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Contents

Automation and Remote Control
Vol. 87, No. 7, 2026

Anniversaries

Linear Systems

Nonlinear Systems

Intellectual Control Systems, Data Analysis

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 7, pp. 569–571.

ANNIVERSARIES

To the 80th Anniversary of Academician S.N. Vassilyev

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GOLDEN JUBILEE OF AN ACADEMICIAN: A LIFE FOR THE BENEFIT OF SCIENCE

On July 5, 2026, Academician of the Russian Academy of Sciences, Doctor of Physical and Mathematical Sciences Stanislav Nikolaevich Vassilyev celebrates his 80th birthday.

The jubilarian is an outstanding scientist, a specialist in the fields of mathematical cybernetics, systems analysis, control theory, and artificial intelligence. His scientific developments, which combine fundamental mathematics with applied problems of dynamics analysis and intellectualization of control of complex systems, have gained recognition both in Russia and far beyond its borders.

Stanislav Nikolaevich was born on July 5, 1946, in the town of Chistyakovo (now Torez) in the Donetsk region. A gold medal at school set the tone for all his further education: in 1970, he graduated with honors and a recommendation for postgraduate studies from the Radio Engineering Faculty of the Kazan Aviation Institute named after A.N. Tupolev.

Beginning his career at his alma mater, in 1975 he followed his teacher, Academician V.M. Matrosov, to Irkutsk. It was in Siberia that his talent fully blossomed. Having progressed from junior researcher to director of the Institute of Dynamics of Systems and Control Theory of the Siberian Branch of the Russian Academy of Sciences (formerly the Irkutsk Computing Center), he established himself as a major organizer of science. In 2006, already a full member of the Russian Academy of Sciences, he took over as director of the V.A. Trapeznikov Institute of Control Sciences of the Russian Academy of Sciences in Moscow, a position he held until 2016. Today, he continues to work there as a chief researcher, passing on his experience to the younger generation.

Academician S.N. Vassilyev is distinguished by his combination of depth in fundamental research and its practical implementation. Having started with the development of a system for automatic pointing of a telescope from the Moon to the geometric center of the Earth of variable phase, he moved on to the development of methods for model reduction (simplification) that allow analyzing the behavior of complex systems. This is reflected in his work on the transformation of dynamical system models and allows control of both mechanical objects and ecological-economic systems of regions.

Together with V.M. Matrosov, he developed the comparison method in mathematical systems theory and made a significant contribution to the qualitative theory of dynamical systems for analyzing stability, controllability, optimality, and other properties of nonlinear systems, including logic-dynamical and hybrid systems that combine continuous dynamics with discrete events or logical conditions. Together with his students, he developed algorithms for the machine derivation of theorems on the dynamical properties of systems, which made it possible to automate this process and obtain hundreds of new theorems on computers. These results were presented in the monographs (co-authored) "The Comparison Method in Mathematical Systems Theory" (1980), "Algorithms for Deriving Theorems Using the Method of Vector Lyapunov Functions" (1981), and a series of his articles in the journal "Differential Equations" (I–IV, 1981–82).

Since the 1990s, S.N. Vassilyev has been actively developing logical methods and artificial intelligence methods for control problems. His approach is based on the formalization of knowledge and the automation of logical inference, which led to the creation of a new direction — intelligent control of dynamical systems. One of the significant results was the creation of a system of automatic theorem proving, which is used for processing dynamical knowledge and control. This direction is devoted to the monograph "Intelligent Control of Dynamical Systems" (2000, co-authored).

An important part of S.N. Vassilyev's activity is the application of theoretical developments to solve real-world problems. The most striking example is comprehensive modeling and analysis for ensuring sustainable development of the Baikal region. Special attention deserves the contribution of Stanislav Nikolaevich to the training of the scientific elite. He founded the Department of "Physical and Mathematical Methods of Control" at the Faculty of Physics of Lomonosov Moscow State University, and for many years headed departments at MIPT, Irkutsk State University, and Buryat State University. Under his supervision, more than 20 candidates and doctors of sciences have been trained. For 15 years, Stanislav Nikolaevich was the editor-in-chief of the journal "Automation and Remote Control." In total, he has authored more than 400 scientific works, including authoritative monographs. Their significance was recognized by the USSR State Prize back in 1984 (for a series of works in the field of control theory of complex systems and mathematical modeling). For his services to the state and science, he has been awarded the titles of Laureate of the Prize of the Government of the Russian Federation in the field of education (2010) and the Prize of the Government of the Russian Federation in the field of science and technology (2012), and has been awarded the Order of the Badge of Honor, the Order of Friendship, a number of medals, including the S.P. Korolev Medal, as well as other state and scientific awards, diplomas, and letters of gratitude.

Eighty years is an age of wisdom and maturity, but colleagues and students know Stanislav Nikolaevich as a person who retains an inexhaustible interest in a wide range of pressing scientific problems, as a person of creative search and creation of original solutions (today, first of all, at the intersection of automation and intelligent control). From the bottom of our hearts, we congratulate the jubilarian and wish him good health, new discoveries, and grateful students.

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An Approach to Finding Points Inside the Reachable Set for a Linear Discrete Large-Order Model with Time-Varying Parameters

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Abstract—A method is proposed for finding reachable values of the output vector, taking into account constraints on the input and output signals during the control process for linear discrete systems with variable parameters. The developed algorithm overcomes the computational limitations of traditional approaches when working with high-order models. The method is based on sequential analysis of the output vector components by reducing them to linear programming problems. Its application is demonstrated using the example of the Globus-M2 spherical tokamak plasma model, where the possibility of determining attainable plasma configurations under constraints on the control voltages and currents is shown.

Keywords: linear systems, digital systems, systems with variable parameters, reachability set, plasma, tokamak

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1. INTRODUCTION

The problem of determining reachability sets (RS) for dynamical systems is central in control theory [1–3]. Classical approaches to determining the RS for systems with controls belonging to a given set, described in [1, 4–8], face significant computational difficulties when analyzing high-order system models typical of modern applications, including plasma control systems in tokamaks. The main limitations of traditional methods stem from the exponential growth of computational complexity, the need to construct convex hulls in high-dimensional spaces, limited applicability to systems with variable parameters and constraints on output signals, which makes accurate determination of the RS computationally challenging for high-order plant models. This paper proposes an iterative approach that enables efficient finding of points inside the RS for systems with tens of state variables, takes into account parametric changes in the model, and obtains practically useful estimates for engineering applications.

The method for computing the reachability set in the problem formulation described below in (6) was proposed in [4–6], where it was shown that using dynamic programming a multi-step estimate of the RS can be reduced to a recurrent computation of one-step estimates. The latter, in turn, reduce to constructing the convex hull of a set of points in an n-dimensional space corresponding to the dimension of the state vector.

In [4–6], modifications of the simplex method are used to construct the convex hull. An alternative approach is based on space partitioning, similar to the quicksort algorithm [9]. Although this method has no fundamental restrictions on the class of models, it requires significant computational resources [4] when the state vector dimension is high. This is because the number of hyperfaces of a d-polytope with N vertices can reach [9, 10]:

$$F(d,N) = \begin{cases} \dfrac{2N}{d} C_{N-d/2-1}^{d/2-1}, & \text{for even } d, \\ 2C_{N-\lfloor d/2 \rfloor -1}^{\lfloor d/2 \rfloor -1}, & \text{for odd } d, \end{cases} \tag{1}$$

where $\lfloor \cdot \rfloor$ denotes the floor function, $C_n^k = \frac{n!}{k!(n-k)!}$ is the binomial coefficient. Thus, $F(d,N) = O(N^{\lfloor d/2 \rfloor})$, which indicates an exponential growth in complexity as the number of vertices increases.

The method of [1, 7, 8] is based on approximating the RS by sets of simple structure, namely ellipsoids. Such a description is efficient for small phase space dimensions ($n = 2, 3$), but for large dimensions this method is laborious and provides a very incomplete representation of the reachable set [1, 7, 11].

This computational complexity makes it impractical to apply the algorithm to models with tens of state variables, including tokamak models and their control systems [12–14]. In this paper, an approach is proposed that overcomes this limitation by sequentially estimating the RS along the components of the output vector using the simplex method for solving linear programming problems [15–17]. Although the simplex method has exponential complexity in the worst case, in practice it exhibits polynomial convergence on average [17]. The price for increased efficiency is obtaining not full information about the RS but only its cross-sections. However, this method allows finding reachable points under given control constraints.

In [18, 19] a different approach is applied, consisting of two ideas: estimating the reachability set in steady-state and dividing the multidimensional RS into upper and lower estimates. The methodology based on considering the steady state is applicable to stationary and quasi-stationary plants, in which the characteristic time of parameter variation is orders of magnitude higher than the characteristic time of transients. The approach of dividing the RS into upper and lower estimates is illustrative and efficient for multidimensional plants. The upper estimate of the RS is described around the real reachability set, and the lower estimate is inscribed inside it. Thus, all points inside the lower estimate are reachable, and no points outside the upper estimate are reachable. However, in the case of strong coupling between channels and asymmetric constraints on the control and/or initial states, such an approach leads to an expansion of the "gray area" between the upper and lower estimates for points about which it is impossible to say whether they are reachable or not. This is illustrated in Section 4 in Fig. 1.

Further, Section 2 presents the problem statement. Section 3 describes preliminary computations allowing one to obtain the matrix relationship between the system outputs, given the inputs and initial conditions. Section 4 provides a sequential upper estimate of the reachability set based on previous results and compares the sequential estimate with the upper and lower estimates from [18, 19]. In Section 5 the sequential estimate is applied to a model example, showing that the results agree with those obtained by the method of [4–6]. Section 6 finds the dynamics of reachability sets with respect to plasma shape for models of the Globus-M2 tokamak with variable parameters and verifies these results in numerical experiments. The Conclusion summarizes the main results of the work.

2. PROBLEM STATEMENT

Consider a discrete-time plant model in the state space of dimension n, with p inputs and two groups of outputs of dimensions m and r, with variable parameters:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k + D_k u_k, \\ \tilde{y}_k &= \tilde{C}_k x_k + \tilde{D}_k u_k, \end{aligned} \tag{2}$$

where $k \in [1, T]$, the matrices A_k, B_k, C_k, $\tilde{C}_k$, D_k, $\tilde{D}_k$ have appropriate dimensions, x_k is the state vector, u_k is the control vector, y_k and $\tilde{y}_k$ are output vectors. The following constraints are considered:

- on the initial state vector

$$X_1 = \{x : x \in \mathbb{R}^n, x_{\min}^{(i)} \leqslant x^{(i)} \leqslant x_{\max}^{(i)}, i \in [1, n]\},\ x_1 \in X_1; \tag{3}$$

- on the controls at each step

$$P = \{u : u \in \mathbb{R}^p, u_{\min}^{(i)} \leqslant u^{(i)} \leqslant u_{\max}^{(i)}, i \in [1, p]\},\ u_k \in P; \tag{4}$$

- on the second group of output vectors at each step

$$F = \{\tilde{y} : \tilde{y} \in \mathbb{R}^r, \tilde{y}_{\min}^{(i)} \leqslant \tilde{y}^{(i)} \leqslant \tilde{y}_{\max}^{(i)}, i \in [1, r]\},\ \tilde{y}_k \in F. \tag{5}$$

The superscript $x^{(i)}$ denotes the i-th element of the vector x.

By the state reachability set in system (2) with constraints (3), (4), (5) at the end of the Tth step, similarly to [2–4], we shall call the set

$$\begin{aligned} &G(x_{\min}, x_{\max}, u_{\min}, u_{\max}, \tilde{y}_{\min}, \tilde{y}_{\max}, T) = \{x_T : x_T \in \mathbb{R}^n \quad \forall k \in [1, T-1], \\ &x_{k+1} = A_k x_k + B_k u_k, \tilde{y}_k = \tilde{C}_k x_k + \tilde{D}_k u_k, x_1 \in X_1, u_k \in P, \tilde{y} \in F\}. \end{aligned} \tag{6}$$

By the output reachability set we shall call the set

$$\begin{aligned} &Y(x_{\min}, x_{\max}, u_{\min}, u_{\max}, \tilde{y}_{\min}, \tilde{y}_{\max}, T) = \{y_T : y_T \in \mathbb{R}^m, \\ &y_T = C_T x_T + D_T u_T, x_T \in G(x_{\min}, x_{\max}, u_{\min}, u_{\max}, \tilde{y}_{\min}, \tilde{y}_{\max}, T), u_T \in P\}. \end{aligned} \tag{7}$$

In [4] it is shown that set (6) under constraints of the form (3), (4) is a convex polyhedron with a finite number of vertices. By similar reasoning one can show that (7) under constraints of the same type is also a convex polyhedron with a finite number of vertices.

3. DERIVATION OF INPUT-OUTPUT MATRIX RELATION

Let us find the matrix relation between the initial state vector x_1, the control actions u_i and the state vector at the kth step for model (2). Since $x_{k+1} = A_k x_k + B_k u_k$, the state vector can be computed according to the formula:

$$\begin{aligned} &x_2 = A_1 x_1 + B_1 u_1, \\ &x_3 = A_2 x_2 + B_2 u_2 = A_2(A_1 x_1 + B_1 u_1) + B_2 u_2, \\ &\quad \ldots \\ &x_k = A_{k-1} \cdots A_1 x_1 + A_{k-1} \cdots A_2 B_1 u_1 + A_{k-1} \cdots A_3 B_2 u_2 + \cdots + B_{k-1} u_{k-1}. \end{aligned} \tag{8}$$

Using the relation $y_k = C_k x_k + D_k u_k$, we obtain the output vector values

$$\begin{aligned} &y_1 = C_1 x_1 + D_1 u_1, \\ &y_2 = C_2 x_2 + D_2 u_2 = C_2(A_1 x_1 + B_1 u_1) + D_2 u_2, \\ &\quad \ldots \\ &y_k = C_k(A_{k-1} \cdots A_1 x_1 + A_{k-1} \cdots A_2 B_1 u_1 + \cdots + B_{k-1} u_{k-1}) + D_k u_k. \end{aligned} \tag{9}$$

Combining the vectors of initial values and control signals at all steps up to the kth inclusive into a column vector

$$U_k = [x_1^\top, u_1^\top, \ldots, u_k^\top]^\top \tag{10}$$

and introducing the notation

$$M_k = [C_k A_{k-1} \cdots A_1, C_k A_{k-1} \cdots A_2 B_1, C_k A_{k-1} \cdots A_3 B_2, \ldots, C_k B_{k-1}, D_k], \tag{11}$$

we obtain the matrix relation between all control signals, the initial state, and the output vector at the kth step:

$$y_k = M_k U_k, \tag{12}$$

where $M_k \in \mathbb{R}^{m\times(kp+n)}$, $U_k \in \mathbb{R}^{(kp+n)}$. Similarly, we find the values of the second group of output signals over the steps $[1, k]$

$$\begin{aligned} \tilde{y}_1 &= \tilde{C}_1 x_1 + \tilde{D}_1 u_1, \\ \tilde{y}_2 &= \tilde{C}_2 x_2 + \tilde{D}_2 u_2 = \tilde{C}_2(A_1 x_1 + B_1 u_1) + \tilde{D}_2 u_2, \\ &\ldots \\ \tilde{y}_k &= \tilde{C}_k(A_{k-1} \cdots A_1 x_1 + A_{k-1} \cdots A_2 B_1 u_1 + \cdots + B_{k-1} u_{k-1}) + \tilde{D}_k u_k. \end{aligned} \tag{13}$$

Introduce the notation

$$\begin{aligned} &\tilde{M}_k = [\tilde{C}_k A_{k-1} \cdots A_1, \tilde{C}_k A_{k-1} \cdots A_2 B_1, \tilde{C}_k A_{k-1} \cdots A_3 B_2, \ldots, \tilde{C}_k B_{k-1}, \tilde{D}_k], \\ &H_k = \begin{bmatrix} \tilde{M}_1,\ 0_{r\times p(k-1)} \\ \tilde{M}_2,\ 0_{r\times p(k-2)} \\ \ldots \\ \tilde{M}_{k-1},\ 0_{r\times p} \\ \tilde{M}_k \end{bmatrix}, \tilde{Y}_k = \begin{bmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \ldots \\ \tilde{y}_{k-1} \\ \tilde{y}_k \end{bmatrix} = H_k U_k, \end{aligned} \tag{14}$$

where $0_{r\times p}$ is a zero matrix of the corresponding dimension.

When computing the dynamics of the RS, a recurrent representation of the formulas for computing the matrix relation is also convenient:

$$\begin{aligned} &M_1 = D_1, H_1 = \tilde{M}_1 = \tilde{D}_1, \\ &M_k = [C_k Q_k, D_k], \tilde{M}_k = [\tilde{C}_k Q_k, \tilde{D}_k], k \in [2, T], \\ &H_k = \begin{bmatrix} H_{k-1},\ 0_{r(k-1)\times p} \\ \tilde{M}_k \end{bmatrix}, \end{aligned} \tag{15}$$

where Q_k is an auxiliary matrix computed iteratively

$$\begin{aligned} &Q_2 = [A_1, B_1], \\ &Q_k = [A_{k-1} Q_{k-1}, B_{k-1}], k \in [3, T]. \end{aligned} \tag{16}$$

4. SEQUENTIAL ESTIMATION OF THE REACHABILITY SET OVER OUTPUT COMPONENTS

Knowing the matrix relation (12), one can perform a multidimensional estimate of the reachability set. One approach is to divide the reachability set into upper and lower estimates, as shown in [18]. In this splitting, the upper estimate is a hyperrectangle circumscribed around set (7), and the lower estimate is a hypercube centered at the origin inscribed into set (7) in the m-dimensional space. This approach is illustrative; however, in the general case a large gap arises between the upper and lower estimates, leading to considerable uncertainty when estimating the reachability set. This is clearly shown in Fig. 1 for the two-dimensional case.

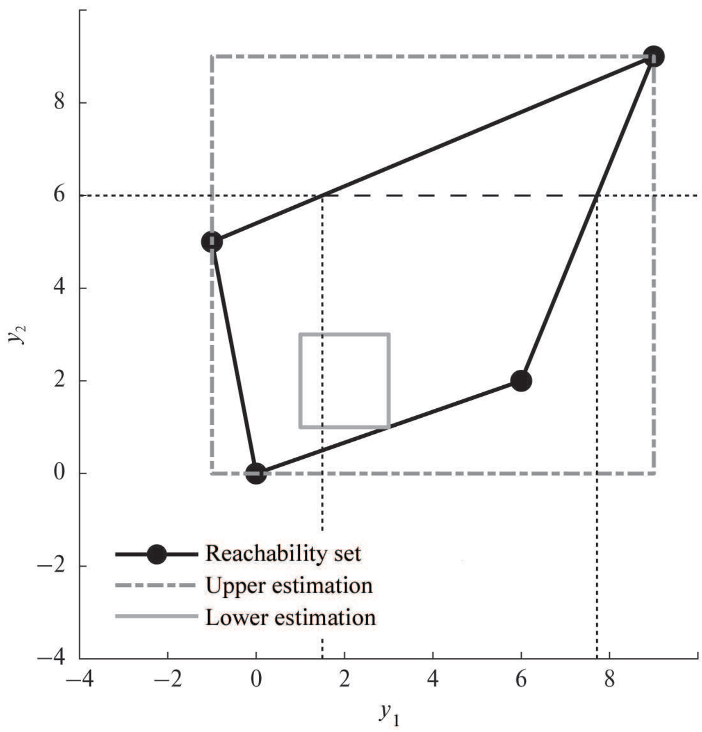

Fig. 1. Illustration of the lower and upper estimates of the reachability set in the two-dimensional case. The dashed line $y_2 = 4$ shows fixing the value of one of the outputs, then the dash-dotted line corresponds to the reachability set with respect to the y_1 coordinate.

A more informative approach can be the sequential finding of an upper estimate followed by fixing the value of one of the outputs within the admissible range, after which a new upper estimate can be found taking into account the fixed output. This procedure can be repeated as many times as there are outputs in the considered system.

Let us consider in more detail the process of constructing a sequential estimate of the reachability set. The first step of the algorithm for finding a sequential estimate of the reachability set is to find an upper estimate of the reachability set over all components of the output vector. This estimate can be found by solving m problems for the maximum

$$\max\{y_T^{(i)} = M_T^{(i)} U_T : \tilde{Y}_T = H_T U_T, x_1 \in X_1, u_k \in P, \tilde{y}_k \in F, k \in [1, T], i \in [1, m]\} \tag{17}$$

and minimum

$$\min\{y_T^{(i)} = M_T^{(i)} U_T : \tilde{Y}_T = H_T U_T, x_1 \in X_1, u_k \in P, \tilde{y}_k \in F, k \in [1, T], i \in [1, m]\}, \tag{18}$$

where $M_k^{(i)}$ is the row of matrix M_k. Problems (17), (18) are linear programming problems that can be solved by the simplex method [15], the interior-point method [20], etc. Generally speaking, the systems of equations (17), (18) may be inconsistent; let us consider the conditions for the existence of a solution.

A solution exists if one of the two statements holds: only constraints on the initial state and controls (3), (4) are given, $r = 0$, or the upper and lower constraints on the initial conditions, controls, and outputs have opposite signs. In the first case, systems (17), (18) always have at least one solution, since there are no constraints on the output signals, and at least one input signal, for example $u_k^{(i)} = u_{\min}^{(i)}$, $i \in [1, n]$, is realizable, and consequently at least one output signal is realizable. In the presence of constraints on a part of the outputs (5) and for $r \neq 0$, a sufficient (but not necessary) condition for the existence of a solution is

$$\begin{cases} x_{\min}^{(i)} \leqslant 0 \leqslant x_{\max}^{(i)}, \ i \in [1, n], \\ u_{\min}^{(i)} \leqslant 0 \leqslant u_{\max}^{(i)}, \ i \in [1, p], \\ \tilde{y}_{\min}^{(i)} \leqslant 0 \leqslant \tilde{y}_{\max}^{(i)}, \ i \in [1, r]. \end{cases} \tag{19}$$

This can be easily verified by substituting the zero vector as the control signal at each step. Such an additional constraint condition is natural for linear systems.

Thus, for each component of the output vector an interval is found in which it can vary ${y_T^{(i)}}_{\min 1} \leqslant y_T^{(i)} \leqslant {y_T^{(i)}}_{\max 1}$, $i \in [1, m]$. The user is offered to choose one of the components and fix its value. Without loss of generality, assume that the value of the first component is fixed: $y_T^{(1)} = y^{(1)}$. Consider the second step of the algorithm. To find the upper estimate for the remaining $(m-1)$ components, it is necessary to solve the following problems: for the maximum

$$\begin{cases} y_T^{(1)} = M_T^{(1)} U_T, \\ \max\{y_T^{(i)} = M_T^{(i)} U_T : \tilde{Y}_T = H_T U_T, x_1 \in X_1, u_k \in P, \tilde{y} \in F, k \in [1, T] i \in [2, m]\} \end{cases} \tag{20}$$

and for the minimum

$$\begin{cases} y_T^{(1)} = M_T^{(1)} U_T, \\ \min\{y_T^{(i)} = M_T^{(i)} U_T : \tilde{Y}_T = H_T U_T, x_1 \in X_1, u_k \in P, \tilde{y} \in F, k \in [1, T] i \in [2, m]\}. \end{cases} \tag{21}$$

Thus, at the second step new estimates will be found ${y_T^{(i)}}_{\min 2} \leqslant y_T^{(i)} \leqslant {y_T^{(i)}}_{\max 2}$, $i \in [2, m]$. Having performed the remaining steps of the algorithm in a similar way, after m steps, we obtain the problem

$$\begin{cases} y_T = M_T U_T, \\ \tilde{Y}_T = H_T U_T, x_1 \in X_1, u_k \in P, \tilde{y}_k \in F, k \in [1, T], \end{cases} \tag{22}$$

where $y_T \in Y(x_{\min}, x_{\max}, u_{\min}, u_{\max}, \tilde{y}_{\min}, \tilde{y}_{\max}, T)$ is a realizable value of the output vector. Problems (20), (21, (22) are also linear programming problems, and they have a solution because the equality-type constraint on the output signal added at each step is consistent with the other equations, since it satisfies the constraints of the analogous problem at the previous step. The solution to problem (22) is the vector $U_T^* \in \mathbb{R}^{(kp+n)}$, containing the initial conditions $x_1^* \in X_1$ and the open-loop control $u_k^* \in P$, $j \in [1, T]$, realizing the output y_T.

5. EXAMPLE

Let us demonstrate the algorithm on the example considered in [6]. The plant model is:

$$\begin{cases} x_{k+1} = & \begin{bmatrix} 0.8877 & -0.012 \\ 0.0258 & 0.4215 \end{bmatrix} \begin{bmatrix} \cos\frac{\pi}{18}(k-1) & \sin\frac{\pi}{36}(k-1) \\ -\sin\frac{\pi}{36}(k-1) & \cos\frac{\pi}{18}(k-1) \end{bmatrix} x_k + \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} u_k; \\ y_k = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k; \end{cases} \tag{23}$$

constraints on the control actions and the initial state vector:

$$|x_1^{(1)}| \leqslant 5, \ |x_1^{(2)}| \leqslant 5, \ |u^{(1)}| \leqslant 1, \ |u^{(2)}| \leqslant 1.5. \tag{24}$$

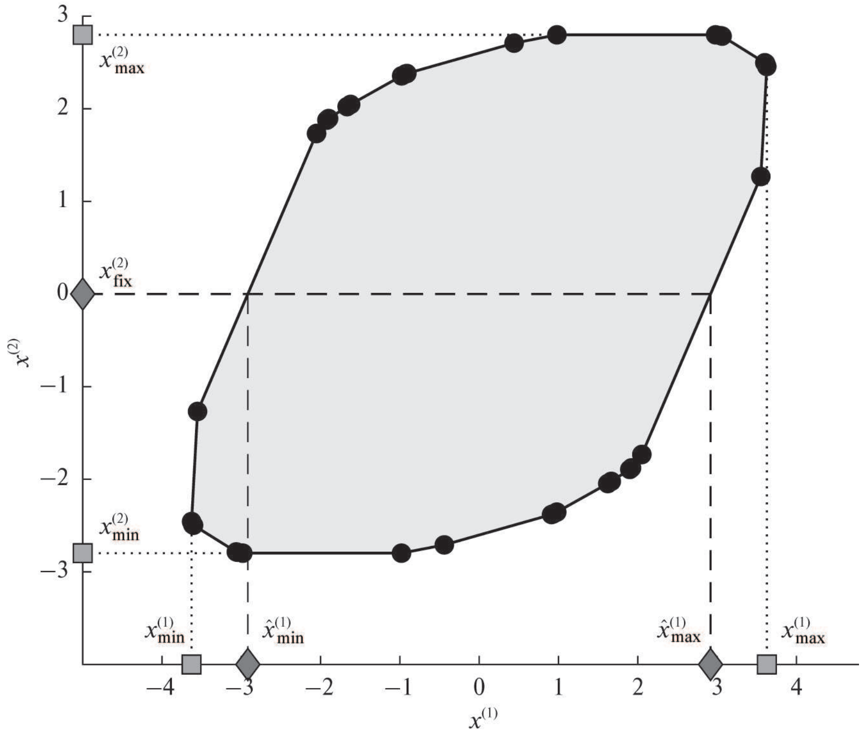

Fig. 2. Reachability set obtained by the method of [6] (dark circles) and by the method described in this paper (squares and diamonds).

