\right)x = b. \tag{26}$$

The model formulated in [19] simultaneously accounts for the effects of strategic complementarity and substitutability: firms compete in markets while cooperating in R&D projects (the so-called Cournot oligopoly game with the spillover effect of R&D collaborations). The player's best response in this model takes the form

$$x_i = b_i - \rho\sum_{j=1}^{n} b_{ij}x_j + \beta\sum_{j=1}^{n} g_{ij}x_j, \tag{27}$$

where $B = \{b_{ij}\}$ is the competition matrix between firms (a link between firms arises only if they compete in the same market), and $G = \{g_{ij}\}$ is the cooperation matrix in R&D. Under rather nontrivial conditions, there exists a unique equilibrium given by

$$x^* = (I + \rho B - \beta G)^{-1}b. \tag{28}$$

Up to this point, we have considered the case of common knowledge, i.e., a situation where each agent has complete information about the individual characteristics of other agents, the structure of links, etc. Reflexive games are one approach to analyzing information asymmetry. A reflexive game is a model of agents' decision-making based on a hierarchy of their beliefs about opponents' behavior [71]. If the awareness structure in such a game has finite complexity (a point-based or pointwise information structure [50] representing a finite hierarchy of beliefs), then a reflexive game graph $\overline{G}$ can be constructed, clearly demonstrating the relationship between the actions of agents participating in equilibrium. In the case of the linear best response of players[6], i.e.,

$$BR_i(x_{-i}) = b_i - \beta \sum_{j=1}^{n} \overline{g}_{ij} x_j, \tag{30}$$

the informational equilibrium can be found using the formula

$$x^* = (I + \beta\overline{G})^{-1} b, \tag{31}$$

where the vector b reflects the awareness of agents, and $\overline{G}$ is the reflexive game graph [72]. The vertices of the graph $\overline{G}$ correspond to real and phantom agents participating in the reflexive game, and the arcs of $\overline{G}$ reflect the mutual awareness of agents.

In this model, solutions of many control problems (particularly in comparative statics) are close to those of problems in the linear quadratic game on a network. For the case of a unique equilibrium in the model, the authors [73] considered the incentive-targeting problem; see subsection 3.2.

2.3. Equilibrium Analysis of Linear Best-Response Games

The existence and uniqueness of equilibrium in the above examples are usually established in two ways: via existence theorems for a solution of a system of linear algebraic equations (as in [39]) or via proofs of the existence of a potential function for a game.

The concept of potential games was pioneered in [74]. The main idea is to show the existence of a function describing the outcome of a strategic interaction of players using a scalar function.[7] A function φ is a potential function (best-reply potential) of a game if, for any x_i, x_i', and x_{-i} and all $i \in N$,

$$\varphi(x_i, x_{-i}) - \varphi(x_i', x_{-i}) = v_i(x_i, x_{-i}) - v_i(x_i', x_{-i}). \tag{32}$$

Linear best-response games possess a potential function (generally, nonunique): for example [77], it can be represented in the matrix form

$$\varphi(x) = x^T 1 - \frac{1}{2} x^T (I \pm \beta G) x. \tag{33}$$

The maximum of a potential function corresponds to a Nash equilibrium. A sufficient condition for a unique solution is the concavity of a potential function. The matrix of second derivatives takes the form $\nabla^2 \varphi(x) = -(I \pm \beta G)$, and $\varphi(x)$ is strictly concave iff the matrix $I \pm \beta G$ is positive definite: for any $y \neq 0$,

$$y^T (I \pm \beta G) y > 0. \tag{34}$$

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[6] The original paper considered the model

$$v_i = x_i \left(b - \sum_{j=1}^{n} \hat{g}_{ij} x_j \right) - \frac{x_i^2}{2} \tag{29}$$

and $g_{ii} = 1$.

[7] As demonstrated in [75], for some classes of games, the potential function is a Lyapunov function of the game dynamics; hence, the Lyapunov function method can be applied for equilibrium analysis [76].

For local aggregation games, depending on the sign of βG, a necessary and sufficient condition for the existence of a unique equilibrium is either $\beta\lambda_{\max}(G) < 1$ in the case of strategic complementarity $(-)$, or $\beta|\lambda_{\min}(G)| < 1$ in the case of strategic substitutability $(+)$, where $\lambda_{\max}(G)$ and $\lambda_{\min}(G)$ are the largest and smallest eigenvalues of the matrix G, respectively. In local averaging games, these conditions are not required: G is a row-stochastic matrix ($\lambda_{\max}(G) = 1$), and the existence condition becomes $\beta < 1$.

A significant distinction of games with strategic substitutability is the existence of a corner equilibrium [78]: the potential function ceases to be concave, and by the game's property of nonnegative strategies x^*, there arises a game situation (strategy profile) where some players—passive agents—are inactive ($x_i = 0$) whereas the others—active ones—choose the maximum admissible action $x_j = b_j$. In this strategy profile, the equilibrium existence condition is related to the presence of a maximal independent set in the graph G, which, like the number of equilibria, may be nonunique: active players must not be connected among themselves.

In the case of an asymmetric matrix G, a sufficient condition for equilibrium existence is the positive definiteness of the Hermitian component of the matrix $G = \frac{G+G^T}{2}$ and, consequently, $1 + \delta\lambda_{\max(\min)}(\frac{G+G^T}{2}) > 0$. However, in the case of strategic substitutability, the analysis of best-response dynamics is more complicated (see the discussion below).

As shown in [63, 100], an equilibrium in a linear best-response game is a solution of the linear complementarity problem (LCP). According to [76], LCP is a subclass of variational inequality problems[8]; the analysis of games in this vein (including the case of nonlinear best response) was presented in [76, 79, 80] and the references therein. Using the formulation of Nash equilibrium as a solution of the variational inequality problem, the authors [76] investigated the relationship between the parameterization of the game and the matrix of partial derivatives of the best-response operator (the Jacobian of the game), as well as the influence of the Jacobian's properties on the existence and uniqueness of equilibrium and the convergence of best-response dynamics in discrete and continuous time.

2.4. Best-Response Dynamics

Above, the interaction between agents has been formulated as a one-stage noncooperative game, but this interaction can be represented alternatively. In [81], system (18) was formulated as a linear affine dynamic system in continuous time:

$$\dot{x} = Ax + b, \tag{35}$$

where $A = \beta G - I$. For each agent i,

$$\dot{x_i} = b_i + \beta \sum_{j=1}^{n} g_{ij} x_j - x_i. \tag{36}$$

The solution of the system, or its steady state, is

$$\begin{aligned} \lim_{t\to\infty} x(t) &= -A^{-1}b \\ &= -(\beta G - I)^{-1}b \\ &= (I - \beta G)^{-1}b. \end{aligned} \tag{37}$$

[8] The formulation of Nash equilibrium as a solution of a variational inequality generalizes the game's potential property [74].

The difference approximation of equation (35) is also of interest[9]: it can be used to establish a relationship between the discretization step δ and the network effect β. The particular value of the factor at the interaction matrix is related both to the direct influence of the environment on the agent's choice and to the discretization step of the model. In addition, the best responses of players may be bounded due to the specifics of various real-life situations: in most cases (effort level in collective behavior models, output in production models of goods or services, etc.), the best response is nonnegative; in the popular analysis model of educational processes, agents choose effort levels often measured in hours, and their best responses are bounded (below by zero and above by some threshold) and are also supposed to be integer. All these aspects restrict the choice of appropriate tools for analyzing dynamics in such games [82]. The relationship between the game-theoretic interpretation and the theory of dynamic systems, particularly for the above model, was discussed in the book [83].

In game-theoretic terminology, the procedure for finding equilibrium by players is called tatonnement, e.g., Nash/Cournot tatonnement (borrowed from French). In games with strategic substitutability, an equilibrium that is stable with respect to discrete Nash tatonnement implies stability of equilibrium with respect to continuous best-response dynamics, but the converse is not true [78]. The convergence of continuous best-response dynamics to Nash equilibrium in such games was shown in [84]; the case of farsighted agents was considered in [85].