In [6] the state reachability set is considered, while in this paper we consider the output reachability set. To enable comparison, in model (23) the output matrix is the identity matrix. Thus, the state vector equals the output vector. In this example there are no constraints on the output signals, $r = 0$. After seventy time steps the state reachability set found by the method of [4, 6] is shown in Fig. 2 with round markers and filling. The algorithm proposed in this paper gives at the first step an estimate of the reachability set over each component: $x^{(i)}_{\min} \leqslant x^{(i)} \leqslant x^{(i)}_{\max}$, $i \in [1, 2]$, thus obtaining a rectangle circumscribed around the reachability set. The first step of finding the reachability set is shown in Fig. 2 with square markers, $-3.6276 \leqslant x^{(1)} \leqslant 3.6276$, $-2.7982 \leqslant x^{(2)} \leqslant 2.7982$. At the second step the user fixes one of the vector components, for example: $x^{(2)} = x^{(2)}_{\text{fix}} = 0$, and the algorithm allows finding the changed estimate of the reachability set over the remaining component of the state vector (which equals the output vector in this example). In Fig. 2 the second step is shown with diamond-shaped markers, $-2.9178 \leqslant x^{(1)} \leqslant 2.9178$. At the final step the user can specify a correct value of the remaining component, and the algorithm will compute one of the possible open-loop controls realizing the obtained output vector at the given time step.

Thus, by successive approximations over each component of the output (state) vector, a point inside the reachability set and an open-loop control realizing the achievement of this point were found.

6. ESTIMATION OF THE REACHABILITY SET FOR THE GLOBUS-M2 TOKAMAK

The method described above for obtaining the reachability set was applied to find possible plasma shape configurations in the Globus-M2 tokamak (Ioffe Institute, St. Petersburg). Linear models with variable parameters and constraints on input and output signals were considered.

6.1. Experimental Signals

In the Globus-M2 tokamak [21, 22] voltages and currents in the control coils are measured, as well as signals from magnetic diagnostic loops during the plasma discharge. The measurement time step is 0.1 ms. The control coils are the central solenoid, the horizontal and vertical field coils, two poloidal field coils, and a correction coil. The magnetic diagnostics consists of twenty-one loops mounted on the vacuum vessel surface and measures the poloidal magnetic flux. Using the coil signals and the magnetic diagnostics, the plasma shape in the divertor discharge phase was reconstructed with the same time step of 0.1 ms [23]. By plasma shape we mean 24 gaps between the tokamak limiter wall and the plasma boundary along directions with a step of 15° (Fig. 3).

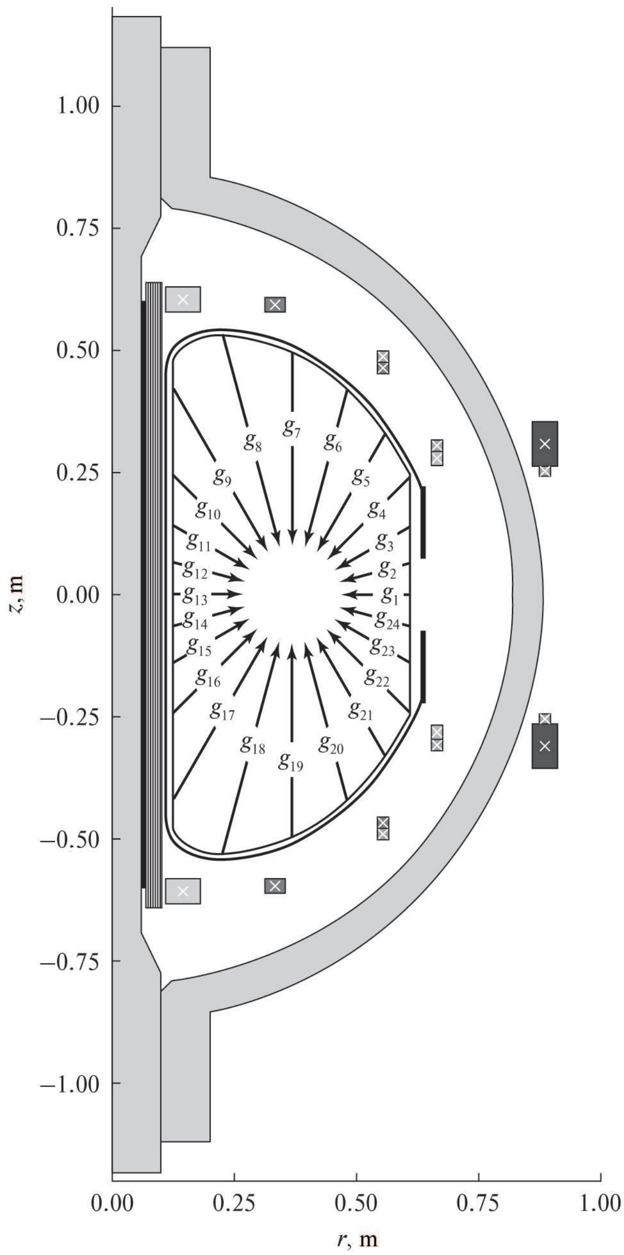

Fig. 3. Vertical cross-section of the Globus-M2 tokamak. Arrows indicate 24 directions along which the 24 gaps g_1–g_{24} between the tokamak limiter and the plasma boundary are measured.

Thus, experimental signals from the control coils were obtained with a 0.1 ms step over the time interval 0.15–0.24 s:

$$U_{PF,\,k} = [U_{HFC,\,k}, U_{VFC,\,k}, U_{CS,\,k}, U_{CC,\,k}, U_{PF1,\,k}, U_{PF3,\,k}]^\top,$$

$$I_{PF\,k} = [I_{HFC,\,k}, I_{VFC,\,k}, I_{CS,\,k}, I_{CC,\,k}, I_{PF1,\,k}, I_{PF3,\,k}]^\top$$

and the gaps between the plasma boundary and the tokamak limiter $G_k = [g_{1,\,k}, \ldots, g_{24,\,k}]^\top$, as well as the vertical position of the plasma column center Z, reconstructed from the magnetic measurements. Hereafter we will refer to these signals as experimental or scenario signals.

6.2. Model

To assess the capability of the tokamak to obtain plasma configurations, its linear model was built. The Globus-M2 tokamak models [12, 13] are constructed from the experimental data of discharges nos. 41 805, 41 835, and 41 867 in deviations from the experimental signals in discrete time with variable parameters:

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k \delta u_k, \\ \delta y_k &= C_k x_k, \\ \delta \tilde{y}_k &= \tilde{C}_k x_k, \end{aligned} \tag{25}$$

where $x \in \mathbb{R}^{17}$, $\delta u = [\delta U_{HFC}, \delta U_{VFC}, \delta U_{CS}, \delta U_{CC}, \delta U_{PF1}, \delta U_{PF3}]^\top \in \mathbb{R}^6$ are deviations from the scenario voltages in the control coils, $\delta y = [\delta g_1, \ldots, \delta g_{24}]^\top \in \mathbb{R}^{24}$ are changes in the shape relative to the scenario signals, $\delta \tilde{y} = [\delta Z, \delta I_{HFC}, \delta I_{VFC}, \delta I_{CS}, \delta I_{CC}, \delta I_{PF1}, \delta I_{PF3}]^\top \in \mathbb{R}^7$ are changes in the vertical plasma position and currents in the control coils. The matrices of the stationary linear models $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ are computed over the time interval from 0.13 to 0.24 s with a step of 1 ms and interpolated linearly:

$$A_k = \frac{10 - (k-1)\%10}{10} \hat{A}_{\lfloor \frac{k-1}{10} \rfloor + 1} + \frac{(k-1)\%10}{10} \hat{A}_{\lfloor \frac{k-1}{10} \rfloor + 2}, \tag{26}$$

where $\lfloor \cdot \rfloor$ denotes rounding down, % is the remainder of division. The matrices B and C are computed analogously.

6.3. Constraints on the Control Signals

Consider deviations from the control signals under the following constraints:

- zero initial conditions $x_1^{(j)} = 0$, $j \in [1, 17]$;
- deviations on the control coil voltages $|\delta u^{(j)}| \leqslant 100$ V, $j \in [1, 6]$;
- deviations on the unstable vertical position of the plasma column center $|\delta Z| \leqslant 1$ cm;
- deviations on the currents in the control coils $|\delta I_{HFC}| \leqslant 500$ A, $|\delta I_{VFC}| \leqslant 500$ A, $|\delta I_{CS}| \leqslant 5$ kA, $|\delta I_{CC}| \leqslant 500$ A, $|\delta I_{PF1}| \leqslant 500$ A, $|\delta I_{PF3}| \leqslant 500$ A.

The voltages in the control coils vary in the range ± 1.6 kV, typical values of currents in the HFC, VFC, CC coils are 0.5 kA, PF1—1.5 kA, PF3—3 kA, the current in the CS coil varies from 0.5 to 2.5 kA. Typical values of the vertical displacement are on the order of 1 cm (Fig. 4). Therefore, one can consider the deviations from the scenario values small and apply linear plasma models for subsequent computations.

Thus, the model equations (25) and the constraints at each step satisfy the problem statement (2), (3), (4), (5) and condition (19). At the same time, the approach to computing the RS described in [4–6] is inapplicable to these models, since it requires constructing the convex hull of points in a space whose dimension equals the number of model states, in this case 17. In such a

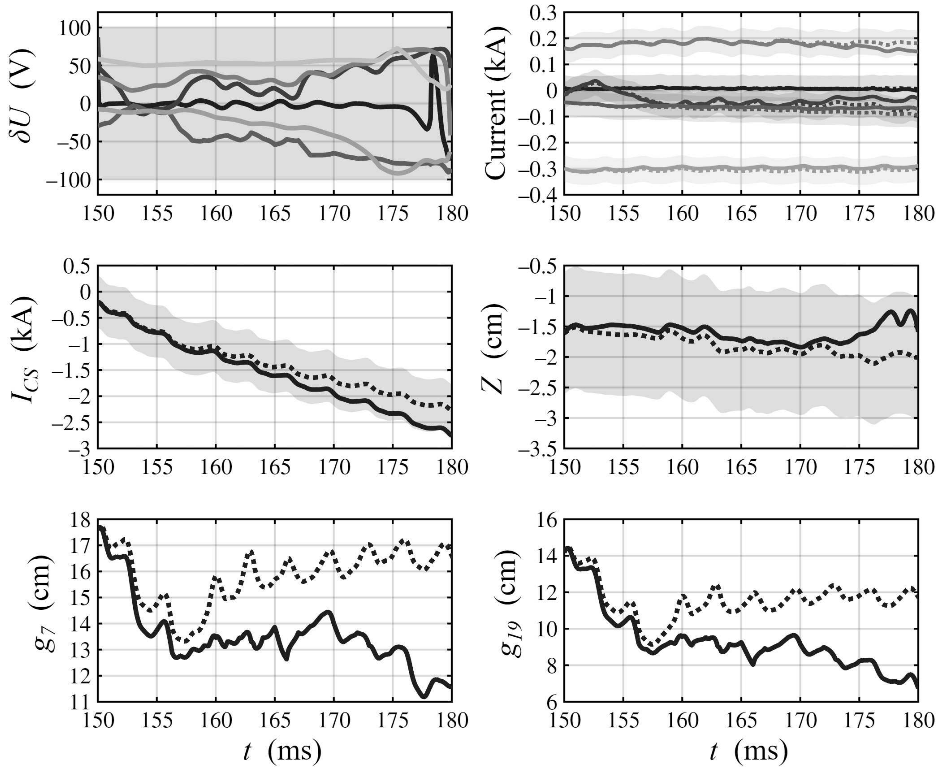

Fig. 4. Simulation result of the plasma with a modified control realizing after 30 ms deviations in the shape components $\delta g_7 = -5$ cm, $\delta g_{19} = -5$ cm, additionally elongating the plasma vertically under constraints on the vertical plasma displacement, voltages and currents in the control coils. Dashed lines—scenario, solid lines—deviated signals, halftone—admissible deviations. Discharge no. 41 805.

space the points form a d-polytope having at least $N = 2^d$ vertices, according to (1) $F(d) \sim 2^{3d/2}$ hyperfaces, and at each step this number increases proportionally to the number of inputs. Thus, the method described in Sections 2, 3, 4 becomes relevant.

6.4. Plasma Shape Reachability Set

Let us analyze the possibility of changing the shape components g_7 and g_{19}—the upper and lower points of the plasma. By changing these quantities one can influence the plasma vertical elongation. Decreasing the gaps between the plasma and the vacuum vessel wall corresponds to increasing elongation. The initial value of the state vector is assumed to be zero.

Using the proposed algorithm, we obtain for the model of discharge no. 41 805 that after 30 ms (300 time steps) the system outputs δg_7 and δg_{19} are constrained within the intervals:

$$-7.9 \leqslant \delta g_7 \leqslant 7.9;\ -8.6 \leqslant \delta g_{19} \leqslant 8.6, \tag{27}$$

where the deviation values are expressed in centimeters. At the second step we fix $\delta g_7 = -5.0$ cm, then we obtain:

$$\delta g_7 = -5.0;\ -8.6 \leqslant \delta g_{19} \leqslant -2.9. \tag{28}$$

At the last step we can fix the value $\delta g_{19} = -5.0$ cm and obtain an open-loop control, the output deviation values of 5 cm. The dynamics of the system with the found open-loop control are shown in Fig. 4.

Thus, it is shown that using the poloidal field coils it is possible to increase the vertical elongation of the plasma in the existing configuration of the Globus-M2 tokamak.

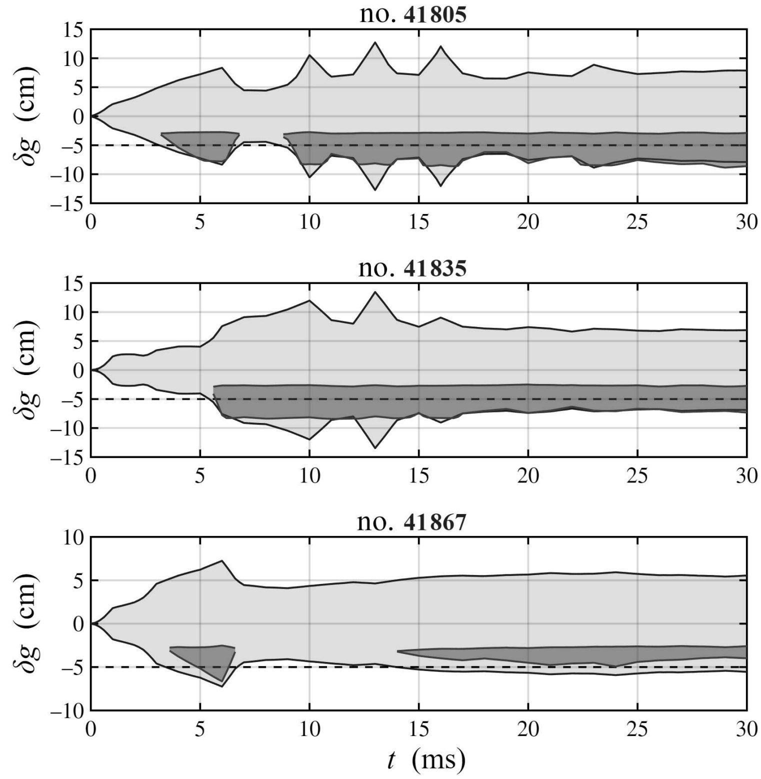

Fig. 5. Change in the reachability set during the divertor discharge phase. Light gray shows the reachability set with respect to the deviation for the gap δg_7, dark gray—for the gap δg_{19}, under the condition $\delta g_7 = -5$ cm, at those time instants where such a condition is feasible. Time is given from the start of the divertor phase.

Let us analyze the change in the reachability sets over these shape components with time. Consider plasma discharges no. 41 805, 41 835, 41 867. Compute the possible deviation in δg_7. If the deviation in δg_7 can be equal to -5 cm, we fix it and find the reachability set with respect to δg_{19}. The result is shown in Fig. 5, the time is given from the beginning of the divertor phase. A strong coupling between the opposite gaps is visible; a deviation of the δg_7 component inevitably leads to a deviation of the δg_{19} component in the same direction.

7. CONCLUSION

The paper proposes a method for finding points lying inside a multidimensional reachability set for linear discrete-time systems with variable parameters. A computationally efficient algorithm has been developed that allows sequentially estimating cross-sections of the reachability set over the output vector components, finding points inside the RS using linear programming problems. The method has been applied to the plasma control problem in the Globus-M2 tokamak, which made it possible to find attainable plasma configurations under given control constraints, to demonstrate the possibility of increasing the vertical elongation of the plasma, and to analyze the change in the RS during the plasma discharge. The numerical experiments performed demonstrate consistency of the results with known methods for low-dimensional cases, practical applicability of the approach for systems with dozens of state variables, and the ability to account for variable parameters.

The proposed approach opens new possibilities for analysis and synthesis of control systems for complex dynamic plants, where traditional methods for constructing the RS become inapplicable due to high computational complexity. The developed algorithm can be useful both for theoretical studies of controllability properties of dynamical systems and for solving applied control problems for complex engineering objects.

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An Approach to Parametrization of Suboptimal Anisotropic Controllers

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Abstract—The problem of parametrization of suboptimal and γ-optimal anisotropic controllers in the form of static state feedback for linear discrete-time invariant systems driven by stochastic disturbances with bounded mean anisotropy is considered. In the controller design problem, the solution is based on applying Finsler lemma to the system of inequalities characterizing the boundedness of the anisotropic norm by some number. The relation of the solution of this problem to the solution of the controller design problem for suboptimal and γ-optimal anisotropic controllers via convex optimization without parametrization is pointed out. An example demonstrating the efficiency of the proposed approach to parametrization of anisotropic controllers is provided.

Keywords: anisotropy-based theory, linear systems, discrete-time systems, suboptimal controllers, parametrization

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1. INTRODUCTION

The anisotropy-based control and filtering theory, the foundations of which were laid in the mid-90s by I.G. Vladimirov [1–3], provides a mathematical framework for the analysis of linear discrete-time control systems with random external disturbances and for the synthesis of controllers which are optimal with respect to the mean-square gain criterion under the worst-case disturbance from a certain class. The main difference of this theory from the well-known $\mathcal{H}_2$- and $\mathcal{H}_\infty$-control theories is the manner of describing external disturbances perturbing on the system – the associated stochastic uncertainty is described in information-theoretic terms such as relative entropy and mutual information. The approach proposed by I.G. Vladimirov allows the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of systems to be considered as special cases of the so-called anisotropic norm, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-controllers as corresponding versions of anisotropic controllers.

Within the framework of anisotropy-based theory, many problems have been solved in various formulations: control and filtering, for stationary (time-invariant) and nonstationary (time-varying) systems, in optimal and suboptimal cases, with multiple criteria, for systems with multiplicative noise and systems with network structure, and many others [4–9]. Along with this, methods for solving these problems have also been developed: if in the first works the results were presented in the form of coupled Riccati and Lyapunov equations general primarily for optimal formulations, then later almost all solutions began to be presented in the form of a convex optimization problem with constraints in the form of linear matrix inequalities and one inequality of a special type. Nevertheless, each of the mentioned cases, when implemented on a computing device, gave only one solution found by a numerical algorithm. In fact, such a controller (even an optimal one) is not always unique, and in the suboptimal formulation the solution is a continuum set. Parametrization of controllers, if possible, is one of the ways to describe the set of solutions.

In this paper, we consider the problem of parametric description of the set of anisotropic controllers in the form of state feedback in suboptimal and γ-optimal formulations. The paper is organized as follows. Section 2 describes the basic concepts of anisotropy-based theory necessary for understanding what follows. Section 3 presents the mathematical statement of the problem to be solved. The main result and accompanying statements are given in Section 4. The proof of the main result is placed in the Appendix. Section 5 gives an example, and Section 6 concludes the paper.

2. PRELIMINARIES

This section provides the basic information necessary for the reader to understand the further material.

2.1. Notations

Let $L \in \mathbb{R}^{m\times n}$ be an arbitrary matrix over the field of real numbers. The transpose of L is denoted by L^{T}. The spectral norm (or the Schatten ∞-norm) of L is defined as $\|L\| = \sigma_{\max}(L)$, where $\sigma_{\max}(L) = \sqrt{\lambda_{\max}(L^{\mathrm{T}}L)}$ is the maximum singular value of L, and $\lambda_{\max}(L^{\mathrm{T}}L)$ is the maximum eigenvalue of $L^{\mathrm{T}}L$. For a symmetric matrix $L = L^{\mathrm{T}}$, the notation $L > 0$ is used to denote its positive definiteness. The notation $L^{\perp}$ is used for the left null-space (or cokernel, or orthogonal complement) of L, and is defined as any matrix satisfying the relations $L^{\perp}L = 0$ and $L^{\perp}L^{\perp\mathrm{T}} \succ 0$, if such exists. The matrix L^{+} denotes the Moore–Penrose pseudoinverse of L.

2.2. Basic Concepts of Anisotropy-Based Theory

Anisotropy-based theory belongs to the branches of automatic control theory dealing with a wide range of control and filtering problems for linear discrete-time stochastic systems, the presence of stochasticity in which is determined mainly by the fact that they operate in the presence of additive random noises and/or external disturbances. Assuming that the statistics of these disturbances are not precisely known, but it is possible to describe in some way the general pattern of information transfer in the process under consideration, a special functional is introduced on the set of spectral densities of the disturbance. Setting the condition that this functional is bounded by some number allows us to isolate a certain class from the set of all external disturbances and further investigate the influence of disturbances from this class on the system, as well as the ability of the control system to attenuate disturbances from such a class.

Let $\{w_k\}_{k\in\mathbb{Z}}$ be a stationary ergodic sequence of n_w-dimensional random vectors. The mean anisotropy of the sequence W is defined as [10]

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N-1})}{N} = -\frac{1}{2}\ln\det\left(\frac{n_w\mathbf{E}[\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}}]}{\mathbf{E}[|w_0|^2]}\right),$$

where $W_{s:t} = (w_s^{\mathrm{T}}, \ldots, w_t^{\mathrm{T}})^{\mathrm{T}}$ is a vector fragment of the sequence, $\widetilde{w}_0 = w_0 - \mathbf{E}[w_0|\{w_k\}_{k<0}]$ is the steady-state prediction error vector, and $\mathbf{A}(w)$ is the anisotropy of a random vector w, defined as

$$\mathbf{A}(w) = \frac{n_w}{2}\ln\left(\frac{2\pi\mathrm{e}}{n_w}\mathbf{E}[|w|^2]\right) - \mathbf{h}(w),$$

where the differential entropy $\mathbf{h}(w)$ of a random vector w with probability density function $f : \mathbb{R}^{n_w} \to \mathbb{R}_+$ is defined as

$$\mathbf{h}(w) = -\mathbf{E}[\ln(f(w))].$$

Now let $\{z_k\}_{k\in\mathbb{Z}}$ be a sequence linearly dependent on $\{w_j\}_{j\leqslant k}$ vectors generated by a system F, i.e. $Z = F * W$. Assume that the transfer function of F is analytic inside the unit disk on the complex plane and has a finite $\mathcal{H}_\infty$-norm

$$\|F\|_\infty = \sup_{z\in\mathbb{C}:\,|z|<1} \sigma_{\max}(F(z)).$$

Then for the system F the anisotropic norm can be introduced as [2]

$$|\!|\!|F|\!|\!|_a = \sup_{G:\,V\to W,\,\overline{\mathbf{A}}(W)\leqslant a} \frac{\|FG\|_2}{\|G\|_2},$$

where W is a stationary Gaussian sequence formed by passing a standard Gaussian white noise V through a stationary linear system G with transfer function $G(z)$ analytic inside the unit disk and having a finite $\mathcal{H}_2$-norm

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}(\widehat{G}(\omega)(\widehat{G}(\omega))^*)d\omega}.$$

3. PROBLEM STATEMENT

Consider a linear discrete-time stationary system F given by the equations

$$\begin{cases} x_{k+1} = Ax_k + B_u u_k + B_w w_k, \\ z_k \quad\;\, = Cx_k, \end{cases} \tag{1}$$

where $k = 0, 1, \ldots$ is the time index, x_k is an n_x-dimensional state vector ($x_0 = 0$), u_k is an n_u-dimensional control vector, w_k is an n_w-dimensional external disturbance vector, and z_k is an n_z-dimensional regulated output vector. All matrices A, B_u, B_w, C of system (1) have appropriate dimensions and are known, the pair (A, B_w) is controllable, and the pair (A, B_u) is stabilizable. Without loss of generality, we assume that $n_u \leqslant n_x$, $\operatorname{rank}(C) = n_z \leqslant n_x$. It is assumed that the system (1) is subject to a random external disturbance represented by the sequence $W = \{w_k\}_{k\geqslant 0}$ with uncertainty related to unknown stochastic characteristics of the vectors w_k, the quantitative description of which is expressed in terms of an upper bound on the mean anisotropy of the sequence W by some number $a > 0$: $\overline{\mathbf{A}}(W) \leqslant a$. Let the control law have the form of static state feedback

$$u_k = Kx_k, \tag{2}$$

where the matrix $K \in \mathbb{R}^{n_u\times n_x}$ of the controller is to be determined. The closed-loop system F_{cl} obtained by closing system (1) with controller (2) has the form

$$\begin{cases} x_{k+1} = A_{cl}x_k + B_{cl}w_k, \\ z_k \quad\;\, = C_{cl}x_k, \end{cases} \tag{3}$$

where $A_{cl} = A + B_uK$, $B_{cl} = B_w$, $C_{cl} = C$.

The standard problem of γ-optimal anisotropic controller design in the form of static state feedback for system (1) consists in finding a matrix K of the control law (2) such that the corresponding controller is stabilizing and the anisotropic norm of the closed-loop system (3) is bounded from above by the minimal possible number $\gamma_{\min}$, i.e. the conditions

$$\rho(A + B_uK) < 1, \quad |\!|\!|F_{cl}|\!|\!|_a \leqslant \gamma, \quad \gamma \to \min \tag{4}$$

are met. This problem, including more general formulations, has been solved in [4, 11, 12]. The difference between the above formulation and the suboptimal one is the minimization of the upper bound γ: if γ is a fixed number ($\gamma > \gamma_{\min}$), then we have a suboptimal formulation.

Problem 1. Parametrize the set of solutions of the standard problems of synthesis of suboptimal and γ-optimal anisotropic controllers in the form of static state feedback.

The case of state feedback (4) is chosen for simplicity. For the cases of static and dynamic output feedback, under a number of additional requirements on the system, parametrization is also possible.

The practical significance of the problem posed consists of several aspects. First, solving the problem of controller parametrization allows one to form an idea of their set, which may be useful in studying the possibility of introducing additional criteria into the problem that do not lead to an empty set of solutions. Second, when the parameters of the problem (for example, the representation of the mean anisotropy of the external disturbance) vary in real time, the standard way of solving is to recompute the controller, and there is no guarantee that as a result of this operation, relatively small changes in the parameters will lead to relatively small changes in the controller coefficients. In other words, since the problem of finding controller coefficients in the classical approach is solved numerically and the found solution may differ significantly from the previous one, when switching from one controller to another, adverse transients may occur. Among other aspects one can mention, for example, the importance of taking into account rounding-off errors, which cannot be taken into account in the classical way, but for which corresponding results can be obtained within the framework of solving the parametrization problem.

4. PARAMETRIZATION OF SUBOPTIMAL ANISOTROPIC STATE-FEEDBACK CONTROLLERS

As noted, the problem of synthesizing a γ-optimal anisotropic controller has already been solved. Its solution (up to some renotation) is described by the following theorem [12, Theorem 1].

Theorem 1. *For given $a \geqslant 0$ and $\gamma > 0$, for the existence of a stabilizing static state-feedback controller* (2) *for system* (1) *guaranteeing that the anisotropic norm of the closed-loop system* (3) *is bounded by γ, it is sufficient that the system of inequalities*

$$P \succ 0, \quad \Psi \succ 0, \quad \eta^2 \geqslant \gamma^2, \tag{5}$$

$$\begin{bmatrix} P & \star & \star & \star \\ 0_{n_w \times n_x} & \eta^2 I_{n_w} & \star & \star \\ AP + B_u\widetilde{K} & B_w & P & \star \\ CP & 0_{n_z \times n_w} & 0_{n_z \times n_x} & I_{n_z} \end{bmatrix} \succ 0, \tag{6}$$

$$\begin{bmatrix} \eta^2 I_{n_w} - \Psi & \star \\ B_w & P \end{bmatrix} \succ 0, \tag{7}$$

$$(\det(\Psi))^{1/n_w} - e^{2a/n_w}(\eta^2 - \gamma^2) \geqslant 0 \tag{8}$$

is feasible with respect to the scalar variable η^2 and real matrices $P \succ 0_{n_x \times n_x}$, $\Psi \succ 0_{n_w \times n_w}$, and $\widetilde{K} \in \mathbb{R}^{n_u \times n_x}$. In this case, the static controller matrix is given by $K = \widetilde{K}P^{-1}$.

Remark 1. If in Theorem 1 γ is considered as a minimized variable, then we obtain formulation (4) for finding a γ-optimal anisotropic state-feedback controller.

Remark 2. The application of a numerical algorithm based on the conditions of Theorem 1 allows one to find suboptimal anisotropic controllers, but leaves no freedom of choice—the result of the algorithm is a fixed point K in the space of matrices $\mathbb{R}^{n_u \times n_x}$.

Remark 2 represents one of the motivation sides for this research. From the set of controllers that are a solution to some problem, the choice of a particular one may be determined by a number of auxiliary conditions not included in the original problem statement, and neglecting this freedom of choice (transferring the "right to choose" to a computing device) may lead to the fact that a number of secondary characteristics of the controller or closed-loop system will be chosen not in the best way. In addition, when an additional condition is introduced into the problem, it will have to be solved anew instead of choosing a suitable one from the already found set of controllers. Finally, describing the boundaries of the entire set of controllers is a more complete mathematical problem than finding some single solution.

To formulate the main result, we need the following auxiliary statement— a version of Finsler lemma [13, Theorem 2.3.12] (given with renotation consistent with that used in this paper).

Theorem 2. *Let matrices $\Theta \in \mathbb{R}^{n\times n}$, $\Lambda \in \mathbb{R}^{n\times m}$, $\Pi \in \mathbb{R}^{k\times n}$ be given. The following statements are equivalent:*

(1) *there exists a matrix $\widetilde{K}$ such that*

$$\Theta + \Lambda\widetilde{K}\Pi + (\Lambda\widetilde{K}\Pi)^{\mathrm{T}} \succ 0; \tag{9}$$

(2) *the following two conditions hold:*

$$\Lambda^{\perp}\Theta\Lambda^{\perp\mathrm{T}} \succ 0 \quad or \quad \Lambda\Lambda^{\mathrm{T}} \succ 0, \tag{10}$$

$$\Pi^{\mathrm{T}\perp}\Theta\Pi^{\mathrm{T}\perp\mathrm{T}} \succ 0 \quad or \quad \Pi^{\mathrm{T}}\Pi \succ 0. \tag{11}$$

Assume that the above statements hold. Let the matrices Λ and Π are factorized as $\Lambda = \Lambda_L\Lambda_R$ and $\Pi = \Pi_L\Pi_R$, where Λ_L, Λ_R, Π_L, Π_R are arbitrary full-rank matrices. Then all solution matrices $\widetilde{K}$ of inequality (9) have the form

$$\widetilde{K} = \Lambda_R^{+}X\Pi_L^{+} + Z - \Lambda_R^{+}\Lambda_R Z\Pi_L\Pi_L^{+}, \tag{12}$$

where Z is an arbitrary matrix, and

$$X = T\Lambda_L^{\mathrm{T}}\Phi\Pi_R^{\mathrm{T}}(\Pi_R\Phi\Pi_R^{\mathrm{T}})^{-1} + S^{1/2}Y(\Pi_R\Phi\Pi_R^{\mathrm{T}})^{-1/2}, \tag{13}$$

$$S = T - T\Lambda_L^{\mathrm{T}}(\Phi - \Phi\Pi_R^{\mathrm{T}}(\Pi_R\Phi\Pi_R^{\mathrm{T}})^{-1}\Pi_R\Phi)\Lambda_L T, \tag{14}$$

where $Y : \|Y\| < 1$ is an arbitrary matrix with bounded spectral norm, and T is an arbitrary positive definite matrix such that

$$\Phi = (\Theta + \Lambda_L T\Lambda_L^{\mathrm{T}})^{-1} \succ 0. \tag{15}$$

The content of this theorem is omitted. We only note that when it is applied in control problems, in addition to the degrees of freedom associated with the arbitrary choice of matrices Z, Y, and T, there will be one more degree of freedom caused by the value taken by the matrix Θ, which depends, generally speaking, on other variables. More on this will be presented after the main result given below.

4.1. Main Result

Theorem 3. *Let system (1) with matrix B_u be given, and let B_u admit a factorization $B_u = B_{uL}B_{uR}$ with full-rank matrices $B_{uL} \in \mathbb{R}^{n_x\times r}$ and $B_{uR} \in \mathbb{R}^{r\times n_u}$, where $r = \operatorname{rank}(B_u)$, and the pairs (A, B_w) and (A, B_u) are controllable and stabilizable, respectively. Let also two numbers $a \geqslant 0$*

and $\gamma > 0$ be chosen. Then for the existence of a stabilizing static state-feedback controller (2) *guarantying $\|F_{cl}\|_a \leqslant \gamma$, it is sufficient that the system of inequalities*

$$\begin{bmatrix} P & \star & \star & \star \\ 0_{n_w\times n_x} & \eta^2 I_{n_w} & \star & \star \\ B_u^{\perp}AP & B_u^{\perp}B_w & B_u^{\perp}PB_u^{\perp\mathrm{T}} & \star \\ CP & 0_{n_z\times n_w} & 0_{n_z\times(n_u-r)} & I_{n_z} \end{bmatrix} \succ 0, \tag{16}$$

$$\begin{bmatrix} \eta^2 I_{n_w} - \Psi & \star \\ B_w & P \end{bmatrix} \succ 0, \tag{17}$$

$$(\det(\Psi))^{1/n_w} - e^{2a/n_w}(\eta^2 - \gamma^2) \geqslant 0, \tag{18}$$

$$P \succ 0, \quad \Psi \succ 0, \quad \eta^2 \geqslant \gamma^2 \tag{19}$$

be feasible with respect to the variables P, Ψ, η^2.

If the statements of the theorem hold, then all matrices K of stabilizing controllers (2) *guaranteeing $\|F_{cl}\|_a \leqslant \gamma$ are of the form $K = \widetilde{K}P^{-1}$, where*

$$\widetilde{K} = B_{uR}^{+}X + (I_{n_u} - B_{uR}^{+}B_{uR})Z, \tag{20}$$

$$X = TB_{uL}^{\mathrm{T}}\Phi_{31}\Phi_{11}^{-1} + S^{1/2}Y\Phi_{11}^{-1/2}, \tag{21}$$

$$S = T - TB_{uL}^{\mathrm{T}}(\Phi_{33} - \Phi_{31}\Phi_{11}^{-1}\Phi_{31}^{\mathrm{T}})B_{uL}T, \tag{22}$$

$$\Phi = \operatorname*{block}_{i,j=\overline{1,4}}(\Phi_{ij}) = \begin{bmatrix} P & \star & \star & \star \\ 0_{n_w\times n_x} & \eta^2 I_{n_w} & \star & \star \\ AP & B_w & P + B_{uL}TB_{uL}^{\mathrm{T}} & \star \\ CP & 0_{n_z\times n_w} & 0_{n_z\times n_x} & I_{n_z} \end{bmatrix}^{-1}, \tag{23}$$

$Z \in \mathbb{R}^{n_u\times n_x}$ is an arbitrary matrix, $Y \in \mathbb{R}^{r\times n_x}$ is an arbitrary matrix satisfying $\|Y\| < 1$, and $T \in \mathbb{R}^{r\times r}$ is any positive definite matrix such that $\Phi \succ 0$.

The proof of the theorem is given in the Appendix.

Note that expressions (16)–(19) of Theorem 3 do not depend on the matrix K of the desired controller, i.e., the sufficient conditions for the existence of a suitable anisotropic controller have a closed form. The second part of the theorem, namely formulas (20)–(23), describe in explicit parametric form all anisotropic controllers that are solutions to the problem.

It can be shown that all matrices T from Theorem 3 associated with $\Phi \succ 0$ given by (23) belong to the cone

$$T \succ B_{uL}^{+}\left(A(P^{-1} - C^{\mathrm{T}}C)^{-1}A^{\mathrm{T}} + \eta^{-2}B_wB_w^{\mathrm{T}} - P\right)B_{uL}^{+\mathrm{T}}, \tag{24}$$

i.e. the boundary (in the sense of the order relation $\succ$ on the set of symmetric matrices) value of T can be found explicitly. We also draw attention to the fact that in addition to the degrees of freedom due to the choice of matrices Z, Y, and T, the conditions of Theorem 3 implicitly contain another degree of freedom related to the values that the variables P and η^2 will take during the solution of inequalities (16)–(19), on which the matrices in (20)–(23) depend. If in the framework of Theorem 3 we consider the case when B_u is a full-rank matrix, i.e. $r = n_u$, then condition (16) takes the simplified form

$$\begin{bmatrix} P & \star \\ CP & I_{n_z} \end{bmatrix} \succ 0.$$

Remark 3. If in Theorem 3 the number γ is not given but is one of the variables, and we solve the problem of minimizing the upper bound of the anisotropic norm

$$\gamma^2 \to \min_{P\succ 0, \Psi\succ 0, \eta^2\geqslant\gamma^2>0} \quad \text{subject to (16)–(19)},$$

then we obtain a formulation for the parametrization of a γ-optimal state-feedback anisotropic controller.

We note once again that the obtained parametrization depends on the variables P, Ψ, η^2 (and in the framework of Remark 3 also on γ^2) found at the stage of solving the system of inequalities (16)–(19). Obtaining a parametrization of this system of inequalities is indeed a relevant problem, but it is difficult due to the presence of condition (18), which describes the relation between the variables η^2 and γ^2 and the spectrum of Ψ.

5. EXAMPLE

For academic purposes, we demonstrate the obtained results on a simple example. To this end, consider a system of the form

$$\begin{cases} x_{k+1} = Ax_k + B_u u_k + w_k, \\ z_k \quad = \quad x_k \end{cases}$$

with matrices

$$A = \begin{bmatrix} 0.98 & 0.2 & 0.02 & 0 \\ -0.2 & 0.98 & 0.2 & 0.02 \\ 0.02 & 0 & 0.98 & 0.2 \\ 0.2 & 0.02 & -0.2 & 0.98 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.02 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}.$$

Assume that the external disturbance $\{w_k\}_{k\geqslant 0}$ acting on the system belongs to the class of sequences of random vectors with mean anisotropy bounded by $a = 0.1$: $\overline{\mathbf{A}}(W) \leqslant 0.1$. It is required to parametrically describe the set of solutions to the problem of constructing a stabilizing control law that ensures $\|F\|_a \leqslant \gamma$. We consider this problem in two cases: in the first case we assume that γ is a given number equal to $\gamma = 6$; in the second, the problem is to synthesize a γ-optimal anisotropic controller, i.e., $\gamma \to \min$. For comparative analysis, we also present the solution of the problems in accordance with Theorem 1 without parametrization. The calculations were performed in Matlab using the Yalmip toolbox for solving convex programming problems. Alternatively, the CVX system for solving convex programming problems can be used.

During the solution in accordance with the formulas of Theorem 1, one of the suboptimal anisotropic controller matrices found numerically at $\gamma = 6$ has the form (here and below, to distinguish controller matrices and other variables, they are assigned to a superscript index in parentheses)

$$K^{(1)} \approx [-7.14 \quad -5.72 \quad 1.95 \quad -5.01],$$

and the γ-optimal anisotropic controller matrix obtained by the same method but in the optimal formulation (see Remark 1) is

$$K^{(2)} \approx [-12.04 \quad -6.23 \quad 3.75 \quad -9.73]. \tag{25}$$

For comparison, the values of the anisotropic norms of the systems closed by the corresponding controllers turned out to be

$$\|F_{cl}(K^{(1)})\|_{0.1} \approx 5.83, \quad \|F_{cl}(K^{(2)})\|_{0.1} \approx 5.39.$$

Next, we successively present the solution of the parametrization problems for suboptimal and γ-optimal anisotropic state-feedback controllers, carried out in accordance with the content of Theorem 3 and Remark 3. The first of them will be presented in sufficient detail, and for the second we will only give the final result.

The found solution of the system of inequalities (16)–(19) with respect to the variables $P \succ 0$, $\Psi \succ 0$, $\eta^2 \geqslant \gamma^2$ are the matrices

$$P^{(3)} \approx \begin{bmatrix} 0.13 & \star & \star & \star \\ -0.09 & 0.93 & \star & \star \\ 0.05 & 0.01 & 0.07 & \star \\ -0.07 & -0.01 & -0.03 & 0.07 \end{bmatrix}, \quad \Psi^{(3)} \approx \begin{bmatrix} 213.52 & \star & \star & \star \\ -2.17 & 233.12 & \star & \star \\ 4.69 & 0.57 & 215.96 & \star \\ -19.68 & -2.1 & -3.7 & 197.32 \end{bmatrix}$$

and the number $\eta^{(3)} \approx 15.37$. Since some of the inverted matrices on the right-hand side of inequality (24) are ill-conditioned, determining the threshold value of T with high accuracy may turn out to be quite laborious. To avoid problems with computing the threshold value of T, in this example we set an overestimated value for it: $T^{(3)} \geq 410$. The required blocks Φ_{ij} of the matrix Φ will be obtained by substituting into (23) the chosen matrix T and the found P and η. Note that due to $B_{uR} = 1$, the variable Z will not participate in the parametrization, i.e., $\widetilde{K} = X$, and the controller matrix itself, given by expression (20) and the substitution $K = \widetilde{K}P^{-1}$, becomes $K = XP^{-1}$. Since it is difficult to visualize the parametrization proposed in Theorem 3 even in the (very simple) example under consideration, we restrict ourselves to the case of variation of the matrix T. To this end, we fix the matrix $\overline{Y} = [0.999 \quad 0 \quad 0 \quad 0]$ and consider the case $Y^{(3)} = \overline{Y}$, the case of the opposite sign matrix (i.e., $Y^{(3)} = -\overline{Y}$), and the case of the zero matrix $Y^{(3)} = 0$ in order to form an idea of the range of admissible values of the controller coefficients. The obtained dependences of the controller coefficients on the parameter T are shown in the figure. For the points "a" ($T^{(3)} = 410$, $Y^{(3)} = \overline{Y}$), "b" ($T^{(3)} = 820$, $Y^{(3)} = -\overline{Y}$), and "c" ($T^{(3)} = 39\,000$, $Y^{(3)} = 0$), we give the coefficients of the corresponding controllers:

$$\begin{aligned} K^{(3,\mathrm{a})} &\approx [-11.79 \quad -6.14 \quad 3.79 \quad -10.19], \\ K^{(3,\mathrm{b})} &\approx [-15.23 \quad -6.47 \quad 4.23 \quad -13.17], \\ K^{(3,\mathrm{c})} &\approx [-13.6 \quad -6.39 \quad 4.11 \quad -11.81]. \end{aligned}$$

The values of the anisotropic norms of the systems closed by them are respectively

$$|||F_{cl}(K^{(3,\mathrm{a})})|||_{0.1} \approx 5.4, \quad |||F_{cl}(K^{(3,\mathrm{b})})|||_{0.1} \approx 5.45, \quad |||F_{cl}(K^{(3,\mathrm{c})})|||_{0.1} \approx 5.41,$$

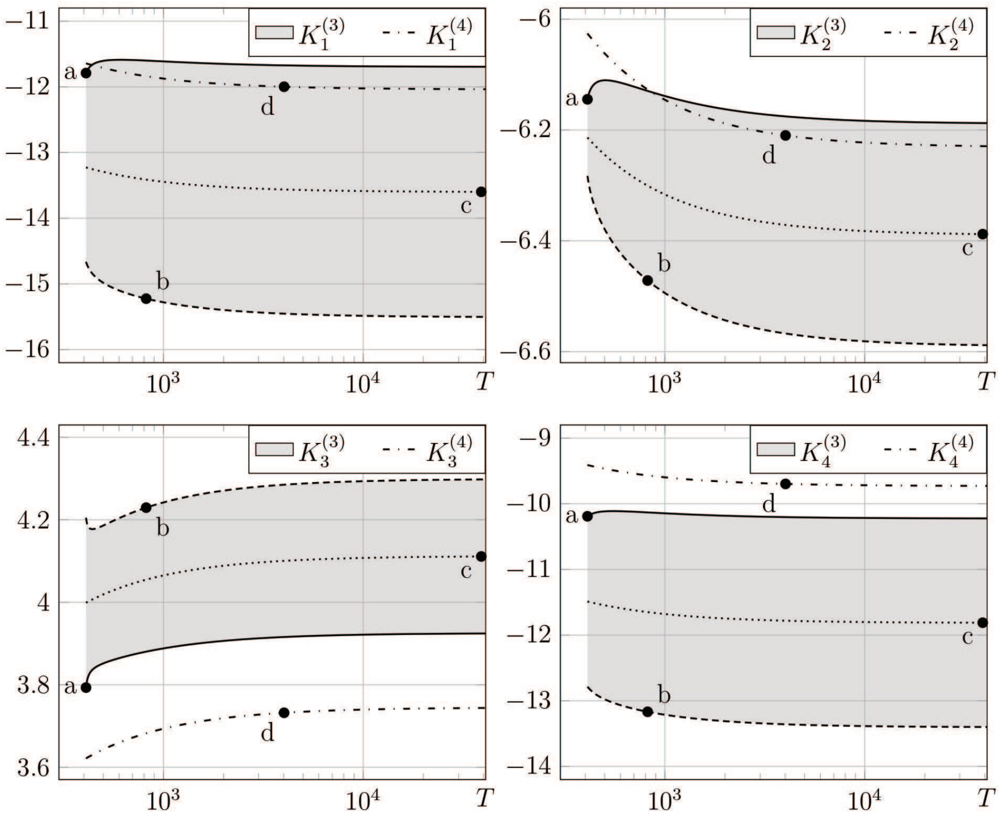
which satisfies the original requirement $|||F_{cl}|||_{0,1} \leqslant 6$.

Now consider the solution of the parametrization problem for a γ-optimal anisotropic state-feedback controller, as described in Remark 3. For this case, the term $S^{1/2} Y \Phi_{11}^{-1/2}$ from (21) is close to zero, i.e., the controller "loses" dependence on the matrix Y; therefore, only one line is shown in the figure, reflecting the direct dependence of $K^{(4)}$ on T. At the selected point "d", corresponding to the value $T^{(4)} = 4000$, the matrix of the γ-optimal anisotropic controller has the form

$$K^{(4,\mathrm{d})} \approx [-12 \quad -6.21 \quad 3.73 \quad -9.7], \tag{26}$$

and the value of the anisotropic norm of the system closed by this controller is

$$|||F_{cl}(K^{(4,\mathrm{d})})|||_{0.1} \approx 5.39.$$



Dependence of the coefficients of suboptimal and γ-optimal anisotropic controllers in the example of Section 5 on the parameter T (logarithmic scale). Solid line corresponds to $Y = Y^{(3)}$; dashed line—to $Y = 0$; shaded area reflects all possible controller coefficients for the intermediate case $Y = (2\lambda - 1)Y^{(3)}$ for arbitrary $\lambda \in [0; 1]$; dash-dotted line reflects the dependence of the γ-optimal controller $K^{(4)}$ on T. All lines and the shaded area are unbounded to the right (in the direction of increasing T). Points "a", "b", "c", "d" mark the coefficients of the controller matrices considered in the example.

This value is consistent with the previously obtained value for the γ-optimal controller $K^{(2)}$, as do (with some accuracy) the coefficients of the controller matrices themselves, which follows from a comparison of (25) and (26). The only difference is the possibility of choosing controller coefficients from some admissible range of values.

In conclusion of the example, it is worth emphasizing once again that the shaded area in the figure corresponds to all suboptimal anisotropic controllers generated by the matrices $P^{(3)}$, $\Psi^{(3)}$ and the number $\eta^{(3)}$ found at the first step of the solution. For another set of solutions (P, Ψ, η), this set will differ from that shown in the figure, and the union of all such sets will give the entire set of suboptimal anisotropic controllers for the chosen values $a \geqslant 0$ and $\gamma > 0$. This, in particular, explains why the line corresponding to the γ-optimal case (dash-dotted line in the figure) lies outside the shaded region for some values of T, although γ-optimal anisotropic controllers for a fixed $a \geqslant 0$ belong to all possible (but nonempty) sets of suboptimal controllers for the same a.

6. CONCLUSION

The paper proposes an approach to parametrization of suboptimal and γ-optimal anisotropic controllers in the form of static state feedback for linear discrete-time invariant systems with stochastic disturbances of bounded mean anisotropy. The relation of the solution of this problem to the solution of the synthesis problem for suboptimal and γ-optimal anisotropic controllers via convex optimization without parametrization is pointed out.

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APPENDIX

A.1. PROOF OF THEOREM 3

The proof of the theorem is based on "crossing" Theorems 1 and 2 and applying one useful property. In the conditions of Theorem 1, inequality (6) is representable in the form (9), where

$$\Theta = \begin{bmatrix} P & \star & \star & \star \\ 0_{n_w \times n_x} & \eta^2 I_{n_w} & \star & \star \\ AP & B_w & P & \star \\ CP & 0_{n_z \times n_w} & 0_{n_z \times n_x} & I_{n_z} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0_{n_x \times n_u} \\ 0_{n_w \times n_u} \\ B_u \\ 0_{n_z \times n_u} \end{bmatrix}, \quad \Pi^{\mathrm{T}} = \begin{bmatrix} I_{n_x} \\ 0_{n_w \times n_x} \\ 0_{n_x \times n_x} \\ 0_{n_z \times n_x} \end{bmatrix}.$$

Since Theorem 2 gives necessary and sufficient conditions for the existence of a solution to inequality (9), its analogue (6) can be replaced by the pair of inequalities corresponding to (10), (11). The first of them, analogous to (10), takes the form (16). The second inequality, analogous to (11), takes the form

$$\begin{bmatrix} \eta^2 I_{n_w} & \star & \star \\ B_w & P & \star \\ 0_{n_z \times n_w} & 0_{n_z \times n_x} & I_{n_z} \end{bmatrix} \succ 0,$$

which can be disregarded in view of the requirement that (7) holds for a positive definite matrix $\Psi \succ 0_{n_w \times n_w}$.

Since adding constraints to an already specified system of conditions can only narrow the corresponding set of solutions, adding inequalities (5), (7), (8) to (10), (11) can only narrow the solution set in terms of the variables P, Ψ, η^2, but does not change the fact that the parametrization of solutions given in the second part of Theorem 2 remains valid. Indeed, all matrices $\widetilde{K}$ given by (12) depend on Θ via (15)—it is this dependence (along with the restrictions on the chosen matrices Y and T) that guarantees the fulfillment of the original inequality (9).

Carrying out the necessary calculations carefully, taking into account that B_u admits the factorization $B_u = B_{uL} B_{uR}$ with full-rank matrices B_{uL} and B_{uR}, we obtain that expressions (20)–(23) are complete analogues of expressions (12)–(15), respectively.

At the end of the proof, the reverse substitution $K = \widetilde{K} P^{-1}$ is motivated by the content of Theorem 1. The theorem is proved.

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On the Relationship between the Average Characteristics of the Input and Output Fuzzy Signals of a Linear Dynamic System

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Abstract—This paper considers the problem of transforming a fuzzy signal by a linear dynamic system described by a linear differential equation of high order with variable real coefficients, a fuzzy-valued inhomogeneity, and fuzzy initial conditions. By assumption, certain numerical average characteristics of the fuzzy-valued input signal (fuzzy means and fuzzy half-ranges) are known. Based on these characteristics and system parameters, the corresponding numerical average characteristics of the fuzzy signal at the output are established. Methods of fuzzy mathematics and the theory of ordinary differential equations are used. As an application, the model of a series resonant radio circuit with a fuzzy-valued input signal is considered.

Keywords: fuzzy-valued functions, means of fuzzy-valued functions, derivatives of fuzzy-valued functions, fuzzy linear differential equations with variable coefficients

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1. INTRODUCTION

The problem of estimating the output signal of dynamic systems with incomplete or weakly formalized information on the input signal is of significant theoretical and applied importance, particularly for the subsequent mitigation of external disturbances (measurement noise, faults, etc.). The use of fuzzy differential equations is an important modern approach to modeling dynamic systems under possible uncertainties.

This paper addresses the problem of transforming a fuzzy signal by a linear dynamic system. The linear dynamic system is described by a linear fuzzy differential equation of high order with variable real coefficients, a fuzzy-valued inhomogeneity, and fuzzy initial conditions.

The foundations of the theory of fuzzy differential equations were laid in [1–5] and other works. This research area has been actively developing in recent decades. A method for transforming fuzzy linear differential equations to integral fuzzy equations (followed by their solution) was proposed in [6, 7]. Several authors considered an approach involving a reduction of original fuzzy differential equations to differential inclusions [8, 9] and an analysis of the latter.

In recent years, the fuzzy Laplace transform has become a widespread method for solving linear fuzzy differential equations of high order [10, 11]. Many researchers study numerical methods for solving fuzzy differential equations [12–14].

As a rule [15, 16], when dealing with linear fuzzy differential equations, a system of equations for the corresponding α-indices is compiled and solved. Then, it is necessary to verify that the derivatives of the resulting α-indices define the derivative of a fuzzy-valued function. According to demonstrative examples [6, 16], this is not always the case.

This paper utilizes the concept of an ultraweak fuzzy solution [17, 18] as a fuzzy-valued Hausdorff-continuous function whose α-indices satisfy the equations for the α-indices arising from a given fuzzy differential equation.

Various definitions of differentiability of fuzzy-valued functions are considered in the literature; for example, see [2–5]. In Section 3 of this paper, we use the Seikkala derivative [3]; in Section 4, generalized Hukuhara derivatives [5–7] and others.

Note that, in particular, the issue of smoothness of ultraweak Seikkala solutions was discussed in [17, 18].

The standard situation in applications is incomplete information on an input fuzzy signal. More precisely put, the exact membership functions of the corresponding fuzzy numbers are unknown, but some estimates can be provided. Here, we adopt an analogy with the theory of random processes: within a conventional approach, the average characteristics (mean and correlation function) of the output random signal of a linear dynamic system are determined based on those of its input random signal and system parameters [19, Ch. 7]. This approach is motivated by the fact that the random variables corresponding to an input random signal (and even more so, those corresponding to the output one) often have unknown distributions.

In this paper, the numerical average characteristics of an input fuzzy signal, i.e., the means of fuzzy numbers (fuzzy means) and the half-ranges of fuzzy numbers (fuzzy half-ranges), are supposed to be known; in particular, they can be obtained using expert appraisals. Then the following practically important problem naturally arises: taking into account the system parameters, it is required to determine the corresponding numerical average characteristics of a fuzzy signal at the output of a linear dynamic system.

Below, we consider the initial value problem for a fuzzy differential equation and actually determine the above numerical characteristics of its solution from the system parameters, fuzzy initial conditions, and the corresponding characteristics of the input data, without finding the fuzzy-valued solution itself. Note that such an approach for fuzzy differential equations has not been presented before. Fuzzy means commonly serve to defuzzify fuzzy numbers, and fuzzy half-ranges are introduced here and have proved fruitful in both theoretical and applied aspects.

Fuzzy linear differential equations are widely used in various applications. (Among contemporary works, we mention [20–22].) In Section 5, as an application, we describe the model of the response of a radio circuit with constant parameters to a fuzzy input signal.

This paper continues the author's research initiated in [23], where a linear dynamic system with constant real parameters was considered, and the relationship between the means and covariance functions of its input and output bounded fuzzy signals was established.

2. FUZZY NUMBERS AND FUZZY-VALUED FUNCTIONS

A fuzzy number is understood as a fuzzy subset of the universal set of real numbers $\mathbb{R}$ that has a compact support and a normal, convex, and upper semicontinuous membership function; for example, see [24, Ch. 5; 25, Chs. 2, 3].