In a game on a directed graph G, best-response dynamics can cycle. Two classes of directed networks for which best-response dynamics converge to a unique equilibrium were introduced in [86]. One class is determined by the possibility of changing the game parameters in a special way so that the dynamics of the new game coincide with those of the original one; the other class of networks is closely related to the Kolmogorov criterion for reversibility of a Markov chain.[10]

2.5. Relation to Other Fields

The main problems and applications in control of networks were considered in [7, 8]. In this field, control issues in networks (in particular, collective, cooperative, and multi-agent control) are studied. Let $x_i(t) \in \mathbb{R}^n$ be the state vector of network nodes at a time instant t, $i = 1, \dots, n$; this vector consists of elements g_{ij} reflecting the link between vertices i and j in a graph G. As noted by the authors, a wide class of multi-agent networks is described by continuous-time models of the form

$$\dot{x_i} = F_i(t, x_i, u_i) + \sum_{j=1}^{n} g_{ij}(t) f_{ij}(x_i, x_j), \tag{38}$$

where functions $F_i(\cdot)$ and $f_{ij}(\cdot)$ characterize the agent's local dynamics and interaction with other agents, respectively. In most widespread networks, the interactions of agents depend only on their disagreements (the divergence of their states); and linear networks with the function

$$f_{ij}(x_i, x_j) = x_i - x_j \tag{39}$$

are the best studied by now. The remainder of this review is devoted precisely to such models. At the same time, a formulation similar to (38) is valid not only for consensus models but also for game-theoretic interaction models, while the function $f_{ij}(x_i, x_j)$ has specific features (see subsections 2.1 and 2.4 of the paper).

Input–output (interindustry) models [89, 90] are closely related to the above ones as well; similar issues were studied in works devoted to dynamic input–output models [91–93] and optimal planning

[9] Formula (19) has at least three interpretations: it is the equilibrium of a game-theoretic model, the equilibrium of a linear affine dynamic system, and also a centrality measure of network nodes.

[10] The relationship between directed graphs and the Kolmogorov criterion was also addressed in [87, 88].

models [94]. Let x_i be the output of (economic) sector i, and let G be the direct cost matrix. Its elements reflect the costs (input) of sector i to provide one unit of goods or services (output) of sector j. Then the total cost matrix is

$$\Lambda = (I - G)^{-1},$$

whose elements reflect the output of sector i required to provide one unit of the final output of sector j. In the literature, a more general case of the matrix Λ, depending on a parameter β (the Leontief inverse), is also considered [89]: $\Lambda = (I - \beta G)^{-1}$.

Various ways of describing game-theoretic interaction on a network were addressed in [95, 96]; control problems in the case of strategic network formation, in [97, 98]; and in models of bounded rationality, in [50, 71]. For details, the reader is referred to subsection 2.2. This review does not cover cooperative games [99], although several cooperative solutions of the games described above were proposed in [100–102].

Note separately the works devoted to control problems in social network models [12, 103] and in models of mixed social influence (the mechanisms of assimilative/dissimilative influence and bounded trust) [104]. As demonstrated above, such models can also be treated as models of strategic interaction.

3. CONTROL PROBLEMS

3.1. Principal's Objective Functions and Budget Constraints

A control authority (also called Principal or central planner in the literature) receives information about the participants' payoffs and can influence their actions, generally speaking, at different time instants and in various ways. Consequently, the outcomes of agent interaction change, becoming more desirable for the Principal. The subject of control can be various elements of the system, i.e., agents' actions, their incentives, the interaction structure, and other characteristics.

In the case of linear best-response games, the control problem is posed as a general optimization problem of the form

$$\max_u W(x, u) \tag{40}$$

subject to the constraints

$$x \geqslant 0, \quad x = BR(x, u), \quad K(u, u_0) \leqslant C, \tag{41}$$

where u is a game parameter influenced during the control process, with u_0 specifying its initial value; $K(u, u_0)$ is a cost function characterizing constraints on parameter variations relative to the initial value; and C is a constant. As shown above, in the analysis of comparative statics and social optimum, the vector x (the players' best responses) can be treated as a control action. Thus, a two-stage game is considered: at the first stage, the Principal applies control, and at the second stage, the agents play out an equilibrium.

Although the same best-response function may correspond to various payoff functions, the solution of the control problem may differ depending on the function maximized by the Principal:

- increasing/decreasing the aggregate outcomes of players:

$$W = \sum_{i=1}^{n} x_i \longrightarrow \max;$$

- increasing the total payoff of all participants, i.e., the Benthamite (utilitarian) social optimum:

$$W = \sum_{i=1}^{n} v_i \longrightarrow \max;$$

- increasing the payoffs of the least well-off participants, i.e., the Rawlsian (maximin) social optimum:

$$W = \min_i v_i \longrightarrow \max.$$

The challenge of choosing the Principal's objective functions in such problems has received little attention so far; for example, see [105]. And the most popular choice is the social welfare function (the sum of agents' payoffs).