Here, the support of a fuzzy number $\tilde{z}$ is defined as $Supp\,\tilde{z} = \{x|\mu_{\tilde{z}}(x) > 0\}$. A fuzzy number $\tilde{z}$ is normal if $\max_x \mu_{\tilde{z}}(x) = 1$ and convex if $\mu_{\tilde{z}}(t) \geqslant \min(\mu_{\tilde{z}}(r), \mu_{\tilde{z}}(s))$ for any real numbers $r \leqslant t \leqslant s$.

Let J denote the set of such fuzzy numbers.

Below we will use the interval representation of fuzzy numbers.

As is known, the α-level intervals (α-intervals) of a fuzzy number $\tilde{z} \in J$ with a membership function $\mu_{\tilde{z}}(x)$ are defined as $[\tilde{z}]_\alpha = \{x|\mu_{\tilde{z}}(x) \geqslant \alpha\}$ $(\alpha \in (0, 1])$, $[\tilde{z}]_0 = cl\{x|\mu_{\tilde{z}}(x) > 0\}$, where cl indicates the closure of an appropriate set. According to the accepted assumptions, all α-levels of a fuzzy number are closed and bounded intervals of the real axis.

Let z_α^- and z_α^+ denote the left and right bounds of an α-interval: $[\tilde{z}]_\alpha = [z_\alpha^-, z_\alpha^+]$. The values z_α^- and z_α^+ are called the left and right α-indices of a fuzzy number, respectively.

The α-indices of a fuzzy number $\tilde{z} \in J$ has the following properties:

1) $z_\alpha^- \leqslant z_\alpha^+ \ \forall\, \alpha \in [0,1]$.

2) The function z_α^- is bounded, nondecreasing, left-continuous on $(0,1]$ and right-continuous at 0.

3) The function z_α^+ is bounded, nonincreasing, left-continuous on $(0,1]$ and right-continuous at 0.

Conversely, a pair of functions on the interval $[0,1]$ with properties 1–3 defines a fuzzy number whose α-interval has the form $[z_\alpha^-, z_\alpha^+]$.

The sum of fuzzy numbers with indices z_α^-, z_α^+ and u_α^-, u_α^+ is understood as a fuzzy number c with the α-level intervals $[z_\alpha^- + u_\alpha^-, z_\alpha^+ + u_\alpha^+]$. Multiplication by a positive number c is characterized by the α-level intervals $[cz_\alpha^-, cz_\alpha^+]$; multiplication by a negative number c, by the α-level intervals $[cz_\alpha^+, cz_\alpha^-]$. Equality for fuzzy numbers is understood as equality for all the corresponding α-indices $\forall\, \alpha \in [0,1]$. A real number r is treated as a fuzzy number whose α-indices coincide with $r \ \forall\, \alpha \in [0,1]$.

According to [26], the mean value (fuzzy mean) of a fuzzy number $\tilde{z}$ with α-indices $z_\alpha^\pm$ is the real number

$$m_{\tilde{z}} = \frac{1}{2}\int_0^1 (z_\alpha^+ + z_\alpha^-)\, d\alpha.$$

As is known [24], the modal value (mode) of a fuzzy number $\tilde{z}$ is a value z_{Mod} for which $\mu(z_{Mod}) = 1$. For unimodal fuzzy numbers $\tilde{z}$, i.e., those having a unique modal value, the right and left widths are sometimes considered (for example, see [27, p. 11]):

$$(r_{\tilde{z}})_l = \int_0^1 (z_\alpha^+ - z_{Mod})\, d\alpha, \quad (r_{\tilde{z}})_r = \int_0^1 (z_{Mod} - z_\alpha^-)\, d\alpha.$$

We define the half-range of a fuzzy number $\tilde{z}(t)$ (fuzzy half-range) as the real number

$$r_{\tilde{z}} = \frac{1}{2}\int_0^1 (z_\alpha^+ - z_\alpha^-)\, d\alpha.$$

This definition does not require knowledge of the modal value of a fuzzy number and is natural, as a generalization of the concept of radius in interval analysis. Note that $r_{\tilde{z}} = \frac{1}{2}\left((r_{\tilde{z}})_l + (r_{\tilde{z}})_r\right)$.

Example 1. Consider a fuzzy triangular number $\tilde{z}$ characterized by a real-valued triple (a,b,c) with $a < b < c$ defining the membership function

$$\mu_{\tilde{z}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in [a,b] \\ \dfrac{x-c}{b-c} & \text{if } x \in [b,c] \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the left and right bounds of the α-interval have the form $z_\alpha^- = (b-a) + a$ and $z_\alpha^+ = -(c-b) + c$, respectively.

As is easily verified, the mean $m_{\tilde{z}}$ of this triangular number is $m_{\tilde{z}} = \frac{1}{4}(2b + a + c)$, and its half-range $r_{\tilde{z}}$ is $r_{\tilde{z}} = \frac{1}{4}(c-a)$.

Below, we will use the Hausdorff distance (metric), defined for fuzzy numbers $\tilde{z}$ and $\tilde{u}$ as [28] $d(\tilde{z},\tilde{u}) = \sup_{0 \leqslant \alpha \leqslant 1} \max\{|z_\alpha^- - u_\alpha^-|,\ |z_\alpha^+ - u_\alpha^+|\}$. Here, $z_\alpha^\pm$ and $u_\alpha^\pm$ are the α-indices of the fuzzy numbers $\tilde{z}$ and $\tilde{u}$, respectively.

Consider a fixed interval $[0,T]$ on the real axis. A mapping $\tilde z : [0,T] \to J$ will be called a fuzzy-valued function.

Let a fuzzy-valued function $\tilde z(t)$, $t \in [0,T]$, be characterized by a membership function $\mu_{\tilde z}(x,t)$. For a fixed number $\alpha \in (0,1]$, consider the α-interval $[\tilde z]_\alpha(t) = \{x \in \mathbb{R} : \mu_{\tilde z}(x,t) \geqslant \alpha\}$ and $z_0(\alpha) = cl\{x \in \mathbb{R} : \mu_{\tilde z}(x,t) > 0\}$. We denote by $z_\alpha^-(t)$ and $z_\alpha^+(t)$ the left and right bounds of the α-interval, respectively: $[\tilde z]_\alpha(t) = [z_\alpha^-(t), z_\alpha^+(t)]$.

The continuity of a function $\tilde z(t)$ in t will be understood in the Hausdorff metric, and its boundedness as the existence of a constant $C > 0$ such that $d(\tilde z(t), \tilde\theta) = \sup_{0\leqslant\alpha\leqslant1} \max\{|z_\alpha^-(t)|,\ |z_\alpha^+(t)|\} \leqslant C$ ($\forall\, t \in [0,T]$), where $\tilde\theta$ is a fuzzy number with the α-indices $\theta_\alpha^\pm = 0\ \forall\, \alpha \in [0,1]$, and d means the Hausdorff metric.

Remark 1. The sum of Hausdorff-continuous fuzzy-valued functions and the product of a fuzzy-valued function by a real number are continuous in the Hausdorff metric.

Remark 2. The indices $z_\alpha^-(t)$ and $z_\alpha^+(t)$ of a continuous fuzzy-valued function $\tilde z(t)$ are continuous in t for any $\alpha \in [0,1]$. Moreover, if $\tilde z(t)$ is bounded for $t \in [0,T]$, then $z_\alpha^\pm(t)$ is bounded in $t \in [0,T]$ for any $\alpha \in [0,1]$.

The (Riemann) integral of a continuous fuzzy-valued function $\tilde z(t)$ over an interval $[0,T]$ is defined [29] as the fuzzy number $\tilde g$ with the α-level intervals $g_\alpha = \int_0^T z_\alpha(t)\, dt$ for any $\alpha \in [0,1]$. The integral is denoted by $\int_0^T \tilde z(t) dt$.

In view of Remark 2, for a continuous function $\tilde z : [0,T] \to J$, we have the interval representation $\left[\int_0^T \tilde z(t)dt\right]_\alpha = \left[\int_0^T z_\alpha^-(t)dt,\ \int_0^T z_\alpha^+(t)dt\right]$ [29].

Recall the definition of the Seikkala derivative of a fuzzy-valued function.

A fuzzy-valued function $\tilde z : [0,T] \to J$ is said to be Seikkala-differentiable (or S-differentiable) at a point $t \in (0,T)$ [3] if its α-indices $z_\alpha^-(t)$ and $z_\alpha^+(t)$ are differentiable and their derivatives $(z_\alpha^-)'(t)$ and $(z_\alpha^+)'(t)\ \forall\, \alpha \in [0,1]$ induce a fuzzy number with the α-interval $[\tilde z'(t)]_\alpha = [(z_\alpha^-)'(t), (z_\alpha^+)'(t)]$.

For example, a fuzzy-valued function $\tilde z(t) = g(t)\tilde r$, where $g(t)$ is a scalar differentiable function and $\tilde r \in J$ is a given fuzzy number, will be S-differentiable at a point t under the condition $g(t)g'(t) \geqslant 0$ [5].

Remark 3. By definition, the fuzzy S-derivative is additive and positively homogeneous. That is, for fuzzy-valued S-differentiable functions $\tilde z(t)$ and $\tilde w(t)$, we have $(\tilde z(t) + \tilde w(t))' = \tilde z'(t) + \tilde w'(t)$ and $(c\tilde z(t))' = c\tilde z'(t)$ for any real constant $c \geqslant 0$. Moreover, $[(\tilde z(t) + \tilde w(t))']_\alpha^\pm = (z_\alpha^\pm)'(t) + (w_\alpha^\pm)'(t)$ and $[(c\tilde z(t))']_\alpha^\pm = c(z_\alpha^\pm)'(t)$.

The second S-derivative $\tilde z''(t)$ at a point $t \in (0,T)$ is defined as the S-derivative of the first one, i.e., as the fuzzy number $\tilde z''(t)$ having the left α-index $(z_\alpha^-)''(t)$ and the right α-index $(z_\alpha^+)''(t)$ $\forall\, \alpha \in [0,1]$.

The definitions of higher-order S-derivatives are introduced by analogy.

Generalized derivatives of fuzzy-valued functions will be defined and used in Section 4.

3. THE INITIAL VALUE PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS WITH A FUZZY-VALUED INHOMOGENEITY AND FUZZY INITIAL CONDITIONS

For a linear inhomogeneous fuzzy differential equation on an interval $[0,T]$, consider the initial value problem

$$\tilde y^{(n)}(t) + p_1(t)\tilde y^{(n-1)}(t) + \cdots + p_n(t)\tilde y(t) = \tilde f(t) \quad (t \in (0,T]), \tag{1}$$

$$\tilde y^{(j)}(0) = \tilde a_j \quad (j = 0,\ldots,n-1) \tag{2}$$

with variable real continuous coefficients $p_i(t)$ $(i = 1, \ldots, n)$, a Hausdorff-continuous fuzzy-valued function $\tilde{f} : [0, T] \to J$, and fuzzy initial conditions $\tilde{a}_j$ $(j = 0, \ldots, n-1)$.

We call an n-times continuously S-differentiable fuzzy-valued function $\tilde{y}(t)$ satisfying (1) and (2) a strong solution of problem (1), (2).

Let $y^{\pm}_{\alpha}(t)$, $f^{\pm}_{\alpha}(t)$ $\forall\, t \in [0, T]$ and $(a_j)^{\pm}_{\alpha}$ $(j = 0, \ldots, n-1)$ denote the α-indices of $\tilde{y}(t)$, $\tilde{f}(t)$, and $\tilde{a}_j$, respectively. Then, according to the definition and properties of S-derivatives, problem (1), (2) $\forall\, \alpha \in [0, 1]$ has the interval representation

$$\Big[(y^-_{\alpha})^{(n)}, (y^+_{\alpha})^{(n)}\Big] + p_1(t)\Big[(y^-_{\alpha})^{(n-1)}, (y^+_{\alpha})^{(n-1)}\Big] + p_n(t)\,[y^-_{\alpha}, y^+_{\alpha}] = [f^-_{\alpha}(t), f^+_{\alpha}(t)] \quad (t \in (0, T]), \tag{3}$$

$$\Big[(y^-_{\alpha})^{(j)}(0), (y^+_{\alpha})^{(j)}(0)\Big] = [(a_j)^-_{\alpha}, (a_j)^+_{\alpha}] \quad (j = 0, \ldots, n-1). \tag{4}$$

Note that $f^{\pm}_{\alpha}(t)$ (the α-indices of the continuous fuzzy-valued function $\tilde{f}(t)$) are, continuous real-valued functions of t $\forall\, \alpha \in [0, 1]$ (see Remark 2).

We call a Hausdorff-continuous fuzzy-valued function $\tilde{y} : [0, T] \to J$ with the α-indices satisfying (3), (4) an ultraweak solution of problem (1), (2) [17, 18].

The following result is immediate from the above definitions.

Lemma 1. *Let the coefficients $p_i(t)$ $(t \in [0, T];\ i = 1, \ldots, n)$ of equation* (1) *be continuous real-valued functions, and let the fuzzy-valued function $\tilde{f} : [0, T] \to J$ be continuous in the Hausdorff metric. Then a strong solution of problem* (1), (2) *is an ultraweak solution.*

An ultraweak solution is strong only under the additional assumption of the existence and continuity of all its fuzzy derivatives appearing in equation (1).

According to demonstrative examples [6, 16], the solutions $y^{\pm}_{\alpha}(t)$ of problem (3), (4) may fail to define fuzzy numbers for some (or all) values of t, i.e., at least one of conditions 1–3 of Section 2 is violated for them. On the other hand, even if $y^{\pm}_{\alpha}(t)$ define a fuzzy-valued function, it may be non-differentiable. In this connection, the fuzzy-valued solutions of problem (1), (2) were classified by the author into strong, weak, and ultraweak [17, 18]. Here, a weak solution is understood as an ultraweak one that additionally satisfies the integral equation corresponding to problem (1), (2). Weak solutions are not considered in this paper.

Lemma 2. *Under the hypotheses of Lemma 1, let $\forall\, t \in [0, T]$ $p_i(t) \geqslant 0$ $(i = 1, \ldots, n)$. Then the following problems hold for the α-indices $y^{\pm}_{\alpha}(t)$ of the strong solution $\tilde{y}(t)$ $\forall\, \alpha \in [0, 1]$:*

—for the right indices,

$$(y^+_{\alpha})^{(n)}(t) + p_1(t)(y^+_{\alpha})^{(n-1)}(t) + \cdots + p_n(t)y^+_{\alpha}(t) = f^+_{\alpha}(t) \quad (t \in (0, T]), \tag{5}$$

$$(y^+_{\alpha})^{(j)}(0) = (a_j)^+_{\alpha}, \quad j = 0, \ldots, n-1; \tag{6}$$

—for the left indices,

$$(y^-_{\alpha})^{(n)}(t) + p_1(t)(y^-_{\alpha})^{(n-1)}(t) + \cdots + p_n(t)y^-_{\alpha}(t) = f^-_{\alpha}(t) \quad (t \in (0, T]), \tag{7}$$

$$(y^-_{\alpha})^{(j)}(0) = (a_j)^-_{\alpha}, \quad j = 0, \ldots, n-1. \tag{8}$$

This fact follows from the linearity of equation (1), the properties of arithmetic operations on fuzzy numbers in interval form, and the additivity and positive homogeneity of derivatives (Remark 3).

Thus, in the case of nonnegative coefficients $p_i(t)$ $(i = 1, \ldots, n)$ of equation (1), for all $t \in [0, T]$ the α-indices of an ultraweak solution $\tilde{y} : [0, T] \to J$ of problem (1), (2) satisfy the relations (5), (6) and (7), (8), respectively. That is, the system for the α-indices splits into two independent problems for the right and left indices, respectively.

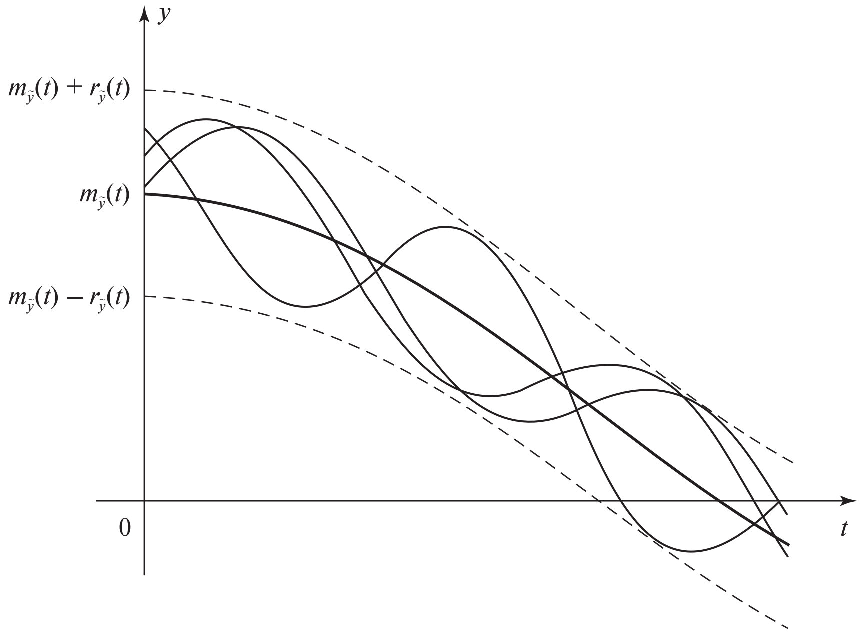

Fig. 1. The geometric interpretation of the mean and half-range of solutions.

Note that according to [30, Ch. 3], under the hypotheses of Lemma 2, for all $\alpha \in [0,1]$ there exist unique solutions of the linear problems (5), (6) and (7), (8).

We denote by

$$m_{\tilde{y}}(t) = \frac{1}{2}\int\limits_0^1 (y_\alpha^+(t) + y_\alpha^-(t))d\alpha, \quad r_{\tilde{y}}(t) = \frac{1}{2}\int\limits_0^1 (y_\alpha^+(t) - y_\alpha^-(t))d\alpha \tag{9}$$

the mean and half-range of an ultraweak solution $\tilde{y}(t)$ of problem (3), (4) for all $t \in (0,T]$. By definition, for any $t \in (0,T]$ and $\alpha \in [0,1]$, an ultraweak solution of problem (1), (2) is associated with the α-interval $[\tilde{y}(t)]_\alpha = [y_\alpha^-(t), y_\alpha^+(t)]$. Consider the interval $\left[\int_0^1 y_\alpha^-(t)d\alpha, \int_0^1 y_\alpha^+(t)d\alpha\right]$. Any element $y(t), t \in (0,T]$, of this interval lies in a band (Fig. 1).

Let us investigate the relationship between the numerical average characteristics of a fuzzy-valued inhomogeneity at the input of a linear fuzzy system and those of the fuzzy-valued function at its output, i.e., the corresponding characteristics of an ultraweak fuzzy-valued solution of problem (1), (2).

Also, we introduce

$$m_{\tilde{f}}(t) = \frac{1}{2}\int\limits_0^1 \left(f_\alpha^+(t) + f_\alpha^-(t)\right) d\alpha, \quad r_{\tilde{f}}(t) = \frac{1}{2}\int\limits_0^1 \left(f_\alpha^+(t) - f_\alpha^-(t)\right) d\alpha, \tag{10}$$

representing the mean and half-range, respectively, of the inhomogeneity $\tilde{f}(t)$ of equation (1) for all $t \in (0,T]$.

Theorem 1. *Under the hypotheses of Lemma 1, the mean* $m_{\tilde{y}}(t)$ (9) *of an ultraweak solution* $\tilde{y}(t)$ *of problem* (1), (2) *satisfies the equation*

$$(m_{\tilde{y}})^{(n)}(t) + p_1(t)(m_{\tilde{y}})^{(n-1)}(t) + \cdots + p_n(t)m_{\tilde{y}}(t) = m_{\tilde{f}}(t) \quad (t \in (0,T]), \tag{11}$$

where $m_{\tilde f}(t)$ is given by formula (10). *Moreover, the following initial conditions hold:*

$$(m_{\tilde y})^{(j)}(0) = m_{\tilde y_j} = \frac{1}{2}\left(\int_0^1 ((a_j)^+_\alpha + (a_j)^-_\alpha)\, d\alpha\right) \quad (j = 0, \ldots, n-1). \tag{12}$$

Proof. First, assume that all coefficients of equation (1) are $p_i(t) \geqslant 0$ $(i = 1, \ldots, n)$, $\forall\, t \in [0, T]$. Consider the arithmetic average $A_{\tilde y}(\alpha, t) = \frac{1}{2}(y^+_\alpha(t) + y^-_\alpha(t))$. We sum the left and right-hand sides of equalities (5), (7) and divide the resulting relation by 2. Due to the linearity of equations (5), (7), the real-valued function $A_{\tilde y}(\alpha, t)$ $\forall\, \alpha \in [0, 1], t \in (0, T]$ satisfies the real differential equation

$$(A_{\tilde y})^{(n)}_t(\alpha, t) + p_1(t)(A_{\tilde y})^{(n-1)}_t(\alpha, t) + \cdots + p_n(t)A_{\tilde y}(\alpha, t) = \frac{1}{2}(f^+_\alpha(t) + f^-_\alpha(t)).$$

In view of the interchangeability of integration with respect to α and differentiation with respect to t for $m_{\tilde y}(t)$, we integrate both sides of this equation with respect to α from 0 to 1 to obtain the differential equation (11).

Concerning the initial conditions, note that according to (6), (8),

$$(A_{\tilde y}(\alpha, t))^{(j)}\Big|_{t=0} = \frac{1}{2}((a_j)^+_\alpha + (a_j)^-_\alpha) \quad (j = 0, \ldots, n-1);$$

consequently, conditions (12) are valid.

Now we analyze the case where some (or all) coefficients of equation (1) are negative. For example, suppose that $p_i(t) < 0$ for some i at some point $t \in (0, T]$. For the sake of definiteness, let the other coefficients be nonnegative. With $p_i(t) = -q_i(t)$, based on the relation (3), the corresponding expression can be written as

$$p_i(t)\left[(y^-_\alpha)^{(n-i)}, (y^+_\alpha)^{(n-i)}\right] = \left[q_i(t)(-y^+_\alpha)^{(n-i)}, q_i(t)(-y^-_\alpha)^{(n-i)}\right] = q_i(t)\left[(-y^+_\alpha)^{(n-i)}, (-y^-_\alpha)^{(n-i)}\right].$$

Therefore, the equations for the right and left α-indices become

$$(y^+_\alpha)^{(n)} + p_1(t)(y^+_\alpha)^{(n-1)} + \cdots + q_i(-y^-_\alpha)^{(n-i)} + \cdots + p_n(t)y^+_\alpha = f^+_\alpha(t) \tag{13}$$

and

$$(y^-_\alpha)^{(n)} + p_1(t)(y^-_\alpha)^{(n-1)} + \cdots + q_i(-y^+_\alpha)^{(n-i)} + \cdots + p_n(t)y^-_\alpha = f^-_\alpha(t), \tag{14}$$

respectively, where $t \in (0, T]$.

We sum both sides of equalities (13) and (14) and divide by 2. By integrating the resulting relation with respect to α from 0 to 1 and using the designation $p_i(t) = -q_i(t)$, we arrive at equation (11) for the mean $m_{\tilde y}(t)$.

Next, it is necessary to study the relationship between the half-range $r_{\tilde f}(t)$ (10) of the inhomogeneity and the half-range (9) of an ultraweak solution $\tilde y(t)$ of equation (1), (2).

Theorem 2. *Under the hypotheses of Lemma 1, the half-range $r_{\tilde y}(t)$* (9) *of an ultraweak solution $\tilde y(t)$ of problem* (1), (2) *satisfies the equation*

$$x^{(n)}(t) + |p_1(t)|x^{(n-1)}(t) + \cdots + |p_i(t)|x^{(n-i)}(t) + \cdots + |p_n(t)|x(t) = r_{\tilde f}(t), \tag{15}$$

where $t \in (0, T]$, and the right-hand side $r_{\tilde f}(t)$ is given by (10). *Moreover, the following initial conditions hold:*

$$(r_{\tilde y})^{(j)}(0) = r_{\tilde a_j} \equiv \frac{1}{2}\left(\int_0^1 ((a_j)^+_\alpha - (a_j)^-_\alpha)\, d\alpha\right) \quad (j = 0, \ldots, n-1). \tag{16}$$

Proof. First, consider the case of nonnegative coefficients $p_i(t)$ $(i = 1, \ldots, n)$. By subtracting equality (7) from equality (5), due to the linear equations for the half-difference $R_{\tilde{y}}(\alpha, t) = \frac{1}{2}(y^+_\alpha(t) - y^-_\alpha(t))$, we obtain for all $\alpha \in [0, 1]$ and $t \in (0, T]$ the real equation

$$(R_{\tilde{y}})^{(n)}_t(\alpha, t) + p_1(t)(R_{\tilde{y}})^{(n-1)}_t(\alpha, t) + \cdots + p_n(t)R_{\tilde{y}}(\alpha, t) = \frac{1}{2}(f^+_\alpha(t) - f^-_\alpha(t)) \tag{17}$$

with the initial conditions

$$(R_{\tilde{y}}(\alpha, t))^{(j)}\Big|_{t=0} = \frac{1}{2}\left((a_j)^+_\alpha - (a_j)^-_\alpha\right) \quad (j = 0, \ldots, n-1). \tag{18}$$

Integrating both sides of equation (17) with respect to α from 0 to 1 yields the following real equation for the half-range $r_{\tilde{y}}(t)$:

$$(r_{\tilde{y}})^{(n)}(t) + p_1(t)(r_{\tilde{y}})^{(n-1)}(t) + \cdots + p_n(t)r_{\tilde{y}}(t) = r_{\tilde{f}}(t), t \in (0, T]. \tag{19}$$

In the case of nonnegative coefficients $p_i(t)$, equation (19) coincides with (13).

Moreover, the initial conditions (16) are valid by virtue of (18).

Considering the case of half-ranges, let $p_i(t) < 0$ for some i at some point $t \in (0, T]$, provided that the other coefficients are nonnegative. By subtracting equality (14) from equality (13) and integrating both sides of the resulting relation with respect to α from 0 to 1, we establish the relation

$$x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots + (-p_i(t))x^{(n-i)}(t) + \cdots + p_n(t)x(t) = r_{\tilde{f}}(t)$$

for the half-ranges (10), where $t \in (0, T]$.

Consequently, in the equation for the half-range, there is a minus sign at the negative coefficient of the original equation. Hence, this coefficient becomes positive. Since $p_i(t)$ can take both positive and negative values at different points $t \in (0, T]$, equation (15) generally holds with the initial conditions (16).

Note that if an ultraweak solution $\tilde{y}(t)$ of problem (1), (2) exists, its α-indices $y^\pm_\alpha(t)$ for any $t \in (0, T]$ will satisfy conditions 1–3 of fuzzy numbers (see Section 2), in particular, $y^-_\alpha(t) \leqslant y^+_\alpha(t)$. Consequently, the half-range (9) of the fuzzy-valued function $\tilde{y}(t)$ is nonnegative. If the solution of problem (15), (16) is a function with negative values at some points, then $\tilde{y}(t)$ is not an ultraweak solution at such points. (In other words, an ultraweak solution does not exist.)

4. THE CASE OF GENERALIZED DERIVATIVES

In recent decades, researchers have actively been using generalized derivatives of fuzzy-valued functions [5–8, 14–16]. Let us formulate the corresponding definition [5].

A set $C = A \ominus B$ is called the Hukuhara difference of sets A and B if $A = B + C$.

Definition [5]. Let $\tilde{f} : (0, T) \to J$ and $t_0 \in (0, T)$. A fuzzy-valued function $\tilde{f}$ is said to be strongly generalized differentiable at a point t_0 if there exists an element $\tilde{f}'(t_0) \in J$ such that either

1) for all sufficiently small $h > 0$, there exist $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$, and the limits (in the Hausdorff metric d)

$$\lim_{h \to 0^+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \tilde{f}'(t_0),$$

or

2) for all sufficiently small $h > 0$, there exist $f(t_0) \ominus f(t_0 + h)$, $f(t_0 - h) \ominus f(t_0)$, and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{(-h)} = \tilde{f}'(t_0).$$

In case (1), one speaks of the generalized (1)-differentiability of $\tilde{f}$ at t_0, denoting the derivative by $D_1^{(1)}\tilde{f}(t_0)$. In case (2), one speaks of the generalized (2)-differentiability of $\tilde{f}$ at t_0, denoting the derivative by $D_2^{(1)}\tilde{f}(t_0)$. As usual, generalized differentiability on an interval is understood as the same property at each point of the interval.

Let a function $\tilde{f}\colon (0,T)\to J\ \forall\,\alpha\in[0,1]$ be associated with the α-intervals $[\tilde{f}]_\alpha(t)=[f_\alpha^-(t), f_\alpha^+(t)]$. Note that by the definition of a Hukuhara difference, the expression $\tilde{f}(t_0+h)\ominus\tilde{f}(t_0)$ is associated with the α-interval $[\tilde{f}(t_0+h)\ominus\tilde{f}(t_0)]_\alpha=[f_\alpha^-(t_0+h)-f_\alpha^-(t_0),\ f_\alpha^+(t_0+h)-f_\alpha^+(t_0)]$. The assumption of the existence of $\tilde{f}(t_0+h)\ominus\tilde{f}(t_0)$ means that these α-intervals induce a fuzzy number, i.e., conditions 1–3 of the α-indices are valid (see Section 2).

Lemma 3 [15]. *Let a fuzzy-valued function $\tilde{f}:(0,T)\to J$ be generalized (1)-differentiable. Then its derivative $D_1^{(1)}\tilde{f}(t)\ \forall\,\alpha\in[0,1]$ is associated with the α-interval $[D_1^{(1)}\tilde{f}(t)]_\alpha=[(f_\alpha^-)'(t),(f_\alpha^+)'(t)]$. Let a fuzzy-valued function $\tilde{f}:(0,T)\to J$ be generalized (2)-differentiable. Then its derivative $D_2^{(1)}\tilde{f}(t)\ \forall\,\alpha\in[0,1]$ is associated with the α-interval $[D_2^{(1)}\tilde{f}(t)]_\alpha=[(f_\alpha^+)'(t),(f_\alpha^-)'(t)]$.*

According to Lemma 3, we arrive at the following.

Corollary 1. *The generalized (1)-differentiability of a fuzzy-valued function $\tilde{f}:(0,T)\to J$ implies its Seikkala differentiability on $(0,T)$.*

A fuzzy-valued function $\tilde{f}:(0,T)\to J$ is said to be twice generalized (n,m)-differentiable (for $n,m=1,2$) at a point t_0 if the first generalized derivative of type n, $D_n^{(1)}\tilde{f}(t)$, exists in a neighborhood of t_0 as a fuzzy-valued function, and it is generalized m-differentiable at t_0. This fact will be denoted by $D_{n,m}^{(2)}\tilde{f}(t_0)$. Higher-order generalized derivatives are defined by analogy.

Due to Lemma 3, we have another result.

Corollary 2. *Let a fuzzy-valued function $\tilde{f}\colon(0,T)\to J$ be characterized by α-intervals $[f_\alpha^-(t), f_\alpha^+(t)]$.*

1. *If $\tilde{f}(t)$ is a twice generalized (1,1)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{1,1}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^-)''(t),(f_\alpha^+)''(t)]$ for its second derivative $D_{1,1}^{(2)}\tilde{f}(t)$.*

2. *If $\tilde{f}(t)$ is a twice generalized (1,2)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{1,2}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^+)''(t),(f_\alpha^-)''(t)]$ for its second derivative $D_{1,2}^{(2)}\tilde{f}(t)$.*

3. *If $\tilde{f}(t)$ is a twice generalized (2,1)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{2,1}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^+)''(t),(f_\alpha^-)''(t)]$ for its second derivative $D_{2,1}^{(2)}\tilde{f}(t)$.*

4. *If $\tilde{f}(t)$ is a twice generalized (2,2)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{2,2}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^-)''(t),(f_\alpha^+)''(t)]$ for its second derivative $D_{2,2}^{(2)}\tilde{f}(t)$.*

Now consider the situation where the derivatives of a fuzzy-valued solution of problem (1), (2) are generalized derivatives of the first or second type. Further analysis deals with a second-order fuzzy differential equation of the form

$$\tilde{y}''(t)+p_1(t)\tilde{y}'(t)+p_2(t)\tilde{y}(t)=\tilde{f}(t)\quad (t\in(0,T]) \tag{20}$$

with continuous real coefficients $p_1,p_2:[0,T]\to\mathbb{R}$, a Hausdorff-continuous fuzzy-valued function $\tilde{f}(t):[0,T]\to J$, and the initial conditions

$$\tilde{y}(0)=\tilde{a},\ \tilde{y}'(t)=\tilde{b}\quad (\tilde{a},\tilde{b}\in J). \tag{21}$$

Note that in this context, four types of equations for the α-indices are possible, corresponding to the types of first and second generalized derivatives.

The following result is immediate from Lemma 3 and Corollary 2.

Lemma 4. *Let the coefficients $p_1, p_2 : [0,T] \to \mathbb{R}$ be continuous real functions nonnegative on $[0,T]$, and let the fuzzy-valued function $\tilde{f}(t) : [0,T] \to J$ be continuous in the Hausdorff metric. Then, depending on the type of the generalized differentiability of a fuzzy-valued solution $\tilde{y}(t)$ of equation* (20), *the α-indices satisfy the following equations:*

—in the case of generalized (1, 1)*-differentiability, the equations*

$$(y_\alpha^+)''(t) + p_1(y_\alpha^+)'(t) + p_2 y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{22}$$

$$(y_\alpha^-)''(t) + p_1(y_\alpha^-)'(t) + p_2 y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]); \tag{23}$$

—in the case of the generalized (1, 2)*-differentiability, the equations*

$$(y_\alpha^-)''(t) + p_1(t)(y_\alpha^+)'(t) + p_2(t) y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{24}$$

$$(y_\alpha^+)''(t) + p_1(t)(y_\alpha^-)'(t) + p_2(t) y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]); \tag{25}$$

—in the case of the generalized (2, 1)*-differentiability, the equations*

$$(y_\alpha^-)''(t) + p_1(t)(y_\alpha^-)'(t) + p_2(t) y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{26}$$

$$(y_\alpha^+)''(t) + p_1(t)(y_\alpha^+)'(t) + p_2(t) y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]); \tag{27}$$

—finally, in the case of the generalized (2, 2)*-differentiability, the equations*

$$(y_\alpha^+)''(t) + p_1(t)(y_\alpha^-)'(t) + p_2(t) y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{28}$$

$$(y_\alpha^-)''(t) + p_1(t)(y_\alpha^+)'(t) + p_2(t) y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]). \tag{29}$$

Moreover, the following initial conditions corresponding to (21) *are valid:*

—in the case of the (1)*-differentiability of $\tilde{y}(t)$,*

$$y_\alpha^\pm(0) = a_\alpha^\pm, (y_\alpha^\pm)'(0) = b_\alpha^\pm; \tag{30}$$

—in the case of the (2)*-differentiability of $\tilde{y}(t)$,*

$$y_\alpha^\pm(0) = a_\alpha^\pm, (y_\alpha^\pm)'(0) = b_\alpha^\mp. \tag{31}$$

A fuzzy-valued function whose α-indices satisfy equations (22), (23), and (30) will be called a (1,1)-ultraweak solution of problem (20), (21); one satisfying equations (24), (25), and (30), a (1,2)-ultraweak solution of problem (20), (21); one satisfying equations (26), (27), and (31), a (2,1)-ultraweak solution of problem (20), (21); finally, one satisfying equations (28), (29), and (31), a (2,2)-ultraweak solution of problem (20), (21).

The following result is true.

Theorem 3. *Under the hypotheses of Lemma 4, regardless of the types of ultraweak solutions of problem* (20), (21), *the mean* (9) *satisfies the same equation*

$$m''_{\tilde{y}}(t) + p_1(t) m'_{\tilde{y}}(t) + p_2(t) m_{\tilde{y}}(t) = m_{\tilde{f}}(t) \quad (t \in (0,T]) \tag{32}$$

with the initial conditions

$$m_{\tilde{y}}(0) = m_{\tilde{a}} = \frac{1}{2}\int_0^1 (a_\alpha^+ + a_\alpha^-) d\alpha, \quad (m_{\tilde{y}})'(0) = m_{\tilde{b}} = \frac{1}{2}\int_0^1 (b_\alpha^+ + b_\alpha^-) d\alpha. \tag{33}$$

Indeed, in each case of (n,m)-ultraweak solutions of problem (20), (21) for $n,m = 1,2$, we sum the corresponding equalities for the right and left α-indices and divide the resulting relation by 2. By integrating both sides of the latter with respect to α from 0 to 1, we obtain equation (32) for the mean $m_{\tilde{y}}(t)$.

Concerning the initial conditions, note that:

1. In the case of the (1)-differentiability of $\tilde{y}(t)$, formulas (12) and (30) imply (33).
2. In the case of the (2)-differentiability of $\tilde{y}(t)$, formulas (12) and (31) imply (33).

Remark 4. Theorem 3 remains valid if the coefficients $p_i(t)$ of equation (20) have arbitrary signs.

For half-ranges, the following result is true.

Proposition 1. *Under the hypotheses of Lemma 4, depending on the type of a generalized ultraweak solution $\tilde{y}(t)$ of equation* (20)*, its half-ranges* (9) *satisfy the following equations:*

—in the case of a generalized $(1,1)$*-ultraweak solution, the equation* $r''_{\tilde{y}}(t) + p_1(t)r'_{\tilde{y}}(t) + p_2(t)r_{\tilde{y}}(t) = r_{\tilde{f}}(t)$ $(t \in (0,T])$;

—in the case of a generalized $(1,2)$*-ultraweak solution, the equation* $-r''_{\tilde{y}}(t) + p_1(t)r'_{\tilde{y}}(t) + p_2(t)r_{\tilde{y}}(t) = r_{\tilde{f}}(t)$ $(t \in (0,T])$;

—in the case of a generalized $(2,1)$*-ultraweak solution, the equation* $-r''_{\tilde{y}}(t) - p_1(t)r'_{\tilde{y}}(t) + p_2(t)r_{\tilde{y}}(t) = r_{\tilde{f}}(t)$ $(t \in (0,T])$;

—finally, in the case of a generalized $(2,2)$*-ultraweak solution, the equation* $r''_{\tilde{y}}(t) - p_1(t)r'_{\tilde{y}}(t) + p_2(t)r_{\tilde{y}}(t) = r_{\tilde{f}}(t)$ $(t \in (0,T])$.

This proposition is obtained by subtracting the corresponding equations for the right and left indices.

In this case, the initial conditions have the form $r_{\tilde{y}}(0) = r_{\tilde{a}} = \frac{1}{2}\int_0^1 (a_\alpha^+ - a_\alpha^-)d\alpha$ and $r'_{\tilde{y}}(0) = r_{\tilde{b}} = \frac{1}{2}\int_0^1 (b_\alpha^+ - b_\alpha^-)d\alpha$ in the case of the generalized (1)-differentiability of the solution, or the second condition is replaced by $r'_{\tilde{y}}(0) = -r_{\tilde{b}}$ in the case of the generalized (2)-differentiability of the solution.

5. TRANSFORMATION OF A FUZZY SIGNAL BY A SERIES RESONANT CIRCUIT

Let us pose the following problem: find the response of a radio circuit (Fig. 2) to an input fuzzy signal $\tilde{g}(t)$.

Here, L is the inductance, C is the capacitance, and R is the resistance; $\tilde{g}(t)$ is treated as a voltage source; finally, $\tilde{z}(t)$ is the voltage taken across the resistance. This circuit is called a series resonant circuit. With $\tilde{y}(t) = \frac{1}{R}\tilde{z}(t)$, the circuit is described (e.g.,see [31]) by a second-order

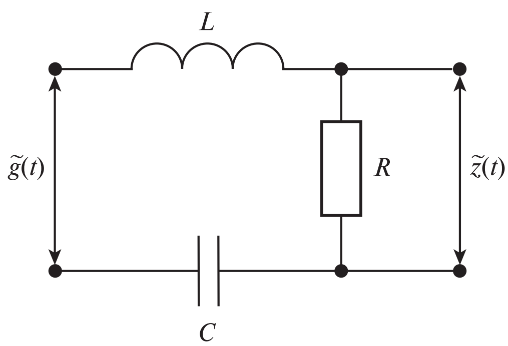

Fig. 2. Series resonant circuit.

differential equation with constant coefficients for the currents $\tilde{y}(t)$ of the form

$$\begin{gathered} a_0\tilde{y}''(t) + a_1\tilde{y}'(t) + a_2\tilde{y}(t) = \tilde{g}'(t), \\ a_0 = L > 0, \ \ a_1 = R > 0, \ \ a_2 = \frac{1}{C} > 0 \quad (t \in (0,T]), \end{gathered} \tag{34}$$

where $\tilde{g} : [0,T] \to J$ is a continuously differentiable (in the Seikkala sense) fuzzy-valued input signal.

Assume that at the initial instant, the system has the state

$$\tilde{y}(0) = \tilde{a}, \quad \tilde{y}'(0) = \tilde{b}, \tag{35}$$

where $a, b \in J$ are fuzzy numbers.

By dividing (34) by $a_0 > 0$, we obtain the equation

$$\begin{gathered} \tilde{y}''(t) + p_1\tilde{y}'(t) + p_2\tilde{y}(t) = \tilde{f}(t) \quad (t \in (0,T]), \\ p_1 := \frac{R}{L}, \quad p_2 := \frac{1}{LC}, \quad \tilde{f}(t) := \frac{1}{L}\tilde{g}'(t). \end{gathered} \tag{36}$$

Proposition 2. *Let the real coefficients p_1 and p_2 be positive, and let the fuzzy-valued function $\tilde{f} : [0,T] \to J$ be continuous in the Hausdorff metric. Let the roots of the characteristic equation $\lambda^2 + p_1\lambda + p_2 = 0$ be real, distinct, and negative, equal to $-\lambda_1$ and $-\lambda_2$, where $\lambda_1 > \lambda_2 > 0$. Then there exists a unique $(1,1)$-ultraweak solution of problem (35), (36) of the form*

$$\tilde{y}(t) = \tilde{a}\gamma_{\tilde{a}}(t) + \tilde{b}\gamma_{\tilde{b}}(t) + \int\limits_0^t K(t-s)\tilde{f}(s)ds, \tag{37}$$

where the functions $\gamma_{\tilde{a}}(t)$ and $\gamma_{\tilde{b}}(t)$ are given by

$$\gamma_{\tilde{a}}(t) = \frac{1}{\lambda_1 - \lambda_2}(\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}), \quad \gamma_{\tilde{b}}(t) = \frac{1}{\lambda_1 - \lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t}), \tag{38}$$

and

$$K(t-s) = \frac{1}{\lambda_1 - \lambda_2}(e^{-\lambda_2(t-s)} - e^{-\lambda_1(t-s)}) \tag{39}$$

is the Cauchy function.

Indeed, under the hypotheses of Proposition 2, the α-indices of an $(1,1)$-ultraweak solution of problem (35), (36) satisfy the equations

$$(y_\alpha^\pm)''(t) + p_1(y_\alpha^\pm)'(t) + p_2 y_\alpha^\pm(t) = f_\alpha^\pm(t) \quad (t \in (0,T]) \tag{40}$$

with the initial conditions

$$y_\alpha^\pm(0) = a_\alpha^\pm, \quad (y_\alpha^\pm)'(0) = b_\alpha^\pm. \tag{41}$$

The solution of problem (40), (41) for all $\alpha \in [0,1]$ can be represented as the sum $y_\alpha^\pm(t) = x_\alpha^\pm(t) + z_\alpha^\pm(t)$, where $x_\alpha^\pm(t)$ is the solution of the corresponding homogeneous differential equation $(x_\alpha^\pm)''(t) + p_1(x_\alpha^\pm)'(t) + p_2 x_\alpha^\pm(t) = 0$ with the initial conditions $x_\alpha^\pm(0) = a_\alpha^\pm$ and $(x_\alpha^\pm)'(0) = b_\alpha^\pm$, and $z_\alpha^\pm(t)$ is the solution of the inhomogeneous differential equation $(z_\alpha^\pm)''(t) + p_1(z_\alpha^\pm)'(t) + p_2 z_\alpha^\pm(t) = f_\alpha^\pm(t)$ with the zero initial conditions $z_\alpha^\pm(0) = 0$ and $(z_\alpha^\pm)'(0) = 0$. Obviously, $x_\alpha^\pm(t) = a_\alpha^\pm\gamma_{\tilde{a}}(t) + b_\alpha^\pm\gamma_{\tilde{b}}(t)$, and $z_\alpha^\pm(t)$ can be expressed via the Cauchy function [30, Ch. 3] as

$$z_\alpha^\pm(t) = \int\limits_0^t K(t-s)f_\alpha^\pm(s)ds.$$

By definition, the Cauchy function $K(t-s)$ here is the solution of the real problem $x''(t)+p_1x'(t)+p_2x(t)=0$ with $x(s)=0$ and $x'(s)=1$. This function is given by equality (39). Then we have the formula

$$y_\alpha^\pm(t) = a_\alpha^\pm \gamma_{\tilde a}(t) + b_\alpha^\pm \gamma_{\tilde b}(t) + \int\limits_0^t K(t-s) f_\alpha^\pm(s)ds. \tag{42}$$

Note that the real functions $\gamma_{\tilde a}(t)$ and $\gamma_{\tilde b}(t)$ (28) in the representation (42) are continuous and positive for $t \geqslant 0$. Furthermore, the Cauchy function (39) is continuous and nonnegative for $t \geqslant s$. Therefore, the functions $y_\alpha^\pm(t)$ (42) satisfy conditions 1–3 on the α-indices of fuzzy numbers (Section 2) for all $t \in [0,T]$. Hence, they define the fuzzy-valued function (37). According to the hypotheses of Proposition 2, based on formula (42) and the definition of the Hausdorff metric, the fuzzy-valued function (37) is continuous. And we conclude that (37) is a $(1,1)$-ultraweak solution of problem (35), (36).

The uniqueness of the $(1,1)$-ultraweak solution of problem (35), (36) follows from the uniqueness of the solutions of problems (40), (41) for all $\alpha \in [0,1]$.

Suppose that only the average characteristics are known for the inhomogeneity $\tilde f(t)$ of equation (36) and the initial conditions $\tilde a$ and $\tilde b$ in (35). The following result is true.

Proposition 3. *Under the hypotheses of Proposition 2, the average characteristics* $m_{\tilde y}(t)$ *and* $r_{\tilde y}(t)$ *of a fuzzy-valued solution of problem* (35), (36) *are expressed through the average characteristics* $m_{\tilde a}$, $m_{\tilde b}$, $m_{\tilde f}$ *and* $r_{\tilde a}$, $r_{\tilde b}$, $r_{\tilde f}$, *respectively, by the formulas*

$$m_{\tilde y}(t) = m_{\tilde a}\gamma_{\tilde a}(t) + m_{\tilde b}\gamma_{\tilde b}(t) + \int\limits_0^t K(t-s) m_{\tilde f}(s)ds, \tag{43}$$

$$r_{\tilde y}(t) = r_{\tilde a}\gamma_{\tilde a}(t) + r_{\tilde b}\gamma_{\tilde b}(t) + \int\limits_0^t K(t-s) r_{\tilde f}(s)ds, \tag{44}$$

where $\gamma_{\tilde a}(t)$ *and* $\gamma_{\tilde b}(t)$ *are given by* (38), *and* $K(t-s)$ *is given by* (39).

Indeed, let us take the mean $m_{\tilde y}(t)$. Due to (40), (41), similar to Theorem 3, the mean $m_{\tilde y}(t)$ is the solution of the equation

$$m_{\tilde y}''(t) + p_1 m_{\tilde y}'(t) + p_2 m_{\tilde y}(t) = m_{\tilde f}(t) \quad (t \in (0,T]) \tag{45}$$

with the initial conditions

$$m_{\tilde y}(0) = m_{\tilde a} \equiv \frac{1}{2}\int\limits_0^1 (a_\alpha^+ + a_\alpha^-)d\alpha, \quad m_{\tilde y}'(0) = m_{\tilde b} \equiv \frac{1}{2}\int\limits_0^1 (b_\alpha^+ + b_\alpha^-)d\alpha. \tag{46}$$

By analogy with the considerations of Proposition 2, we arrive at formula (43).

According to Proposition 1, the half-range $r_{\tilde y}(t)$ satisfies the equation

$$r_{\tilde y}''(t) + p_1 r_{\tilde y}'(t) + p_2 r_{\tilde y}(t) = r_{\tilde f}(t) \quad (t \in (0,T]) \tag{47}$$

with the initial conditions

$$r_{\tilde y}(0) = r_{\tilde a} \equiv \frac{1}{2}\int\limits_0^1 (a_\alpha^+ - a_\alpha^-)d\alpha, \quad r_{\tilde y}'(0) = r_{\tilde b} \equiv \frac{1}{2}\int\limits_0^1 (b_\alpha^+ - b_\alpha^-)d\alpha. \tag{48}$$

And formula (44) is immediate.

Note that by definition, the function $r_{\tilde{y}}(t)$ is nonnegative. In fact, this follows from formula (44) due to the nonnegativity of, first, the numbers $r_{\tilde{a}}$ and $r_{\tilde{b}}$, second, the functions $\gamma_{\tilde{a}}(t)$ and $\gamma_{\tilde{b}}(t) \forall\, t \geqslant 0$, and, third, the functions $r_{\tilde{f}}(s) \forall\, s \geqslant 0$ and $K(t-s) \forall\, t \geqslant s \geqslant 0$.

Consider the case of an infinite interval $[0, \infty)$ of the real axis.

Proposition 4. *Under the hypotheses of Proposition 2, assume also that the right-hand side of equation* (36), $\tilde{f} : [0, \infty) \to J$, *is continuous and bounded in the Hausdorff metric. Then the means* (43) *and* (44) *are bounded for* $t \in [0, \infty)$.

Indeed, let us estimate, e.g., the absolute value $|m_{\tilde{y}}(t)|$ based on formula (33). Taking into account (28), the first term on the right-hand side of (43) can be estimated as $\frac{1}{\lambda_1-\lambda_2}[|m_{\tilde{a}}|(\lambda_1-\lambda_2)+2|m_{\tilde{b}}|]$. To estimate the second term, we make the change of variables $t-s=\sigma$ in the integral; then $\int_0^t K(t-s)m_{\tilde{f}}(s)ds = \int_0^t K(\sigma)m_{\tilde{f}}(t-\sigma)d\sigma$. By assumption and Remark 2, there exists a constant $c_{\tilde{f}}$ such that $|m_{\tilde{f}}(s)| \leqslant c_{\tilde{f}} \forall\, s \geqslant 0$. Then, due to formula (39),

$$\left|\int\limits_0^t K(\sigma)m_{\tilde{f}}(t-\sigma)d\sigma\right| \leqslant \int\limits_0^t |K(\sigma)||m_{\tilde{f}}(t-\sigma)|d\sigma \leqslant c_{\tilde{f}}\int\limits_0^t |K(\sigma)|d\sigma \leqslant \frac{c_{\tilde{f}}}{\lambda_1\lambda_2}.$$

And the desired statement for $m_{\tilde{y}}(t)$ follows immediately. The considerations for $r_{\tilde{y}}(t)$ are analogous.

Remark 5. Under the hypotheses of Proposition 4, the solution (43) of equation (45), as well as the solution (44) of equation (47), are Lyapunov stable (with respect to the initial conditions) according to [32, Ch. II]. Moreover, they are asymptotically stable, and stability holds under persistent excitation [32, Ch. II].

Physically, stability is conditioned by the fact that the series resonant circuit with the positive coefficients p_1 and p_2 induces a passive (causal) dynamic system [31].

It should be emphasized that Propositions 2–4 and Remark 5 actually deal with the case of a (1,1)-generalized ultraweak solution of problem (35), (36).

Let us turn to the case of generalized ultraweak solutions of other types in problem (35), (36). For the sake of definiteness, we take a (1,2)-ultraweak twice-generalized solution of problem (35), (36). In this case, the mean $m_{\tilde{y}}(t)$ is a solution of problem (32), (33) (Theorem 3). Consequently, it has the form (43). The other types of ultraweak generalized solutions are considered by analogy.

Concerning the half-range $r_{\tilde{y}}(t)$, note that in the case of a (1,1)-ultraweak solution of problem (47), (48), the function (44) yields a bounded solution as $t \to +\infty$, which is Lyapunov stable.

For example, in the case of a (1,2)-ultraweak solution of problem (35), (36), the equation for $r_{\tilde{y}}(t)$ has the form $r''_{\tilde{y}}(t) - p_1 r'_{\tilde{y}}(t) - p_2 r_{\tilde{y}}(t) = r_{\tilde{f}}(t)$ (Proposition 1). For the corresponding characteristic equation $\lambda^2 - p_1\lambda - p_2 = 0$, by Vieta's theorem, the roots are real and of opposite signs. This means instability (i.e., physical unrealizability) of the corresponding solution. Other cases are analyzed by analogy. The situation would differ if the original system were replaced, e.g., by a system with feedback or a signal amplification system.

Consider now an important special case of equation (36), i.e., the harmonic oscillator equation (no resistance in the oscillatory circuit, see Fig. 2). We have the following fuzzy differential equation:

$$\tilde{y}''(t) + \omega^2\tilde{y}(t) = \tilde{f}(t). \tag{49}$$

Here, $\omega > 0$ is a constant determining the oscillation frequency of the solutions of the homogeneous equation

$$y''(t) + \omega^2 y(t) = 0. \tag{50}$$

As is known, the general solution of the homogeneous equation (50) has the form $c_1 \cos \omega t + c_2 \sin \omega t$, where c_1, c_2 are arbitrary constants. In this case, the Cauchy function is given by $K(t-s) = \frac{1}{\omega} \sin \omega(t-s)$. Then, similar to Proposition 3, the average characteristics $m_{\tilde{y}}(t)$ and $r_{\tilde{y}}(t)$ of problem (49), (35) have the form

$$m_{\tilde{y}}(t) = m_{\tilde{a}} \cos \omega t + m_{\tilde{b}} \sin \omega t + \frac{1}{\omega} \int_0^t \sin \omega(t-s) m_{\tilde{f}}(s)\, ds,$$

$$r_{\tilde{y}}(t) = r_{\tilde{a}} \cos \omega t + r_{\tilde{b}} \sin \omega t + \frac{1}{\omega} \int_0^t \sin \omega(t-s) r_{\tilde{f}}(s) ds$$

(with the same designations as in Proposition 3).

6. CONCLUSIONS

In the theory of fuzzy numbers, a widespread method is to estimate fuzzy numbers by real numbers (defuzzification) or by real intervals. In this paper, we have solved the following problem: from an estimate of a fuzzy signal at the input of a linear dynamic system, find the corresponding estimate of the fuzzy signal at its output.

The main estimation parameters have been the fuzzy mean and the fuzzy half-range. We emphasize a fundamental fact established above, i.e., the universal role of the fuzzy mean $m_{\tilde{y}}$ (9) (Theorem 3), which is the same for various types of generalized derivatives of the differential problem describing a given dynamic system. This holds despite the various equations for the α-indices for various differentiability types.

The results of this work can be further developed in the following directions: consideration of the relationship between the modal values of fuzzy signals at the input and output of a linear dynamic system, and elaboration of an interval approach to estimating input and output fuzzy signals.

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LINEAR SYSTEMS

The Detectability of Time-Varying Differential-Algebraic Equations with Scalar Output

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Abstract—This paper considers time-varying linear and nonlinear systems of differential-algebraic equations with a scalar observable output. In the linear case, sufficient detectability conditions are established, and a state observer is designed. For systems with a controlled input of a special form, a stabilization procedure using dynamic output feedback is proposed. Based on a linear approximation, detectability conditions are derived for a nonlinear system.