There exist several classifications of control problems applicable to the models under consideration [31, 95, 106, 107]. In [31], the following network intervention strategies were presented, each with different tactical alternatives:

- individual (the identification of key network participants based on some network characteristics, see Table 2);
- segmentation (network division into groups of nodes for further intervention);
- alteration (the change of links between participants or the entire interaction network, see Table 2);
- induction (network excitation for activation of new interactions (cascades) between participants, which can be informational/behavioral, etc.; for example, see [108]);

Table 2. Control problems in linear best-response games

Problem type	Description	Publications
Individual	Key network participants are identified based on some network characteristics	Galeotti and Goyal, 2009 [109]; Candogan, Bimpikis, and Ozdaglar, 2012 [110]; Bloch and Querou, 2013 [111]; Fainmesser and Galeotti, 2016 [112]; Demange, 2017 [113]; Galeotti, Golub, and Goyal, 2020 [73]; Kor and Zhou, 2023 [114]; Belhaj, Deroian, and Safi, 2023 [116]; Jeong and Shin, 2024 [115]; Dasaratha, Golub, and Shah, 2024 [117]
Alternation	Links between participants or the entire interaction network are changed	Borgatti, 2006 [118]; Ballester, Calvo-Armengol, and Zenou, 2006 [39]; Corbo, Calvo-Armengol, and Parkes, 2006 [119]; Corbo, Calvo-Armengol, and Parkes, 2007 [63]; Golub and Lever, 2010 [120]; Konig, Tessone, and Zenou, 2014 [64]; Belhaj, Bervoets, and Deroian, 2016 [121]; Matveenko and Korolev, 2016 [122]; Hiller, 2017 [123]; Harkins, 2020 [124]; Li, 2023 [125]; Sun, Zhao, and Zhou, 2023 [126]

Several authors distinguished institutional interventions, emphasizing the specifics of changing the rules of interaction [95, 106]. In [95], staff control (control of the composition of participants) was singled out into a class of problems; in [31], this class was attributed to alteration problems (control of the interaction structure, structural control).

In works devoted to control in linear best-response games, u can be the staff of players $i \in N$, the incentives b, or the interaction structure G. Solutions of some problems are described below; and the table lists the related publications according to the above classification.

Control of agents' incentives

In such control problems, the Principal changes the individual characteristics of agents' activities. Incentive problems in which the Principal seeks to maximize social welfare in models with strategic complementarity were considered in [73, 113] (including the case of the coordination game [115]). Monopolist's pricing problems under local consumption externalities were studied in [110, 111], particularly the relationship between consumer centrality in the network and the prices and quantities offered by the monopolist.

Structural control

Structural control adjusts the level of activity by changing the network structure. Creating or removing links with such control affects the centrality of agents, thereby changing the equilibrium. Researchers investigate optimal networks from the Principal's standpoint. According to [64], the class of nested split graphs (NSGs) stands out among simple networks maximizing a Principal's concave objective function. In any network that is not an NSG, links can be created between agents to improve welfare in the linear quadratic game on this network [121].

Control under uncertainty

Many authors resort to analyzing the stability of control methods with respect to exogenous disturbances or probabilistic uncertainty; for example, see [73, 78, 127, 128]. Importantly, data on the network structure are often difficult to obtain [129, 130]; this aspect has motivated the appearance of models for both probabilistic and game-theoretic uncertainty regarding the network structure.

For instance, in [112], the Principal chooses the optimal price level for agents, having only information about the degree distribution of network nodes. In another case considered, the network structure is observable, but the location or identity of agents is private information [131]. In other words, the Principal cannot distinguish networks that are identical up to node permutation. An essential feature of this game design is that the vector x^* of equilibrium responses becomes a contract offered by the Principal to the agents. The authors identified classes of network structures for which the Principal's optimal contracts are incentive-compatible for agents and their coalitions.

Network identification

One of the most important problems is network identification. There are several methods for solving this problem based on data from a series of game outcomes, involving a statistical approach (graph regularization) [132], optimization [133], and machine learning [134]. A separate line of research is the solution of bilevel optimization problems[11] [136], where the observed equilibrium responses of players are supposed to arise from optimizing the interaction structure (by the agents) in terms of the social welfare function.

Another network identification approach is based on information about the best-response dynamics of players and the solution of a control problem [137]. The Principal does not know the network structure in the game but observes the players' best responses and manipulates the actions of some players. As demonstrated by the authors, the network structure identifiability criterion is equivalent to the rank controllability criterion of the system reduced to a special form; and they applied the algorithm [138], originally developed for identifying stable linear systems.

3.2. Control in the Linear Quadratic Game on a Network

Perhaps the linear quadratic game on a network is the first and most thoroughly studied game from the perspective of control problems; for this game, there are many statements of control problems, see [39, 118] and the references above. Several problems will be considered below to demonstrate the features and specifics of control problems in linear best-response games. The

[11] Bilevel optimization problems stem from Stackelberg games, where the outer (or inner) problem optimizes the Leader's (or Follower's) actions [135].

payoff of players in this model has the form

$$v_i = x_i\left(b_i + \beta\sum_{j=1}^{n} g_{ij}x_j\right) - \frac{x_i^2}{2}, \tag{42}$$

and the equilibrium in the game is

$$x^* = (I - \beta G)^{-1}b.$$

Historically, the key player problem was the first. A key player is a player whose removal has the greatest impact on the aggregate outcome. Let G be an original (symmetric) adjacency matrix and G^{-i} be the new matrix obtained from G by zeroing out the row and column corresponding to the ith vertex. Then the Principal's problem takes the form $\max_{i\in N}\left[\sum_j x_j^*(G) - \sum_j x_j^*(G^{-i})\right]$ or equivalently,

$$\min_{i\in N}\sum_{j=1}^{n} x_j^*(G^{-i}). \tag{43}$$

Let $M = \{m_{ij}\} = (I - \beta G)^{-1}$, and $k = \{k_i\} = (I - \beta G)^{-1}1$. The authors introduced a special intercentrality measure to capture both the individual centrality of players and their contribution to the centrality of others:

$$c_i = \frac{k_i^2}{m_{ii}}.$$

As shown, the vertex with the largest value of c_i solves problem (43).

The incentive-targeting problem was considered in [73]: the Principal seeks to maximize the social welfare function in equilibrium:

$$W = \sum_{i=1}^{n} v_i = \frac{1}{2}x^T x \longrightarrow \max_b. \tag{44}$$

Thus, the Principal changes the standalone marginal return $\widehat{b} \to b$ of players, which is interpreted as a variation in players' incentives (e.g., monetary subsidies to firms). The Principal's budget constraint has the form

$$K(b,\widehat{b}) = \sum_{i=1}^{n}(b_i - \widehat{b}_i)^2 \leqslant C \sim n. \tag{45}$$

The solution of (44) is the efficient network heuristic

$$\widehat{b}_{nh} = b + \sqrt{C}\mu_{1(n)}, \tag{46}$$

where $\mu_{1(n)}$ denotes the eigenvector of the matrix G corresponding to the largest (smallest) eigenvalue in the case of positive (negative, respectively) values of the coefficient β.

As indicated above, the choice of the Principal's objective function significantly affects the outcome. If, in this problem, we pass from maximization of the total payoff (44) to

$$W = \sum_{i=1}^{n} x_i \longrightarrow \max_b$$

(maximization of the aggregate outcome of participants), the solution will be the budget allocation proportional to the individual contribution of players:

$$\widehat{b}_{nh} = b + \sqrt{C}x^*.$$

The distinction between the functions is that the sum of payoffs accounts for both the actions and their differences: $\sum_i u_i = \frac{1}{n}\sum_i x_i^2 = \hat{x}^2 + \sigma^2$, where $\hat{x}$ and σ^2 are the mean and variance, respectively.

Due to several properties of the cost function (45), the authors extended the results to some other functions [139]. One notable case is the Principal's linear costs:

$$K(b, \widehat{b}) = \sum_{i=1}^{n} |b_i - \widehat{b}_i| \leqslant C \sim n.$$

Here, the solution is to transfer the entire budget to a single player i^*, $\widehat{b}_{i^*} = b_{i^*} + C$, chosen based on the players' contributions to the outcome of the Principal's intervention.

It is of interest to compare the solutions of the incentive problem for local aggregation and local averaging models. To achieve the social optimum without budget constraints in the linear quadratic game

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \left[(b_i + s_i)x_i + \beta \sum_{j=1}^{n} g_{ij} x_i x_j - \frac{x_i^2}{2} \right] \longrightarrow \max_{s}, \tag{47}$$

the Principal needs to choose $s_i^O = \beta \sum_j g_{ij} x_j$. Due to the properties of the game, this value is always positive, and the Principal will subsidize more central players in the network.