Keywords: differential-algebraic equations, detectability, stabilizability

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1. INTRODUCTION

Consider a system of ordinary differential equations of the form

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) + U(t)u(t) = 0, \tag{1.1}$$

$$y(t) = c(t)^\top x(t), \quad t \in T = [0, +\infty), \tag{1.2}$$

where $A(t)$ and $B(t)$ are known matrices of dimensions $(n \times n)$; $U(t)$ is a given matrix of dimensions $(n \times l)$; $x(t)$ and $c(t)$ are the desired and given n-dimensional vector functions, respectively; $u(t)$ denotes the l-dimensional controlled input, and $y(t)$ is the scalar observable output; finally, $^\top$ indicates transpose. By assumption, the elements of the matrices $A(t)$, $B(t)$, and $U(t)$, as well as the components of the vectors $u(t)$ and $c(t)$, are continuously differentiable on T sufficiently many times. The case

$$\det A(t) \equiv 0, \quad t \in T,$$

is allowed.

In particular, such systems are called differential-algebraic equations (DAEs) or descriptor systems.

The classes of DAEs studied in this paper include applied problems from various fields of science and technology: the theory of electrical circuits [1], biology [2], mechanics [3], chemistry [4], economics [5], information security [6], etc.

Since the 1990s, there has been a significantly growing interest of researchers in the design of observers for differential-algebraic equations, in both linear [1, 3, 4, 7, 8] and nonlinear [9–20] settings.

Most works on linear systems analyze the time-invariant case. There exist relatively few results on nonlinear systems. Moreover, as a rule, a quasilinear formulation of the problem is considered under assumptions ensuring a special structure of the system or other rather rigid constraints. Here are some examples of such requirements: the linear part does not depend on time [9–11]; the system has a linear dominant [12, 13] or a triangular form [14]; the nonlinearity of the system satisfies a strict global Lipschitz condition [10, 12, 13].

A literature review on the design of state observers for nonlinear descriptor systems was provided in [11], including a discussion of the types of nonlinearities. The local stability of an observer for a semi-explicit system with a simple structure was considered in [9]. DAEs with rectangular descriptors were studied in [6, 15]. Observer design using linearization and methods for simplifying the internal structure of the system was described in [16]. The most common approach is to establish the existence of observers via the solvability of certain linear matrix inequalities (LMIs) [17–20].

Currently, the issue of obtaining detectability conditions and constructing state observers for time-varying DAEs in the general case remains open. This paper is devoted to linear and nonlinear systems satisfying fairly general requirements on the internal structure. In Sections 2–4, we deal with linear systems. The structural form from Section 2 is employed to derive detectability conditions and design a state observer in Section 3. This observer is used in Section 4 to propose a stabilization procedure for systems with a special controlled input. Section 5 considers the internal structure of nonlinear DAEs of the form

$$f\left(t, x(t), \frac{d}{dt}x(t)\right) = 0,$$

where $\det \frac{\partial f(t,\alpha,\beta)}{\partial \beta} = 0$ in the domain of definition. Based on these results, in Section 6, we establish sufficient detectability conditions under the linear approximation and also the form of the state observer in the nonlinear case.

2. AUXILIARY RESULTS ON THE STRUCTURE OF LINEAR DAES

For some $r : 0 \le r \le n$, let us associate with the matrix coefficients of system (1.1) the following objects: the matrices

$$\mathcal{B}_r(t) = \mathrm{col}\left(B(t),\ B'(t),\ \ldots,\ B^{(r)}(t)\right), \tag{2.1}$$

$$\mathcal{A}_r(t) = \mathrm{col}\left(C_0^0 A(t),\ C_1^0 A'(t) + C_1^1 B(t),\ \ldots,\ C_r^0 A^{(r)}(t) + C_r^1 B^{(r-1)}(t)\right) \tag{2.2}$$

of dimensions $(n(r+1) \times n)$, the matrix

$$\Lambda_r(t) = \begin{pmatrix} O & O & \ldots & O \\ C_1^1 A(t) & O & \ldots & O \\ C_2^1 A'(t) + C_2^2 B(t) & C_2^2 A(t) & \ldots & O \\ \vdots & \vdots & \ldots & \vdots \\ C_r^1 A^{(r-1)}(t) + C_r^2 B^{(r-2)}(t) & C_r^2 A^{(r-2)}(t) + C_r^3 B^{(r-3)}(t) & \ldots & C_r^r A(t) \end{pmatrix} \tag{2.3}$$

of dimensions $n(r+1) \times nr$, and the matrix

$$\mathcal{D}_r(t) = \begin{pmatrix} \mathcal{B}_r(t) & \mathcal{A}_r(t) & \Lambda_r(t) \end{pmatrix} \tag{2.4}$$

of dimensions $n(r+1) \times n(r+2)$. Throughout this paper, $C_i^j = \frac{i!}{j!(i-j)!}$ are the binomial coefficients, and

$$\phi'(t) = \frac{d}{dt}\phi(t), \quad \phi^{(j)}(t) = \left(\frac{d}{dt}\right)^j \phi(t).$$

We make the following assumptions:

A1) $\operatorname{rank}\Lambda_r(t) = \lambda = \text{const} \ \ \forall t \in T$.

A2) For all $t \in T$, the matrix $\mathcal{D}_r(t)$ has an invertible submatrix $\mathcal{M}_r(t)$ of order $n(r+1)$ containing λ columns of the matrix $\Lambda_r(t)$ and all n columns of the matrix $\mathcal{A}_r(t)$.

Let Q denote a column permutation matrix such that

$$\mathcal{B}_r(t)Q = \begin{pmatrix} \bar{\mathcal{B}}_1(t) & \bar{\mathcal{B}}_2(t) \end{pmatrix}, \tag{2.5}$$

where the block $\bar{\mathcal{B}}_2(t)$ ($\bar{\mathcal{B}}_1(t)$) is included (not included, respectively) in the matrix $\mathcal{M}_r(t)$. Let d denote the number of columns in $\bar{\mathcal{B}}_2(t)$. Accordingly,

$$B(t)Q = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}, \ \ A(t)Q = \begin{pmatrix} A_1(t) & A_2(t) \end{pmatrix}, \tag{2.6}$$

where the blocks $B_1(t)$, $A_1(t)$ and $B_2(t)$, $A_2(t)$ have pairwise dimensions $n \times (n-d)$ and $n \times d$, respectively.

Under Assumptions A1 and A2, there exists an operator of the form

$$\mathcal{R} = \sum_{j=0}^{r} R_j(t) \left(\frac{d}{dt}\right)^r, \tag{2.7}$$

with the coefficients given by

$$\begin{pmatrix} R_0(t) & R_1(t) & \dots & R_r(t) \end{pmatrix} = \begin{pmatrix} E_n & O & \dots & O \end{pmatrix} \mathcal{M}_r^{-1}(t), \tag{2.8}$$

where E_n stands for an identity matrix of order n; for details, see [21].

Operator (2.7) transforms system (1.1) to

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1(t) & E_d \\ J_2(t) & O \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \mathcal{R}[U(t)u(t)] = 0, \ \ t \in T, \tag{2.9}$$

where $x_1(t) \in \mathbf{R}^{n-d}$, $x_2(t) \in \mathbf{R}^d$,

$$\operatorname{col}(x_1(t), x_2(t)) = Q^{-1}x(t), \tag{2.10}$$

$$\operatorname{col}(J_1(t),\ J_2(t)) = \sum_{j=0}^{r} R_j(t) B_1^{(j)}(t), \tag{2.11}$$

$$\mathcal{R}[U(t)u(t)] = \sum_{j=0}^{r} R_j(t)\,(U(t)u(t))^{(j)} = \begin{pmatrix} R_0(t) & R_1(t) & \dots & R_r(t) \end{pmatrix} \mathcal{U}_r(t)\bar{u}_r(t), \tag{2.12}$$

$$\bar{u}_r(t) = \operatorname{col}\left(u(t), u'(t), \dots, u^{(r)}(t)\right), \tag{2.13}$$

$$\mathcal{U}_r(t) = \begin{pmatrix} C_0^0 U(t) & O & O & \dots & O \\ C_1^0 U'(t) & C_1^1 U(t) & O & \dots & O \\ C_2^0 U^{(2)}(t) & C_2^1 U'(t) & C_2^2 U(t) \dots & O & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_r^0 U^{(r)}(t) & C_r^1 U^{(r-1)}(t) & C_r^2 U^{(r-2)}(t) & \dots & C_r^r U(t) \end{pmatrix}. \tag{2.14}$$

Definition 1. By a solution of system (1.1) we mean an n-dimensional vector function $x(t) \in \mathbf{C}^1(T)$ turning (1.1) into an identity upon substitution.

Obviously, for sufficiently smooth input data, any solution of system (1.1) is a solution of system (2.9). To prove the converse, additional requirements should be imposed.

Let us supplement A1 and A2 with the following assumptions:

A3) $\operatorname{rank}\Lambda_{r+1}(t) = \operatorname{rank}\Lambda_r(t) + n \ \ \forall t \in T$.

A4) For all $t \in T$, the matrix $\mathcal{D}_{r+1}(t)$ has an invertible submatrix $\mathcal{M}_{r+1}(t)$ of order $n(r+2)$ containing all columns of the matrix $\mathcal{M}_r(t)$ and also n columns of the matrix $\Lambda_{r+1}(t)$.

Then the operator $\mathcal{R}$ has the left inverse [21]

$$\mathcal{L} = L_0(t) + L_1(t)\frac{d}{dt},$$

where $L_0(t)$ and $L_1(t)$ are matrices of dimensions $(n \times n)$. This means that the operator $\mathcal{L}$, together with a permutation of rows of the desired vector (2.10), transforms system (2.9) to (1.1). In this case, any solution of system (2.9) is a solution of system (1.1).

Thus, Assumptions A1–A4 ensure the equivalence of systems (1.1) and (2.9) in the sense of solutions (up to permutations of components). Obviously, to extract a unique solution of system (2.9) or (1.1), it is necessary to specify initial data in the form

$$x_1(0) = x_0, \tag{2.15}$$

where $x_0 \in \mathbf{R}^{n-d}$.

Under certain conditions, the operator $\mathcal{R}$ (2.7) may have a right inverse.

Lemma 1. *Let:*

1) $A(t)$, $B(t) \in \mathbf{C}^{r+1}(T)$.
2) *Assumptions A1 and A2 be valid.*
3) *For all* $t \in T$,

$$\begin{pmatrix} R_1(t) & R_2(t) \dots R_r(t) \end{pmatrix} \begin{pmatrix} C_1^1 B_1(t) & O & \dots & O \\ C_2^1 B_1'(t) & C_2^2 B_1(t) & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^1 B_1^{(r-1)}(t) & C_r^2 B_1^{(r-2)}(t) & \dots & C_r^r B_1(t) \end{pmatrix} = O. \tag{2.16}$$

Then the operator (2.7) *has the right inverse*

$$\mathcal{P} = P_0(t) + P_1(t)\frac{d}{dt}, \tag{2.17}$$

where

$$P_0(t) = \begin{pmatrix} B_2(t) & A_1(t) \end{pmatrix}, \quad P_1(t) = \begin{pmatrix} A_2(t) & O \end{pmatrix},$$

and the matrices $B_1(t)$, $B_2(t)$, $A_1(t)$, *and* $A_2(t)$ *are given by* (2.6).

The proof is postponed to the Appendix.

3. STATE OBSERVER DESIGN FOR THE LINEAR SYSTEM

Let Assumptions A1–A4 be valid. In the output representation (1.2), we make the change of variables

$$x(t) = Q\operatorname{col}\left(x_1(t), x_2(t)\right),$$

where Q is the permutation matrix from (2.5). Then

$$y(t) = c_1^\top(t)x_1(t) + c_2^\top x_2(t), \tag{3.1}$$

$c_1(t) \in \mathbf{R}^{n-d}$, and $c_2(t) \in \mathbf{R}^d \ \forall t \in T$.

Definition 2. A system of the form

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} \hat{x}_1'(t) \\ \hat{x}_2'(t) \end{pmatrix} + \begin{pmatrix} \hat{J}_1(t) & E_d \\ \hat{J}_2(t) & O \end{pmatrix} \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} + \hat{U}(t)\bar{u}_r(t)$$
$$+ g(t)\left(y(t) - c_1^\top(t)\hat{x}_1(t) - c_2^\top \hat{x}_2(t)\right) = 0,$$
$$\hat{y}(t) = H(t)\mathrm{col}\,(\hat{x}_1(t),\, \hat{x}_2(t)) + \nu(t)y(t) + F(t)\bar{u}_r(t), \quad t \in T,$$

where the matrices $\hat{J}_1(t)$, $\hat{J}_2(t)$, $\hat{U}(t)$, $H(t)$, and $F(t)$ and the vectors $g(t)$ and $\nu(t)$ have the corresponding dimensions, is called a *state observer* for (2.9), (3.1), if

$$\lim_{t\to\infty}\left(\hat{y}(t) - \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right) = 0.$$

We will seek a state observer for system (2.9), (3.1) in the form

$$\hat{x}_2(t) + J_1(t)\hat{x}_1(t) + \mathcal{V}_{r,1}(t)\bar{u}_r(t) = 0, \tag{3.2}$$
$$\hat{x}_1'(t) + J_2(t)\hat{x}_1(t) + \mathcal{V}_{r,2}(t)\bar{u}_r(t) + g(t)\left(y(t) - c_1^\top \hat{x}_1(t) - c_2^\top \hat{x}_2(t)\right) = 0, \tag{3.3}$$
$$\hat{y}_1(t) = \hat{x}_1(t), \quad \hat{y}_2(t) = \hat{x}_2(t), \tag{3.4}$$

where $g(t) : T \to \mathbf{R}^{n-d}$;

$$\mathrm{col}\,(\mathcal{V}_{r,1}(t),\, \mathcal{V}_{r,2}(t)) = \begin{pmatrix} R_0(t) & R_1(t) & \dots & R_r(t) \end{pmatrix} \mathcal{U}_r(t),$$

with $\mathcal{U}_r(t)$ defined by (2.14) and the matrices $\mathcal{V}_{r,1}(t)$ and $\mathcal{V}_{r,2}(t)$ having dimensions $d \times l(r+1)$ and $(n-d) \times l(r+1)$, respectively; finally, $J_1(t)$ and $J_2(t)$ are the matrix coefficients from (2.9) that are calculated using formulas (2.6), (2.8), and (2.11).

Then the residuals

$$e_1(t) = \hat{x}_1(t) - x_1(t), \quad e_2(t) = \hat{x}_2(t) - x_2(t) \tag{3.5}$$

satisfy the system

$$e_2(t) + J_1(t)e_1(t) = 0, \tag{3.6}$$
$$e_1'(t) + J_2(t)e_1(t) - g(t)\left(c_1^\top(t)e_1(t) + c_2^\top(t)e_2(t)\right) = 0. \tag{3.7}$$

From (3.6) it follows that

$$e_2(t) = -J_1(t)e_1(t). \tag{3.8}$$

Substituting (3.8) into (3.7) yields the equation for $e_1(t)$:

$$e_1'(t) + \left(J_2(t) - g(t)\left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)\right)e_1(t) = 0. \tag{3.9}$$

Suppose the existence of a vector function $g(t)$ such that $\lim\limits_{t\to\infty} e_1(t) = 0$ for any solution of system (3.9). In this case, the nonsingular system

$$x_1'(t) + J_2(t)x_1(t) = 0, \tag{3.10}$$
$$y(t) = \left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)x_1(t) \tag{3.11}$$

is said to be *detectable* [22, p. 313], and there exists its observer of the form

$$\hat{x}_1'(t) + J_2(t)\hat{x}_1(t) + g(t)\left(y(t) - \left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)\hat{x}_1(t)\right) = 0, \tag{3.12}$$

$$\hat{y}(t) = \hat{x}_1(t). \tag{3.13}$$

Under an additional assumption that the matrix $J_1(t)$ has a nonpositive characteristic (Lyapunov) exponent, i.e.,

$$\chi[J_1(t)] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|J_1(t)\| \le 0, \tag{3.14}$$

equality (3.8) implies the limit relation $\lim_{t\to\infty} e_2(t) = 0$. Thus, system (3.2)–(3.4) is a state observer for system (2.9), (3.1) as well.

Under Assumptions A1–A4, systems (1.1) and (2.9) have the same solutions. This means that system (3.2)–(3.4) is a state observer also for system (1.1), (1.2). Thus, the following result is true.

Proposition 1. *Under Assumptions A1–A4 and* (3.14), *there exists a state observer of the form* (3.2)–(3.4) *for the DAE* (1.1), (1.2) *iff system* (3.10), (3.11) *has an observer of the form* (3.12), (3.13).

Definition 3. The observation system (1.1), (1.2) is said to be detectable if there exists a vector function $g(t)$ such that system (3.9) is asymptotically stable.

Proposition 2. *Under Assumptions A1–A4 and* (3.14), *system* (1.1), (1.2) *is detectable iff system* (3.10), (3.11) *is detectable.*

To proceed, we formulate a detectability condition for system (3.10), (3.11) [22, p. 314]. To this end, it is necessary to define the observability matrix

$$\mathcal{S}(t) = \operatorname{col}\left(s_0(t),\, s_1(t),\, \dots, s_{n-d-1}(t)\right), \tag{3.15}$$

where $s_0(t) = c_1^\top(t) - c_2^\top(t)J_1(t)$ and $s_i(t) = s_{i-1}'(t) - s_{i-1}(t)J_2(t)$, $i = \overline{1, n-d}$.

For an invertible matrix $\mathcal{S}(t)$, we introduce the functions

$$(\omega_1(t),\, \omega_2(t),\, \dots,\, \omega_{n-d}(t)) = s_{n-d}(t)\mathcal{S}^{-1}(t). \tag{3.16}$$

Lemma 2. *Let:*
1) $J_2(t) \in \mathbf{C}^{n-d-1}(T)$ *and* $c_1^\top(t) - c_2^\top(t)J_1(t) \in \mathbf{C}^{n-d}(T)$.
2) $\mathcal{S}(t)$ *be a Lyapunov matrix.*
3) *Each function* $\omega_i(t) \in \mathbf{C}^{i-1}(T)$ $(i = \overline{1, n-d})$ *be bounded on* T *together with its derivatives up to order* $i-1$ *inclusive.*

Then system (3.10), (3.11) *is detectable.*

Remark 1. According to the proof of the detectability theorem for a linear system [22, p. 314], the vector function $g(t)$ in (3.12) can be chosen so that it is bounded on T, and system (3.12) is reducible in the Lyapunov sense.

Based on Propositions 1 and 2 and Lemma 2, we obtain sufficient conditions for the detectability of the DAE (1.1), (1.2).

Theorem 1. *Let:*
1) $A(t), B(t) \in \mathbf{C}^{n+r-d}(T)$ $(n-d \ge 1)$, $c(t) \in \mathbf{C}^{n-d}(T)$, *and* $U(t), u(t) \in \mathbf{C}^r(T)$.
2) *Assumptions A1–A4 and* (3.14) *be valid.*
3) *Conditions 2 and 3 of Lemma 2 hold.*

Then system (1.1), (1.2) *is detectable and has an observer of the form* (3.2)–(3.4).

4. STABILIZABILITY OF THE LINEAR SYSTEM USING A STATE OBSERVER

Within the hypotheses of Lemma 1, consider the DAE

$$A(t)x'(t) + B(t)x(t) + P_0(t)b(t)u(t) + P_1(t)\left(b(t)u(t)\right)' = 0, \tag{4.1}$$

where $P_0(t)$ and $P_1(t)$ are the coefficients of the operator (2.17), which is right inverse to the operator (2.7); $b(t)$ is a given n-dimensional vector function; finally, $u(t)$ is a scalar controlled input.

Under definite conditions, the system

$$A(t)x'(t) + B(t)x(t) + \beta_1(t)u(t) + \beta_2(t)u(t)' = 0 \tag{4.2}$$

with sufficiently smooth n-dimensional vector functions $\beta_1(t)$ and $\beta_2(t)$ can be represented in the form (4.1).

Lemma 3. *Let:*

1) $A(t), B(t) \in \mathbf{C}^{r+1}(T)$.
2) *Assumptions 2 and 3 of Lemma 1 be valid.*
3) $\beta_1(t) \in \mathbf{C}^{r}(T)$.
4) *For all* $t \in T$,

$$\beta_2(t) = P_1(t)\mathcal{R}\left[\beta_1(t)\right]. \tag{4.3}$$

Then system (4.2) *can be written in the form* (4.1), *where* $b(t) \in \mathbf{C}^1(T)$ *is given by*

$$b(t) = \mathcal{R}\left[\beta_1(t)\right], \tag{4.4}$$

and the operator $\mathcal{R}$ *is defined in* (2.7) *and* (2.8).

The proof is provided in the Appendix.

Let us introduce the observable output (1.2) for system (4.1) and design a stabilizing control for (4.1).

By applying the operator (2.7) to both sides of equation (4.1), we obtain the system

$$x_2(t) + J_1(t)x_1(t) + b_1(t)u(t) = 0, \tag{4.5}$$
$$x_1'(t) + J_2(t)x_1(t) + b_2(t)u(t) = 0 \tag{4.6}$$

with the output (3.1), where $\operatorname{col}\left(b_1(t), b_2(t)\right) = b(t)$, $b_1(t) \in \mathbf{R}^d$, and $b_2(t) \in \mathbf{R}^{n-d}$.

A state observer for (4.5), (4.6) will be constructed in the form (3.2)–(3.4), where

$$\mathcal{V}_{r,1}(t)\bar{u}_r(t) = b_1(t)u(t), \quad \mathcal{V}_{r,2}(t)\bar{u}_r(t) = b_2(t)u(t).$$

A stabilizing control for the DAE (4.5), (4.6) will be found in the class of dynamic output feedback

$$u(t) = k^\top(t)\hat{x}_1(t); \tag{4.7}$$

consequently,

$$x_2(t) + J_1(t)x_1(t) + b_1(t)k^\top(t)\hat{x}_1(t) = 0,$$
$$x_1'(t) + J_2(t)x_1(t) + b_2(t)k^\top(t)\hat{x}_1(t) = 0.$$

By introducing the residuals (3.5), we arrive at the system

$$x_2(t) = -\left(J_1(t) + b_1(t)k^\top(t)\right)x_1(t) - b_1(t)k^\top(t)e_1(t), \tag{4.8}$$

$$x_1'(t) + \left(J_2(t) + b_2(t)k^\top(t)\right)x_1(t) + b_2(t)k^\top(t)e_1(t) = 0. \tag{4.9}$$

$$e_2(t) = -J_1(t)e_1(t), \tag{4.10}$$

$$e_1'(t) + \left(J_2(t) - g(t)\left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)\right)e_1(t) = 0. \tag{4.11}$$

The control (4.7) will be stabilizing if system (4.8)–(4.11) is asymptotically stable. The latter condition particularly implies the stabilizability of system (4.6) and the detectability of system (3.10), (3.11).

Definition 4. System (4.6) is said to be *stabilizable* if there exists a feedback control

$$u(t) = k^\top(t)x_1(t) \tag{4.12}$$

such that the closed-loop system

$$x_1'(t) + \left(J_2(t) + b_2(t)k^\top(t)\right)x_1(t) = 0 \tag{4.13}$$

is asymptotically stable.

To formulate stabilizability conditions for system (4.6), we introduce the vectors

$$\varphi_0(t) = b_2(t), \quad \varphi_i = J_2(t)\varphi_{i-1}(t) - \varphi_{i-1}'(t), \quad i = \overline{1, n-d},$$

and the controllability matrix

$$\Phi(t) = \begin{pmatrix} \varphi_0(t) & \varphi_1(t) & \ldots & \varphi_{n-d-1}(t) \end{pmatrix}.$$

If $\Phi(t)$ is invertible, it is possible to define the function

$$\psi(t) = \Phi^{-1}(t)\varphi_{n-d}(t) = \operatorname{col}\left(\psi_1(t),\, \psi_2(t), \ldots,\, \psi_{n-d}(t)\right).$$

Lemma 4 [22, p. 302]. *Let:*
1) $J_2(t) \in \mathbf{C}^{n-d-1}(T)$ *and* $b_2(t) \in \mathbf{C}^{n-d}(T)$.
2) $\Phi(t)$ *be a Lyapunov matrix.*
3) *Each function* $\psi_i(t) \in \mathbf{C}^i(T)$ $(i = \overline{1, n-d})$ *be bounded on* T *together with its derivatives up to order* $i-1$ *inclusive.*

Then there exists a row

$$k^\top(t) = \left(k_1(t),\, \ldots,\, k_{n-d}(t)\right)K(t) \tag{4.14}$$

such that the closed-loop system (4.13) *is asymptotically stable. In* (4.14), $K(t)$ *is a Lyapunov matrix, and each function* $k_i(t) \in \mathbf{C}^{i-1}(T)$ *is bounded on* T *together with its derivatives up to order* $i-1$ *inclusive.*

Remark 2. According to Lemma 4, the vector function $k(t)$ in the feedback control (4.12) is bounded, hence

$$\chi[k(t)] = 0. \tag{4.15}$$

Remark 3. If $J_2(t) \in \mathbf{C}^{n-d}(T)$ and $b_2(t) \in \mathbf{C}^{n-d+1}(T)$ in system (4.6), then the stabilizing control (4.12) can be designed so that $k(t) \in \mathbf{C}^1(T)$ [22, pp. 166, 248, 299].

Theorem 2. *Let:*

1) $A(t), B(t) \in \mathbf{C}^{n+r-d}(T)$ $(n-d \geq 1)$, $c(t) \in \mathbf{C}^{n-d}(T)$, $b_2(t) \in \mathbf{C}^{n-d+1}(T)$, *and* $b_1(t) \in \mathbf{C}^1(T)$.
2) *Assumptions A1–A4 and* (3.14) *be valid.*
3) *Conditions 1 and 3 of Lemma 1, as well as conditions 2 and 3 of Lemma 2, hold.*
4) *Conditions 2 and 3 of Lemma 4 hold.*
5) $\chi\,[b_1(t)] \leq 0$.

Then system (4.1) *is stabilizable by the feedback control* (4.7).

The proof can be found in the Appendix.

5. NECESSARY INFORMATION ON THE STRUCTURE OF NONLINEAR DAES

In this section, we consider the nonlinear system with a scalar observable output of the form

$$f(t, x(t), x'(t)) = 0, \tag{5.1}$$

$$y(t) = c^\top(t)x(t) + \rho(t, x(t)) \ \ t \in T, \tag{5.2}$$

where the components of the function $f(t,\alpha,\beta) : \mathcal{W} \to \mathbf{R}^n$ are continuously differentiable with respect to each argument sufficiently many times in the domain $\mathcal{W} = \{(t,\alpha,\beta) \in \mathbf{R}^{2n+1} : t \in T, \|\alpha\| < K_0, \|\beta\| < K_1\}$; $\rho(t,\alpha) : \hat{\mathcal{W}} \to \mathbf{R}$, $\hat{\mathcal{W}} = \{(t,\alpha) \in \mathbf{R}^{n+1} : t \in T, \|\alpha\| < K_0\}$, and $c(t) : T \to \mathbf{R}^n$. By assumption,

$$\begin{gathered} f(t,0,0) = 0, \\ \rho(t,0) = 0 \ \ \forall t \in T, \\ \det\frac{\partial f(t,\alpha,\beta)}{\partial \beta} = 0 \ \ \forall (t,\alpha,\beta) \in \mathcal{W}. \end{gathered} \tag{5.3}$$

In the nonlinear case, a nonsingular system equivalent to (5.1) in the sense of solutions has to be sought as part of the components of an implicit function satisfying the so-called r-derivative array equations. They consist of system (5.1) and its r total derivatives with respect to t :

$$\mathcal{F}_r\left(t, \bar{x}_{r+1}\right) = \begin{pmatrix} f(t, x(t), x'(t)) \\ \dfrac{d}{dt} f(t, x(t), x'(t)) \\ \cdots \\ \left(\dfrac{d}{dt}\right)^r f(t, x(t), x'(t)) \end{pmatrix} = 0, \tag{5.4}$$

where

$$\bar{x}_{r+1} = \left(x,\, x',\, \ldots,\, x^{(r+1)}\right). \tag{5.5}$$

In (5.4), $t,\, x,\, x',\, \ldots,\, x^{(r+1)}$ should be treated as independent variables.