As shown above, in the social norms model, agents create both positive and negative externalities for their neighbors. As a result, in an incentive problem [46] of the form

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} \left[(b_i + s_i)x_i - \frac{\theta}{2} \left(x_i - \sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j \right)^2 - \frac{x_i^2}{2} \right] \longrightarrow \max_{s}, \tag{48}$$

the solution is the vector s^O composed of

$$s_i^O = \beta \sum_{j=1}^{n} \frac{g_{ij}}{d_i} \left(x_j - \sum_{k=1}^{n} \frac{g_{jk}}{d_j} x_k \right) : \tag{49}$$

the Principal subsidizes (taxes) the agents whose neighbors exert efforts above (below, respectively) their social norms. (This contrasts with the linear quadratic game, where the Principal is forced to incentivize agents because of their positive externalities.) In other words, the Principal needs to subsidize agents exerting efforts below the average of their neighbors and to tax those exerting efforts above their neighbors.

We emphasize that till recently, changing the initial vector b of individual characteristics has been supposed to be a problem with a nonzero budget; in real applications, however, the Principal redistributes resources among agents without additional costs. The balanced budget condition (Pigouvian taxes) was addressed in [20].

The problem of finding the interaction structure maximizing the Principal's objective function was considered in [63, 119]. As β grows, the solutions of the two Principal's problems—maximizing the aggregate outcomes and the sum of payoffs—coincide, and among all networks, the maximum is attained for those with the largest spectral radius. The case of controlling a combination of multiple activities (when x_i becomes a vector) was addressed in [114]. The cases of simultaneously using different control strategies were studied, e.g., in [140, 141].

4. CONCLUSIONS

This paper has presented a detailed review of control problems in the strategic interaction of agents on a network when the agent's dependence on the actions of others is described by the linear best response function. The Nash equilibrium in such games is not the social optimum due to externalities, which motivates the development of control mechanisms (e.g., identifying key agents, incentivizing agents, and creating or removing links) to increase the efficiency of collective interaction.

Some directions for further research are being actively developed at present and deserve special attention.

4.1. Nonlinear Best Response

Control problems in nonlinear best-response games were considered in [142–144]. A general model of network effects was proposed in [34]. In this model, the nonlinear best response includes the local averaging model as a special case; the nonlinearity is implemented via a CES function with an elasticity parameter β :

$$\overline{x} = \left(\sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j^{\beta} \right)^{\frac{1}{\beta}}; \tag{50}$$

and the player's best response takes the form

$$BR_i = (1-\lambda_2) b_i + (\lambda_1+\lambda_2) \left(\sum_{j=1}^{n} \frac{g_{ij}}{d_i} x_j^{\beta} \right)^{\frac{1}{\beta}}, \tag{51}$$

where λ_1 and λ_2 correspond to the intensity of the spillover effect and conformism, respectively. For $\beta = 1$, the LIM model arises; if β is very large, the model $\overline{x}_{-i} = \max_j\{x_j\}$, where the agent is oriented toward the network neighbor with the greatest effort; and when β takes large negative values, the dominating model $\overline{x}_{-i} = \min_j\{x_j\}$, where the agent is oriented toward the neighbor with the smallest effort.

4.2. Games on Multiplex Networks

In many cases, the relationship among agents was considered within a single network. However, agents often maintain multiple types of relations, such as cooperation, mutual assistance, borrowing, etc. [145]. One way to describe such interaction is multiplex networks, i.e., multilayer networks without links between vertices of different layers that reflect the coexistence of various relations among agents [146]. Such games were analyzed in [147, 148].

4.3. Large Networks and Graphons

Some networks encountered in applied research are so large that analysis becomes difficult or even impossible [149]. A tool for analyzing large networks and the asymptotic behavior of sequences of graphs with a growing number of vertices was proposed in [150]. A graph function (graphon) is an object that generalizes discrete graphs to the case of large networks in the continuous-limit representation. Centrality measures for such objects were introduced in [151]. In [127], the above games were extended to the case of a limit object: players were represented as a population on the interval $[0, 1]$, and the probability of a link between them was described by a graphon $W(i, j)$ defined on the unit square. Dynamics in such games were investigated in [152], and control problems were considered in [141, 153].

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