The situation is complicated by the fact that the classical theorem [23, p. 68] ensures the existence of an implicit function only in some neighborhood of the initial point, whereas detectability analysis requires the definition of this function at least for all $t \in T$. To satisfy this requirement, we employ a global version of the implicit function theorem [24].

Let us introduce the matrices

$$\bar{\mathcal{B}}_r\left(t, \bar{x}_{r+1}\right) = \left(\frac{\partial \mathcal{F}_r\left(t, \bar{x}_{r+1}\right)}{\partial x}\right), \quad \bar{\mathcal{A}}_r\left(t, \bar{x}_{r+1}\right) = \left(\frac{\partial \mathcal{F}_r\left(t, \bar{x}_{r+1}\right)}{\partial x'}\right)$$

of dimensions $(n(r+1)\times n)$, the matrix

$$\bar{\Lambda}_r\,(t,\bar{x}_{r+1}) = \left(\frac{\partial \mathcal{F}_r\,(t,\bar{x}_{r+1})}{\partial x''} \quad \ldots \quad \frac{\partial \mathcal{F}_r\,(t,\bar{x}_{r+1})}{\partial x^{(r+1)}} \right)$$

of dimensions $(n(r+1)\times nr)$-matrix, and the matrix

$$\bar{\mathcal{D}}_r\,(t,\bar{x}_{r+1}) = (\bar{\mathcal{B}}_r\,(t,\bar{x}_{r+1}) \quad \bar{\mathcal{A}}_r\,(t,\bar{x}_{r+1}) \quad \bar{\Lambda}_r\,(t,\bar{x}_{r+1}))$$

of dimensions $(n(r+1)\times n(r+2))$.

Condition (5.3) implies

$$\mathcal{F}_r(0,0) = 0.$$

We assume the following conditions:

B1) rank $\bar{\Lambda}_r\,(t,\bar{x}_{r+1}) = \lambda = \text{const}$ everywhere in the domain of definition.

B2) At the point $(t,\bar{x}_{r+1}) = 0$, the matrix $\bar{\mathcal{D}}_r\,(t,\bar{x}_{r+1})$ has an invertible submatrix $\bar{\mathcal{M}}_r\,(t,\bar{x}_{r+1})$ of order $n(r+1)$ containing λ columns of the matrix $\bar{\Lambda}_r\,(t,\bar{x}_{r+1})$ and all n columns of the matrix $\bar{\mathcal{A}}_r\,(t,\bar{x}_{r+1})$. Moreover,

$$\bar{\mathcal{M}}_r\,(t,\bar{x}_{r+1}) = \left(\frac{\partial \mathcal{F}_r}{\partial x_2} \;\; \frac{\partial \mathcal{F}_r}{\partial x'} \quad \frac{\partial \mathcal{F}_r}{\partial X_1} \right),$$

where

$$x = Q\text{col}\,(x_1,x_2)\,, \tag{5.6}$$

$$\text{col}\left(x'',\ldots,x^{(r+1)}\right) = Q_r\text{col}\,(X_1,X_2)\,, \tag{5.7}$$

Q and Q_r are the corresponding row permutation matrices, $x_1\in\mathcal{W}_1\subseteq\mathbf{R}^{n-d}$, $x_2\in\mathcal{W}_2\subseteq\mathbf{R}^{d}$, $X_1\in\mathcal{X}_1\subseteq\mathbf{R}^{\lambda}$, $X_2\in\mathcal{X}_2\subseteq\mathbf{R}^{d}$, $d = nr-\lambda$, Q is constructed by analogy with the linear case (see (2.5)), and $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{X}_1$, and $\mathcal{X}_2$ denote neighborhoods of zero in the corresponding spaces.

Under conditions B1 and B2 with some additional assumptions [25], an implicit function of the form

$$x_2 = f_1(t,x_1), \tag{5.8}$$

$$x_1' = f_2(t,x_1), \tag{5.9}$$

$$x_2' = f_3(t,x_1), \tag{5.10}$$

$$X_1 = f_4(t,x_1,X_2) \tag{5.11}$$

defined in the domain $T\times\mathcal{W}_1\times\mathcal{X}_2$ turns system (5.4) into an identity upon substitution.

The $(r+1)$-derivative array equations for (5.1) have the form

$$\mathcal{F}_{r+1}(t,\bar{x}_{r+2}) = 0.$$

Let us supplement B1 and B2 with the following conditions:

B3) rank $\bar{\Lambda}_{r+1}\,(t,\bar{x}_{r+2}(t)) = \lambda + n = \text{const}$ everywhere on the domain of definition of this matrix.

B4) At the point $(t,\bar{x}_{r+2}) = 0$, the matrix $\bar{\mathcal{D}}_{r+1}\,(t,\bar{x}_{r+2}(t))$ has an invertible submatrix $\bar{\mathcal{M}}_{r+1}\,(t,\bar{x}_{r+2})$ of order $n(r+2)$ containing all columns with $\bar{\mathcal{M}}_r\,(t,\bar{x}_{r+1})$ and n columns of the matrix $\bar{\Lambda}_{r+1}\,(t,\bar{x}_{r+2})$.

Then

$$\bar{\mathcal{M}}_{r+1}(t, \bar{x}_{r+2}) = \begin{pmatrix} \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_2} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x'} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_1} \end{pmatrix},$$

$$\bar{\mathcal{D}}_{r+1}(t, \bar{x}_{r+2}) \begin{pmatrix} Q & O & O \\ O & Q & O \\ O & O & \hat{Q}_r \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_1} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_2} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_1'} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_2'} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_1} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_2} \end{pmatrix},$$

$$\operatorname{col}\left(x'', \ldots, x^{(r+2)}\right) = \hat{Q}_r \operatorname{col}\left(\hat{X}_1,\, \hat{X}_2\right),$$

$\hat{Q}_r$ is the corresponding row permutation matrix, $\hat{X}_1(t) \in \hat{\mathcal{X}}_1 \subseteq \mathbf{R}^{\lambda+n}$, and $\hat{X}_2(t) \in \hat{\mathcal{X}}_2 \subseteq \mathbf{R}^d$.

For a matrix H of dimensions $(p \times q)$ and a q-dimensional vector h, we introduce

$$\Omega(H) = \max_{\|h\|_{\mathbf{R}^q}=1} \|Hh\|_{\mathbf{R}^p}, \quad \omega(H) = \min_{\|h\|_{\mathbf{R}^q}=1} \|Hh\|_{\mathbf{R}^p}.$$

Hereinafter, the norm of a vector is understood as its Euclidean norm, and the norm of a matrix as its spectral norm.

Theorem 3 [25]. *Let:*
1) *The function $f(t, \alpha, \beta)$ have continuous partial derivatives with respect to its arguments up to order $r+2$ inclusive in the domain $\mathcal{W}$: $f(t, \alpha, \beta) \in \mathbf{C}^{r+2}(\mathcal{W})$.*
2) *Conditions B1–B4 hold.*
3) *There exist a continuous function $v(s) : T \to T$ such that*

$$v(s) > 0, \quad s \in (0, \infty), \quad \int_1^\infty \frac{ds}{v(s)} = \infty,$$

and for all $(t, x, x', \hat{X}_1, \hat{X}_2) \in \mathcal{W} \times \hat{\mathcal{X}}_1 \times \hat{\mathcal{X}}_2$,

$$\omega\left(\bar{\mathcal{M}}_{r+1}\right) > 0,$$

$$\Omega\begin{pmatrix} \dfrac{\partial \mathcal{F}_{r+1}}{\partial t} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_1} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_2} \end{pmatrix} / \omega\left(\bar{\mathcal{M}}_{r+1}\right) \le v\left(\left\|\left(x_2, x', \hat{X}_1\right)\right\|\right).$$

Then any solution of system (5.1) *defined in a sufficiently small neighborhood of zero is a solution of the system*

$$x_2(t) = f_1(t, x_1(t)), \tag{5.12}$$

$$x_1'(t) = f_2(t, x_1(t)), \quad t \in T, \tag{5.13}$$

and vice versa.

The functions $f_1(t, x_1(t))$ and $f_2(t, x_1(t))$ are found as some components of the implicit function (5.8)–(5.11) satisfying system (5.4). Moreover,

$$f_i(t, x_1) \in \mathbf{C}^1(T \times \mathcal{W}_1), \tag{5.14}$$

$$f_i(t, 0) = 0, \quad t \in T, \quad i = 1, 2. \tag{5.15}$$

Obviously, to extract a unique solution of system (5.12), (5.13) or (5.1), one should specify the initial data (2.15).

6. THE DETECTABILITY OF NONLINEAR DAES

First, consider a nonlinear system of the form (5.12), (5.13) with the properties (5.14), (5.15) and the observable output

$$y(t) = c_1^\top(t)x_1(t) + c_2^\top(t)x_2(t) + \tilde{\rho}(t, x_1(t), x_2(t)), \tag{6.1}$$

where $c_1(t) : T \to \mathbf{R}^{n-d}$, $c_2(t) : T \to \mathbf{R}^d$, and

$$\tilde{\rho}(t, 0, 0) \equiv 0 \ \ t \in T.$$

We linearize system (5.12), (5.13) to

$$x_2(t) = G_1(t)x_1(t) + r_1(t, x_1(t)), \tag{6.2}$$

$$x_1'(t) = G_2(t)x_1(t) + r_2(t, x_1(t)), \tag{6.3}$$

where $G_1(t) = \frac{\partial f_1}{\partial x_1}(t, 0)$ and $G_2(t) = \frac{\partial f_2}{\partial x_1}(t, 0)$.

Substituting (6.2) into (6.1) yields the output representation

$$y(t) = \Big(c_1^\top(t) + c_2^\top(t)G_1(t)\Big)\, x_1(t) + c_2^\top(t)r_1(t, x_1(t)) + \tilde{\rho}\Big(t, x_1(t), G_1(t) + r_1(t, x_1(t))\Big). \tag{6.4}$$

For system (6.2)–(6.4), we construct a state observer of the form

$$\hat{x}_2(t) = G_1(t)\hat{x}_1(t) + r_1(t, \hat{x}_1(t)), \tag{6.5}$$

$$\hat{x}_1'(t) = G_2(t)\hat{x}_1(t) + r_2(t, \hat{x}_1(t)) + g(t)\Big(y(t) - (c_1^\top(t) + c_2^\top G_1(t))\hat{x}_1(t)$$
$$- c_2^\top(t)r_1(t, \hat{x}_1(t)) - \tilde{\rho}\Big(t, \hat{x}_1(t), G_1(t)\hat{x}_1(t) + r_1(t, \hat{x}_1(t))\Big)\Big), \tag{6.6}$$

$$\hat{y}_1(t) = \hat{x}_1(t), \ \ \hat{y}_2(t) = \hat{x}_2(t). \tag{6.7}$$

Then the residuals $e_1(t) = \hat{x}_1(t) - x_1(t)$ and $e_2(t) = \hat{x}_2(t) - x_2(t)$ satisfy the equations

$$e_2(t) = G_1(t)e_1(t) + r_1(t, \hat{x}_1(t)) - r_1(t, x_1(t)), \tag{6.8}$$

$$e_1'(t) = \Big(G_2(t) - g(t)(c_1^\top(t) + c_2^\top G_1(t))\Big)\, e_1(t) + r_2(t, \hat{x}_1(t)) - r_2(t, x_1(t))$$
$$- c_2^\top(t)\left(r_1(t, \hat{x}_1(t)) - r_1(t, x_1(t))\right) + \tilde{\rho}\Big(t, x_1(t), G_1(t)x_1(t) + r_1(t, x_1(t))\Big)$$
$$- \tilde{\rho}\Big(t, \hat{x}_1(t), G_1(t)\hat{x}_1(t) + r_1(t, \hat{x}_1(t))\Big). \tag{6.9}$$

Definition 5. System (6.2)–(6.4) is said to have a *state observer* of the form (6.5)–(6.7) if

$$\lim_{t\to\infty} (\hat{y}_i(t) - x_i(t)) = 0, \ \ i = 1, 2.$$

Definition 6. System (6.2)–(6.4) is said to be *detectable* if there exists a vector function $g(t) : T \to \mathbf{R}^{n-d}$ such that system (6.8)–(6.9) is asymptotically stable.

Obviously, the system

$$e_2(t) = G_1(t)e_1(t), \tag{6.10}$$

$$e_1'(t) = \Big(G_2(t) - g(t)(c_1^\top(t) + c_2^\top G_1(t))\Big)\, e_1(t), \ \ t \in T, \tag{6.11}$$

is the first approximation system for (6.8), (6.9). The following stability conditions for (6.10), (6.11) are straightforward from Lemma 2.

Proposition 3. *Let:*
1) $G_1(t) \in \mathbf{C}(T)$, $G_2(t) \in \mathbf{C}^{n-d-1}(t)$, *and* $c_1^\top(t) + c_2^\top(t)G_1(t) \in \mathbf{C}^{n-d}(T)$ $(n - d \geq 1)$.
2) $\|G_1(t)\| \leq \kappa$ $\forall t \in T$, $\kappa = \text{const} > 0$.
3) *Assumptions 2 and 3 of Lemma 2 hold, with*

$$s_0(t) = c_1^\top + c_2^\top(t)G_1(t), \quad s_i = s_{i-1}'(t) + s_{i-1}(t)G_2(t), \quad i = \overline{1, n-d}, \tag{6.12}$$

in (3.15) *and* (3.16).

Then there exists an $(n-d)$*-dimensional vector function* $g(t)$ *such that system* (6.10), (6.11) *is asymptotically stable.*

Using Proposition 3, we derive sufficient conditions for the detectability of system (5.12), (5.13), (6.1).

Theorem 4. *Let:*
1) *The properties* (5.14), (5.15), *and* $\tilde{\rho}(t, x_1, x_2) \in \mathbf{C}(\hat{\mathcal{W}})$ *hold.*
2) *The assumptions of Proposition 3 be valid.*
3) $|\tilde{\rho}(t, \hat{x}_1, \hat{x}_2) - \tilde{\rho}(t, x_1, x_2)| \leq \eta_\rho\left(\|\hat{x}_1 - x_1\| + \|\hat{x}_2 - x_2\|\right)$, *where* $\eta_\rho > 0$ *is a sufficiently small constant for* x_i *and* $\hat{x}_i - x_i$ *sufficiently close to zero* $(i = 1, 2)$.
4) $\|c_2(t)\| \leq c$, $t \in T$, $c = \text{const} > 0$.
5) $\|r_i(t, \hat{x}_1) - r_i(t, x_1)\| \leq \eta_i \|\hat{x}_1 - x_1\|$ $(i = 1, 2)$, *where* $\eta_i > 0$ *are sufficiently small constants for* $\|x_1\|$ *and* $\|\hat{x}_1 - x_1\|$ *sufficiently close to zero.*

Then system (5.12), (5.13), (6.1) *is detectable and has a state observer of the form* (6.5)–(6.7).

The proof is postponed to the Appendix.

Addressing system (5.1), (5.2), we linearize (5.1) in a neighborhood of the point $(x, x') = (0, 0)$:

$$A(t)x'(t) + B(t)x(t) + h(t, x(t), x'(t)) = 0, \tag{6.13}$$

where

$$A(t) = \frac{\partial f(t, x, x')}{x'}(t, 0, 0), \quad B(t) = \frac{\partial f(t, x, x')}{x}(t, 0, 0).$$

Under Assumptions A1 and A2, there exists the operator (2.7) transforming system (1.1) to (2.9)–(2.14). Applying this operator to equation (6.13) gives

$$x_2(t) + J_1(t)x_1(t) + h_1(t, \bar{x}_{r+1}(t)) = 0, \tag{6.14}$$

$$x_1'(t) + J_2(t)x_1(t) + h_2(t, \bar{x}_{r+1}(t)) = 0, \tag{6.15}$$

where the designations (2.10), (5.5) are used and

$$\begin{pmatrix} h_1(t, \bar{x}_{r+1}(t)) \\ h_2(t, \bar{x}_{r+1}(t)) \end{pmatrix} = \mathcal{R}\left[h(t, x(t), x'(t))\right].$$

Under the conditions of Theorem 3, any solution of system (5.1) defined in a sufficiently small neighborhood of zero is a solution of system (5.12), (5.13), and vice versa. Equations (6.2), (6.3) linearize system (5.12), (5.13) in a neighborhood of the point $x_1 = 0$.

Lemma 5. *Let all conditions of Theorem 3 hold. Then in systems* (6.14), (6.15) *and* (6.2), (6.3),

$$J_1(t) = -G_1(t), \quad J_2(t) = -G_2(t) \ \ \forall t \in T,$$

and the functions $-r_1(t, x_1(t))$ *and* $-r_2(t, x_1(t))$ *are obtained from* $h_1(t, \bar{x}_{r+1}(t))$ *and* $h_2(t, \bar{x}_{r+1}(t))$, *respectively, by substituting the implicit function* (5.8)–(5.11) *that satisfies the* r*-derivative array equations* (5.4).

A similar result was proved in [26].

Theorem 5. *Let all conditions of Theorem 3, as well as conditions 2–4 of Theorem 4, hold. Moreover, let*

$$\lim_{\hat{x}_2 \to x_2,\, \hat{x}' \to x',\, \hat{X}_1 \to X_1,\, \hat{X}_2 \to X_2} \|h_i(t, \hat{\bar{x}}_{r+1}) - h_i(t, \bar{x}_{r+1})\| \le \mu_i \|\hat{x}_1 - x_1\|, \quad i = 1, 2, \tag{6.16}$$

where $\mu_i = \text{const} > 0$ *are sufficiently small for* $\|\hat{x}_1 - x_1\|$ *and* $\|\bar{x}_{r+1}\|$ *from the corresponding small neighborhoods of zero. Let the variables* $\bar{x}_{r+1}$, x_1, x_2, X_1, *and* X_2 *be given by* (5.5)–(5.7).

Then system (5.1), (5.2) *is detectable and has an observer of the form* (6.5)–(6.7).

The proof is provided in the Appendix.

7. AN ILLUSTRATIVE EXAMPLE

Based on Theorem 5, we analyze the detectability of the nonlinear system

$$2x_{(1)}(t) + x_{(3)}(t) + e^{-t} x_{(1)}(t) x_{(2)}(t) = 0, \tag{7.1}$$

$$\Big(1 + x_{(1)}(t)\Big)\, x'_{(1)}(t) = 0, \tag{7.2}$$

$$x'_{(2)}(t) + x'_{(3)}(t) + 4x_{(2)}(t) + 2x_{(2)}(t) x_{(3)}(t) = 0, \tag{7.3}$$

$$y(t) = \left(1 + \frac{1}{4}\cos t\right) x_{(2)}(t) + 2x_{(3)}(t) + e^{-2t} x_{(1)}(t) x_{(3)}(t), \quad t \in T, \tag{7.4}$$

where

$$x(t) = \operatorname{col}\Big(x_{(1)}(t),\, x_{(2)}(t),\, x_{(3)}(t)\Big).$$

To verify Assumptions B1–B4, we compile the matrix

$$\mathcal{D}_2(t, x, x', x'', x''') = \left(\begin{array}{ccc|ccc|ccc||ccc}
v_1 & v_2 & \boxed{1} & 0 & 0 & 0 & & & & & & \\
x'_{(1)} & 0 & 0 & \boxed{1 + x_{(1)}} & 0 & 0 & & & & & & \\
0 & v_3 & v_4 & 0 & \boxed{1} & 1 & & & & & & \\
\hline
v'_1 & v'_2 & 0 & v_1 & v_2 & \boxed{1} & 0 & 0 & 0 & & & \\
x''_{(1)} & 0 & 0 & 2x'_{(1)} & 0 & 0 & \boxed{1 + x_{(1)}} & 0 & 0 & & & \\
0 & v'_3 & v'_4 & 0 & v_3 & v_4 & 0 & \boxed{1} & 1 & & & \\
\hline\hline
v''_1 & v''_2 & 0 & 2v'_1 & 2v'_2 & 0 & v_1 & v_2 & \boxed{1} & 0 & 0 & 0 \\
x'''_{(1)} & 0 & 0 & 3x''_{(1)} & 0 & 0 & 3x'_{(1)} & 0 & 0 & \boxed{1 + x_{(1)}} & 0 & 0 \\
0 & v''_3 & v''_4 & 0 & 2v'_3 & 2v'_4 & 0 & v_3 & v_4 & 0 & \boxed{1} & 1
\end{array}\right),$$

where $v_1(t) = 2 + e^{-t} x_{(2)}(t)$, $v_2(t) = e^{-t} x_{(1)}(t)$, $v_3(t) = 4 + 2x_{(3)}(t)$, and $v_4(t) = 2x_{(2)}(t)$.

Clearly, Assumption B1 is valid for $r = 1$, i.e.,

$$\operatorname{rank} \Lambda_1(t, x, x') = \operatorname{rank} \begin{pmatrix} 1 + x_{(1)} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \lambda = 2,$$

when $x_{(1)}$ satisfies the inequality $|x_{(1)}| < 1$.

The matrix $\mathcal{D}_1(t, x, x', x'')$ of dimensions (6×9), located above the horizontal and to the left of the vertical double lines, has an invertible submatrix $\mathcal{M}_1(t, x, x', x'')$ of order 6, whose columns are marked by boxed elements. Therefore, Assumption B2 is valid as well.

The matrix $\Lambda_2(t, x, x', x'')$ consists of the last six columns of the matrix $\mathcal{D}_2$. Obviously, $\operatorname{rank} \Lambda_2(t, x, x', x'') = \lambda + n = 5$ for $|x_{(1)}| < 1$, so Assumption B3 is valid.

The matrix $\mathcal{D}_2(t, x, x', x'', x''')$ also has a submatrix $\mathcal{M}_2(t, x, x', x'', x''')$ of order 9 with the properties formulated in Assumption B4; this submatrix consists of the columns containing the boxed elements. Thus, for $r = 1$, Assumptions B1–B4 are valid.

For the first approximation system

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} x'(t) + \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} x(t) = 0,$$

the operator

$$\mathcal{R} = R_0(t) + R_1(t)\frac{d}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \frac{d}{dt}$$

is defined; it has the left inverse operator $\mathcal{L} = L_0(t) + L_1(t)$, where $L_0(t) = R_0^{-1}(t)$ and $L_1(t) = -R_1(t)$.

The operator $\mathcal{R}$ transforms system (7.1)–(7.3) to

$$\begin{pmatrix} O & O \\ E_2 & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1(t) & 1 \\ J_2(t) & O \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} h_1(t, x, x', x'') \\ h_2(t, x, x', x'') \end{pmatrix} = 0,$$

where $d = 1$, $Q = E_3$,

$$x_2(t) = x_{(3)}(t), \quad x_1(t) = \begin{pmatrix} x_{(1)}(t) \\ x_{(2)}(t) \end{pmatrix}, \tag{7.5}$$

$$J_1(t) = \begin{pmatrix} 2 & 0 \end{pmatrix}, \quad J_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix},$$

$$h_1(t, x, x', x'') = e^{-t} x_{(1)}(t) x_{(2)}(t), \tag{7.6}$$

$$h_2(t, x, x', x'') = \begin{pmatrix} x_{(1)}(t) x'_{(1)}(t) \\ e^{-t}\left(x_{(1)}(t)x_{(2)}(t) - x'_{(1)}(t)x_{(2)}(t) - x_{(1)}(t)x'_{(2)}(t)\right) \\ + 2x_{(1)}(t)x'_{(1)}(t) + 2x_{(2)}(t)x_{(3)}(t) \end{pmatrix}. \tag{7.7}$$

Next, we verify the conditions of Proposition 3. According to Lemma 5, $G_1(t) = \begin{pmatrix} -2 & 0 \end{pmatrix}$ and $G_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$. Obviously, Condition 2 holds.

In view of (7.5), from (7.4) we obtain $c_1^\top(t) = \begin{pmatrix} 0 & 1 + \frac{1}{4}\cos t \end{pmatrix}$, $c_2^\top(t) = 2$. According to formulas (6.12) and (3.15),

$$S(t) = \begin{pmatrix} -4 & 1 + \dfrac{1}{4}\cos t \\ 0 & -\left(4 + \dfrac{1}{4}\sin t + \cos t\right) \end{pmatrix}.$$

Since $\det S(t) = 16 + \sin t + 4\cos t \neq 0 \ \forall t \in T$, it is possible to find the inverse matrix

$$S^{-1}(t) = -\frac{1}{16 + \sin t + 4\cos t} \begin{pmatrix} 4 + \dfrac{1}{4}\sin t + \cos t & 1 + \dfrac{1}{4}\cos t \\ 0 & 4 \end{pmatrix}$$

and the functions (3.16)

$$\omega_1(t) = 0, \quad \omega_2(t) = -\frac{64 + 8\sin t + 15\cos t}{16 + \sin t + 4\cos t}.$$

Clearly, conditions 2 and 3 of Lemma 2, as well as condition 3 of Proposition 3, hold.

Let us verify conditions 3 and 4 of Theorem 4, taking the relations (7.5) into account. Obviously, $\|c_2(t)\| = 2$. Since $\tilde{\rho}(t, x_1, x_2) = e^{-2t} x_{(1)} x_{(3)}$,

$$|\tilde{\rho}(t, \hat{x}_1, \hat{x}_2) - \tilde{\rho}(t, x_1, x_2)| \le |x_{(3)}| \, \|\hat{x}_1 - x_1\| + \left(|\hat{x}_{(1)} - x_{(1)}| + |x_{(1)}|\right) \, \|\hat{x}_2 - x_2\|,$$

with yields the estimate from condition 3 of Theorem 4 with $\eta_\rho = \max\{\delta_2, \delta_1 + \delta_3\}$, where $|x_{(1)}| \le \delta_1 < 1$, $|x_{(3)}| \le \delta_2$, and $|\hat{x}_{(1)} - x_{(1)}| \le \delta_3$, $\delta_i = \text{const} > 0$, $i = 1, 2, 3$.

Now we address Theorem 5, in particular, condition (6.16). Due to (7.6) and (7.7),

$$\lim_{\hat{x}_2 \to x_2,\, \hat{x}' \to x',\, \hat{X}_1 \to X_1,\, \hat{X}_2 \to X_2} \|h_1(t, \hat{\bar{x}}_{r+1}) - h_1(t, \bar{x}_{r+1})\| = e^{-t} \left|\hat{x}_{(1)} \hat{x}_{(2)} - x_{(1)} x_{(2)}\right|$$
$$\le \left(|\hat{x}_{(1)} - x_{(1)}| + |x_{(1)}| + |x_{(2)}|\right) \|\hat{x}_1 - x_1\| \le \eta_1 \|\hat{x}_1 - x_1\|,$$

where $\eta_1 = \delta_1 + \delta_3 + \delta_4$, $|x_{(2)}| \le \delta_4$, and $\delta_4 = \text{const} > 0$.

In turn,

$$\lim_{\hat{x}_2 \to x_2,\, \hat{x}' \to x',\, \hat{X}_1 \to X_1,\, \hat{X}_2 \to X_2} \|h_2(t, \hat{\bar{x}}_{r+1}) - h_2(t, \bar{x}_{r+1})\|$$
$$\le \left(|x_{(1)}| + |x_{(2)}| + 2|x_{(3)}| + |\hat{x}_{(1)} - x_{(1)}| + 4|x'_{(1)}| + |x'_{(2)}|\right) \|\hat{x}_1 - x_1\| \le \eta_2 \|\hat{x}_1 - x_1\|,$$

where $\eta_2 = \delta_1 + 2\delta_2 + \delta_3 + \delta_4 + 4\delta_5 + \delta_6$, $|x'_{(1)}| \le \delta_5$, and $|x'_{(2)}| \le \delta_6$ with positive constants and δ_5 and δ_6.

Thus, all conditions of Theorem 5 are satisfied, so system (7.1)–(7.4) is detectable.

For the example under consideration, the first four equations of the 2-derivative array ones can be used to find the functions (5.12), (5.13):

$$x_{(3)}(t) = -x_{(1)}(t) \left(2 + e^{-t} x_{(2)}(t)\right),$$

$$x'_{(1)}(t) = 0, \quad x'_{(2)}(t) = \frac{x_{(2)}(t) \left(e^{-t} x_{(1)}(t)(2x_{(2)}(t) - 1) + 4(x_{(1)}(t) - 1)\right)}{1 - e^{-t} x_{(1)}(t)}.$$

Therefore, it is possible to design the following state observer with the output (6.7) for system (7.1)–(7.4):

$$\hat{x}_{(3)}(t) = -\hat{x}_{(1)}(t) \left(2 + e^{-t} \hat{x}_{(2)}(t)\right),$$

$$\begin{pmatrix} \hat{x}'_{(1)}(t) \\ \hat{x}'_{(2)}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -4\hat{x}_{(2)}(t) + \dfrac{\hat{x}_{(1)}(t)\hat{x}_{(2)}(t)\left(2e^{-t}\hat{x}_{(2)}(t) - 5e^{-t} - 4\right)}{1 - e^{-t}\hat{x}_{(1)}(t)} \end{pmatrix}$$
$$+ g(t) \left(y(t) + 4\hat{x}_{(1)}(t) - \left(1 + \frac{1}{4}\right) \hat{x}_{(2)}(t) + 2e^{-t} \hat{x}_{(1)}(t) \hat{x}_{(2)}(t) + e^{-2t} \hat{x}^2_{(1)}(t) \left(2 + e^{-t} \hat{x}_{(2)}(t)\right)\right).$$

The vector function $g(t)$ is intended to ensure the detectability of system (3.10), (3.11) or, equivalently, the asymptotic stability of system (3.9). According to the book [22, pp. 255–257, 283–284, 312–314], one such function is given by

$$g(t) = \Gamma(t) \begin{pmatrix} g_1(t) - \gamma_1 \\ g_2(t) - \gamma_2 \end{pmatrix},$$

where $\gamma_1 = -\xi_1\xi_2$, $\gamma_2 = -(\xi_1 + \xi_2)$, ξ_1 and ξ_2 are specified negative numbers,

$$g_1(t) = \frac{\sin t + 4\cos t}{q(t)} + \left(\frac{\cos t - 4\sin t}{q(t)}\right)^2, \quad g_2(t) = 4 + \frac{4\sin t - \cos t}{q(t)},$$

$$\Gamma(t) = \frac{1}{q(t)} \begin{pmatrix} 1 + \frac{1}{4}\cos t & g_2(t)\left(1 + \frac{1}{4}\cos t\right) - 4 - \cos t - \frac{1}{4}\sin t \\ 4 & 4g_2(t) \end{pmatrix},$$

$$q(t) = 16 + 4\cos t + \sin t.$$

To verify the asymptotic stability of system (3.9), we make the change of variables $e_1(t) = \Gamma(t)\tilde{e}_1(t)$ in (3.9). Then direct substitution yields the system $\tilde{e}_1'(t) = \begin{pmatrix} 0 & -\gamma_1 \\ -1 & -\gamma_2 \end{pmatrix} \tilde{e}_1(t)$, whose matrix has negative eigenvalues ξ_1 and ξ_2. Since $\Gamma(t)$ is a Lyapunov matrix, system (3.9) will also be asymptotically stable.

8. CONCLUSIONS

This paper has been devoted to linear and nonlinear time-varying DAEs of observation under general assumptions imposed on their internal structure.

For the linear system (1.1), we have established existence conditions for the structural form (2.9) obtained by applying the linear differential operator (2.7) to (1.1). This operator has a left inverse, so systems (1.1) and (2.9) share the same set of solutions. Moreover, in (2.9), the differential and algebraic components are already separated, and each component mentioned is nonsingular. Conditions ensuring the existence of the right inverse operator (2.17) for (2.7) have been derived (Lemma 1).

For the structural form (2.9) with a scalar output, a state observer of the form (3.2)–(3.4) has been designed. It can be used to estimate the state of system (1.1), (1.2) as well (Theorem 1). In fact, it suffices to construct an observer for the differential component of the DAE (1.1).

In addition, we have considered system (4.1) with scalar control, which also includes the derivative of the control. What is essential, the right inverse operator (2.17) is employed to construct such a system. Conditions have been justified under which system (4.2) can be represented in the form (4.1) (Lemma 3). For the DAE (4.1), stabilizability conditions based on the state observer (3.2)–(3.4) have been established (Theorem 2). Moreover, the stabilizing feedback control (4.7) includes only the solution of the differential subsystem of the observer.

The nonlinear system (5.1) with the scalar output (5.2) has been analyzed using the structural form (5.12), (5.13): any of its solutions starting in a sufficiently small neighborhood of zero is also a solution of system (5.1), and vice versa. Being already a nonsingular system, this structural form consists of some components of an implicit function satisfying the r-derivative array equations (5.4) (Theorem 3). For the DAE (5.1), (5.2), we have utilized the linear approximation system to obtain sufficient detectability conditions and design a state observer of the form (6.5)–(6.7) (Theorem 5). Note that to construct the observer, it is necessary to find the above implicit function. At the same time, verifying the detectability conditions does not require this procedure: only the properties of the linear approximation are used here. Important auxiliary results are the detectability conditions and the method for designing a state observer for the quasilinear system of the form (5.12), (5.13), (6.1) (Theorem 4).

The above approach can be applied to analyze detectability and construct observers for linear and nonlinear discrete descriptor systems, as well as singular systems with continuous and discrete time.

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APPENDIX

Proof of Lemma 1. We rearrange the columns of the matrix $\mathcal{D}_r(t)$ (2.1)–(2.4) by multiplying it on the right by the permutation matrix $\operatorname{diag}\{Q, \ldots, Q\}$, where Q is given by (2.5). In doing so, it is necessary to consider the partition structure (2.6). By deleting the columns included in the block $B_1(t)$ from $\mathcal{D}_r(t)$, we obtain the matrix $\bar{\mathcal{D}}_r(t)$ (omitted here due to cumbersomeness).

Note that condition 1 of the lemma ensures $P_0(t), P_1(t) \in \mathbf{C}^r(T)$.

By the construction procedure, the coefficients of the operator $\mathcal{R}$ satisfy the equation

$$\begin{pmatrix} R_0(t) & R_1(t) & \ldots & R_r(t) \end{pmatrix} \bar{\mathcal{D}}_r(t) = \begin{pmatrix} E_n & O & \ldots & O \end{pmatrix} \tag{A.1}$$

for all $t \in T$. On the other hand, by definition, the coefficients of the operators (2.7) and (2.17) are related through the identity

$$\begin{pmatrix} R_0 & R_1 & \ldots & R_r \end{pmatrix} \begin{pmatrix} P_0 & C_0^0 P_1 & O & \ldots & O \\ P_0' & C_1^0 P_1' + C_1^1 P_0 & C_1^1 P_1 & \ldots & O \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ P_0^{(r)} & C_r^0 P_1^{(r)} + C_r^1 P_0^{(r-1)} & C_r^1 P_1^{(r-1)} + C_r^2 P_0^{(r-2)} & \ldots & C_r^r P_1 \end{pmatrix}$$
$$= \begin{pmatrix} E_n & O & \ldots & O \end{pmatrix}. \tag{A.2}$$

Letting $P_0(t) = \begin{pmatrix} B_2(t) & A_1(t) \end{pmatrix}$, $P_1(t) = \begin{pmatrix} A_2(t) & O \end{pmatrix}$ in (A.2) makes equalities (A.1) and (A.2) coincident under identity (2.16).

Proof of Lemma 3. Obviously, for the desired result, the vectors $b(t)$, $\beta_1(t)$, and $\beta_2(t)$ shall satisfy the relations

$$\beta_1(t) = P_0(t)b(t) + P_1(t)b'(t), \tag{A.3}$$

$$\beta_2(t) = P_1(t)b(t), \quad t \in T. \tag{A.4}$$

According to Lemma 1, the operator $P_0(t) + P_1(t)\frac{d}{dt}$ has the left inverse $\mathcal{R}$. By applying this operator to both sides of identity (A.3), we arrive at the representation (4.4). Condition (4.3) is finally obtained by substituting (4.4) into equality (A.4).

Proof of Theorem 2. First, we show that under the conditions of this theorem, there exists a feedback control (4.7) stabilizing system (4.5), (4.6).

Consider equations (4.10), (4.11). Due to assumptions 1 and 3, all conditions of Lemma 2 are satisfied, and there exists a function $g(t) \in \mathbf{C}(T)$ such that

$$\lim_{t\to\infty} e_1(t) = 0. \tag{A.5}$$

In this case, in view of the estimate (3.14), equation (4.10) implies the limit relation $\lim\limits_{t\to\infty} e_2(t) = 0$. Thus, for an appropriate choice of $g(t)$, subsystem (4.10), (4.11) is asymptotically stable.

Now we turn to equations (4.8), (4.9). Under assumptions 1 and 4, Lemma 4 is valid: there exists a vector function $k(t) \in \mathbf{C}(T)$ such that system (4.9) is asymptotically stable. This means that for any continuous function $b_2(t)k^\top(t)e_1(t)$, each solution $x_1(t)$ of system (4.9) has the property

$$\lim_{t\to\infty} x_1(t) = 0. \tag{A.6}$$

In turn, assumption 5 together with conditions (3.14), (4.15) lead to the estimates

$$\chi\left[J_1(t) + b_1(t)k^\top(t)\right] \le 0, \ \ \chi\left[b_1(t)k^\top(t)\right] \le 0;$$

in view of (A.5) and (A.6), it follows that in (4.8), $x_2(t) \to 0$ as $t \to +\infty$.

Based on the above considerations, we conclude that the control (4.7) is stabilizing for system (4.5), (4.6).

The supposed smoothness of the input data of system (4.1) ensures the inclusions $J_2(t) \in \mathbf{C}^{n-d}(T)$ and $b_2(t) \in \mathbf{C}^{n-d+1}(T)$. According to Remark 3, for the control (4.7), the function $k(t)$ can be constructed in the class $\mathbf{C}^1(T)$. Under Assumptions A1–A4, the DAEs (4.1) and (A.5), (4.5) are equivalent in the sense of their solutions; therefore, the feedback control (4.7) will also stabilize system (4.1).

Proof of Theorem 4. According to Proposition 3, there exists a function $g(t) : T \to \mathbf{R}^{n-d}$ such that system (6.11) is asymptotically stable. Moreover, by Remark 1, $g(t)$ can be assumed bounded and system (6.11) reducible in the Lyapunov sense.

Under the above assumptions, in equation (6.9),

$$\Big\| r_2(t,\hat{x}_1(t)) - r_2(t,x_1(t)) - c_2^\top(t)\Big(r_1(t,\hat{x}_1(t)) - r_1(t,x_1(t))\Big)$$
$$+ \tilde{\rho}\Big(t, x_1(t), G_1(t)x_1(t) + r_1(t,x_1(t))\Big) - \tilde{\rho}\Big(t,\hat{x}_1(t), G_1(t)\hat{x}_1(t) + r_1(t,\hat{x}_1(t))\Big)\Big\|$$
$$\le (\eta_2 + \eta_1(1+c) + \eta_\rho(1+\kappa))\, \|e_1\|.$$

According to the stability theorem by the first approximation [22, p. 340] (also known as Lyapunov's indirect method in the literature), the trivial solution of system (6.9) is asymptotically stable for sufficiently small constants η_1, η_2, and η_ρ under which $e_1(t)$ varies in a small neighborhood of zero.

In turn, due to the bounded matrix $G_1(t)$ and the estimate $\|r_1(t,\hat{x}_1) - r_1(t,x_1)\| \le \eta_1 \|e_1(t)\|$, the vector function $e_2(t)$, as a solution of equation (6.8), vanishes as $e_1(t) \to 0$.

Thus, the trivial solution of system (6.8), (6.9) is asymptotically stable for an appropriate choice of $g(t)$. This means that system (6.2)–(6.4), and hence system (5.12), (5.13), (6.1), is detectable and has an observer of the form (6.5)–(6.7).

Proof of Theorem 5. According to Theorem 3, for sufficiently small initial data (2.15), system (5.1) has a nontrivial solution satisfying equations (5.12) and (5.13). In turn, any solution of system (5.12), (5.13) lying in a sufficiently small neighborhood of zero is a solution of system (5.1). Consequently, systems (5.1), (5.2) and (5.12), (5.13), (6.1) are detectable simultaneously. Recall that (6.1) is another representation for the output (5.2).

Under the conditions of Theorem 3, there exists an implicit function (5.8)–(5.11) satisfying the r-derivative array equations (5.4). Due to assumption 1 of Theorem 3 and condition (5.3), this implicit function has continuous partial derivatives with respect to each argument on its domain of definition and $f_j(t,0) \equiv 0$, $f_4(t,0,0) \equiv 0$, $t \in T$, $j = 1,2,3$.

We substitute the functions (5.8)–(5.11) into condition (6.16). Unlike the constants η_i from assumption 5 of Theorem 4, the constants μ_i may depend on the sizes of the corresponding neighborhoods of zero covering the variables x_2, x_1', x_2', and X_1. As a result of the substitution, these variables are replaced by the functions on the right-hand side of equalities (5.8)–(5.11). According to the aforementioned properties of these functions, one can choose $\|x_1\|$ and $\|X_2\|$ sufficiently close to zero so that the values $|f_j(t, x_1)|$ and $|f_4(t, x_1, X_2)|$ will belong to given small neighborhoods of zero for all $t \in T$. In view of Lemma 5, the estimates of assumption 5 of Theorem 4 hold.

Thus, all assumptions of Theorem 4 are satisfied. According to this theorem, system (5.12), (5.13), (5.2) is detectable and has a state observer of the form (6.5)–(6.7). The same property holds for system (5.1), (5.2).

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NONLINEAR SYSTEMS

Reachable Configurations of a Two-Link Manipulator under Time-Optimal Control with Angular Velocity Constraints

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Abstract—This paper considers the problem of constructing reachable domains for a planar two-link manipulator with a statically balanced second link under time-optimal control of its motion with angular velocity constraints for the links. The optimal controls belong to the class of relay regimes and contain the minimum total number of switchings sufficient to satisfy the boundary conditions: the system is at rest at the initial and final time instants. A method is developed to find reachable domains in the manipulator's configuration plane under the proposed controls with angular velocity constraints.

Keywords: reachable domain, two-link manipulator, optimal control, angular velocity constraints

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1. INTRODUCTION

Two-link manipulators are robots widely used in various branches of modern industry, independently and as part of multi-link systems where two links perform the main motions during industrial operations. The development of efficient programmed control regimes for such manipulators remains a topical problem. If reducing the manipulator's cycle time accelerates the industrial process, it is reasonable to design time-optimal control regimes. Much research has been devoted both to mechanical models of two-link manipulators [1–9] and to their dynamic interaction with drives [10–13]. Note that when designing such regimes, one should consider not only the speed of operations but also the constraints imposed on the angular velocities of the links. Exceeding the permissible limits of angular velocities may cause mechanical damage, loss of system stability, and reduced accuracy of task fulfilment. The issues of optimizing time-optimal controls with angular velocity constraints were addressed in [14, 15]. At the same time, accounting for angular velocity constraints possibly increases the number of switchings in the optimal control structure and leads to time intervals during which the manipulator or the motor operates at limiting angular velocities. To implement such optimal controls, it is necessary to calculate the switching instants of the control action precisely. However, due to inevitable computational errors, an increase in the number of switchings can significantly affect control accuracy.

Thus, a key problem is to construct reachable domains on the manipulator's configuration plane that are implemented under optimal controls with a minimum number of switchings. At the same time, limiting angular velocities should be achieved only at the control switching instants, depending on the final configuration. Below, we solve this problem for the two-link manipulator model with a statically balanced second link [16–18]. Note that in the context of time-optimal control, simplified and complete models of manipulators with two degrees of freedom were also considered in [19–25].

2. MANIPULATOR MODEL AND PROBLEM STATEMENT

Consider a mechanical system consisting of two absolutely rigid equal-length links G_1 and G_2 connected by a joint O_2 (a two-link manipulator, see Fig. 1). Link G_1 is connected to a fixed base by joint O_1. The joints are ideal and cylindrical, and their axes are parallel to each other. A gripper is attached to the end of the second link at point O_3. By assumption, the linear dimensions of the gripper are much smaller than the lengths of the links; in the analysis of transport motions, the gripper will be treated as a material point. The manipulator is controlled by two independent drives D_1 and D_2. Drive D_1 carries out the interaction of the first link with the base; drive D_2, the interaction between links G_1 and G_2. The control functions in this manipulator model, denoted by M_1 and M_2, are the torques about axes O_1 and O_2 developed by drives D_1 and D_2, respectively.

The above system performs plane-parallel motion in a horizontal plane perpendicular to the axes of joints O_1 and O_2.

In the case where the center of mass C of the second link G_2 lies on the axis of the second joint (i.e., the second link of the manipulator is statically balanced), the system dynamics are described by the Lagrange equations [1]

$$\left(I_1 + m_2 L^2\right)\ddot{\varphi}_1 = M_1 - M_2, \qquad I_2\,\ddot{\varphi}_2 = M_2. \tag{2.1}$$

Here, φ_1 and φ_2 stand for the angles between the axis O_1x and the lines O_1O_2 and O_2O_3, respectively; $L = |O_1O_2| = |O_2O_3|$ is the length of the first/second link; I_1 and I_2 are the moments of inertia of links G_1 and G_2 about the axes of joints O_1 and O_2, respectively; finally, m_2 is the mass of link G_2.

By assumption, the manipulator's links are designed to make over one full revolution in both positive and negative directions. Depending on the industrial purpose, this allows, e.g., moving the end O_3 of the manipulator along a spiral trajectory in the workspace.

Note that in time-optimal problems, equations (2.1) are considered not only as a basic model for studying the full equations of a manipulator with a displaced center of mass of the second link [2–9] but also separately [16–18].

Problem. It is required to determine the laws of variation of the control torques $M_1 = M_1(t)$ and $M_2 = M_2(t)$, and construct the reachable domains of the manipulator's final rest states, under

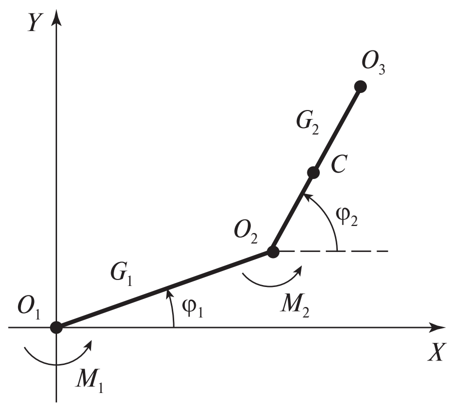

Fig. 1. Two-link manipulator model.

which system (2.1) will move from the initial rest state

$$\varphi_1(0) = \varphi_1^0, \quad \dot\varphi_1(0) = 0, \qquad \varphi_2(0) = \varphi_2^0, \quad \dot\varphi_2(0) = 0 \tag{2.2}$$

to any final rest state

$$\varphi_1(T) = \varphi_1^T, \quad \dot\varphi_1(T) = 0, \qquad \varphi_2(T) = \varphi_2^T, \quad \dot\varphi_2(T) = 0 \tag{2.3}$$

within the constructed domain in the minimum time T while satisfying

$$|M_1(t)| \le M_1^0, \quad |M_2(t)| \le M_2^0 \tag{2.4}$$

(control constraints) and

$$|\dot\varphi_1(t)| \le a_1, \quad |\dot\varphi_2(t)| \le a_2 \tag{2.5}$$

(angular velocity constraints) during the entire motion.

In the dimensionless variables (with the primes subsequently omitted)

$$\begin{aligned} &t' = \left[M_2^0/\left(m_2L^2\right)\right]^{1/2} t, \qquad I_i' = I_i/\left(m_2L^2\right), \quad M_i' = M_i/M_2^0, \\ &a_i' = \left[M_2^0/\left(m_2L^2\right)\right]^{-1/2} a_i, \quad \varphi_i' = \varphi_i - \varphi_i^0, \qquad {\varphi'}_i^{0,T} = \varphi_i^{0,T} - \varphi_i^0, \quad i = 1, 2, \end{aligned} \tag{2.6}$$

the relations (2.1)–(2.5) take the simplified form

$$(I_1 + 1)\,\ddot\varphi_1 = M_1 - M_2, \quad I_2\,\ddot\varphi_2 = M_2, \tag{2.7}$$

$$\varphi_i(0) = \dot\varphi_i(0) = 0, \quad i = 1, 2, \tag{2.8}$$

$$\varphi_i(T) = \varphi_i^T, \quad \dot\varphi_i(T) = 0, \qquad i = 1, 2, \tag{2.9}$$

$$|M_1(t)| \le M_1^0, \quad |M_2(t)| \le 1, \tag{2.10}$$

$$|\dot\varphi_1(t)| \le a_1, \quad |\dot\varphi_2(t)| \le a_2. \tag{2.11}$$

3. OPTIMAL CONTROLS WITHOUT ANGULAR VELOCITY CONSTRAINTS

Using the results of [1, 9], the solution of the time-optimal problem for system (2.7)–(2.10) without the angular velocity constraints (2.11)) can be represented as follows:

$$M^* = \begin{cases} (M_1^*, M_2) & \text{if } (\varphi_1^T, \varphi_2^T) \in \Phi_1 = \bigcup\limits_{\alpha=0}^{1} \Phi_1^{(\alpha)} \\ (M_1, M_2^*) & \text{if } (\varphi_1^T, \varphi_2^T) \in \Phi_2 = \bigcup\limits_{\alpha=0}^{1} \Phi_2^{(\alpha)}, \end{cases} \tag{3.1}$$

$$M^* = (M_1^*, M_2^*) \quad \text{if } (\varphi_1^T, \varphi_2^T) \in \Phi_0 = \bigcup_{\alpha,k=0}^{1} \Phi_0^{(\alpha,k)}.$$

According to (3.1), we have:

1. If $(\varphi_1^T, \varphi_2^T) \in \Phi_1 = \bigcup\limits_{\alpha=0}^{1} \Phi_1^{(\alpha)}$, where

$$\Phi_1^{(\alpha)} = \left\{ \begin{aligned} &(\varphi_1^T, \varphi_2^T)\colon (-1)^\alpha\ (I_1+1)\left[I_2\left(M_1^0 + (-1)^\alpha\right)\right]^{-1} \varphi_1^T \\ &< \varphi_2^T < (-1)^\alpha\ (I_1+1)\left[I_2\left(M_1^0 - (-1)^\alpha\right)\right]^{-1} \varphi_1^T, \\ &(-1)^\alpha\,\varphi_1^T > 0 \end{aligned} \right\}, \ \alpha = 0, 1, \tag{3.2}$$

then the optimal control M_1^* is a relay function with one switching point of the form

$$\begin{gathered} M_1^*(t) = M_1^0 \operatorname{sign}\left\{(t_0 - t)\left[(I_1+1)\,\varphi_1^T + I_2\,\varphi_2^T\right]\right\}, \\ t_0 = T/2, \quad T = 2\left(\left|(I_1+1)\,\varphi_1^T + I_2\,\varphi_2^T\right|/M_1^0\right)^{1/2}, \end{gathered} \tag{3.3}$$

and $M_2(t)$ is any admissible control, in particular, a relay function with two switching points of the form

$$M_2(t) = \begin{cases} (-1)^\gamma, & t \in [0, t_1) \cup [t_2, T], \\ (-1)^{\gamma+1}, & t \in [t_1, t_2), \end{cases} \qquad \gamma = 0, 1, \tag{3.4}$$

$$\begin{gathered} t_1 = (-1)^\gamma I_2 \varphi_2^T/T + T/4, \qquad t_2 = (-1)^\gamma I_2 \varphi_2^T/T + 3T/4, \\ T = 2\left(\left|(I_1+1)\varphi_1^T + I_2\varphi_2^T\right|/M_1^0\right)^{1/2}, \quad T = 2(t_2 - t_1), \quad 0 \le t_1 \le t_2 \le T. \end{gathered}$$

2. If $(\varphi_1^T, \varphi_2^T) \in \Phi_2 = \bigcup\limits_{\alpha=0}^{1} \Phi_2^{(\alpha)}$, where

$$\Phi_2^{(\alpha)} = \left\{ \begin{aligned} &(\varphi_1^T, \varphi_2^T) \colon -(-1)^\alpha \left[I_2 (M_1^0 + (-1)^\alpha\right] (I_1+1)^{-1} \varphi_2^T \\ &< \varphi_1^T < (-1)^\alpha (I_1+1) \left[I_2 \left(M_1^0 - (-1)^\alpha\right)\right]^{-1} \varphi_2^T, \\ &(-1)^\alpha \varphi_2^T > 0 \end{aligned} \right\}, \ \alpha = 0, 1, \tag{3.5}$$

then, conversely,

$$M_2(t) = M_2^*(t) = \operatorname{sign}\left[(t_0 - t) I_2 \varphi_2^T\right], \quad t_0 = T/2, \quad T = 2\left(\left|I_2 \varphi_2^T\right|\right)^{1/2}, \tag{3.6}$$

and $M_1(t)$ is any admissible control, in particular, a relay function with two switching points of the form

$$M_1(t) = \begin{cases} (-1)^\gamma, & t \in [0, t_1) \cup [t_2, T], \\ (-1)^{\gamma+1}, & t \in [t_1, t_2), \end{cases} \qquad \gamma = 0, 1, \tag{3.7}$$

$$\begin{gathered} t_1 = (-1)^\gamma \left[(I_1+1)\varphi_1^T + I_2 \varphi_2^T\right] / \left(M_1^0 T\right) + T/4, \\ t_2 = (-1)^\gamma \left[(I_1+1)\varphi_1^T + I_2 \varphi_2^T\right] / \left(M_1^0 T\right) + 3T/4, \\ T = 2\left(\left|I_2 \varphi_2^T\right|\right)^{1/2}, \ T = 2(t_2 - t_1), \ 0 \le t_1 \le t_2 \le T. \end{gathered}$$

3. If $(\varphi_1^T, \varphi_2^T) \in \Phi_0 = \bigcup\limits_{\alpha,k=0}^{1} \Phi_0^{(\alpha,k)}$, where

$$\Phi_0^{(\alpha,k)} = \left\{ \begin{aligned} &(\varphi_1^T, \varphi_2^T) \colon \varphi_2^T = (-1)^k (I_1+1) \left[I_2 \left(M_1^0 - (-1)^k\right)\right]^{-1} \varphi_1^T, \\ &(-1)^\alpha \varphi_1^T > 0 \end{aligned} \right\}, \tag{3.8}$$

$$\alpha, k = 0, 1,$$

then $M_1^*(t)$ and $M_2^*(t)$ are relay functions with one switching point of the form

$$\begin{gathered} M_1^*(t) = (-1)^\alpha M_1^0 \operatorname{sign}(t_0 - t), \quad M_2^*(t) = (-1)^k \operatorname{sign}(t_0 - t), \\ t_0 = T/2, \ \alpha, k = 0, 1. \end{gathered} \tag{3.9}$$

In (3.9), T is determined by the equivalent formulas (3.3) or (3.6).

4. CONSTRUCTION OF REACHABLE DOMAINS OF THE TWO-LINK MANIPULATOR UNDER OPTIMAL CONTROLS WITH ANGULAR VELOCITY CONSTRAINTS

We introduce the following notation:

$$R(M^*) = \bigcup_{i=0}^{2} R_i(M^*), \tag{4.1}$$

$$R_0(M^*) = \bigcup_{\alpha,k=0}^{1} R_0^{(\alpha,k)}(M^*), \quad R_i(M^*) = \bigcup_{\alpha=0}^{1} R_i^{(\alpha)}(M^*),$$
$$R_0^{(\alpha,k)}(M^*) = \left\{(\varphi_1^T, \varphi_2^T) \in \Phi_0^{(\alpha,k)}:\ |\dot\varphi_1| \le a_1,\ |\dot\varphi_2| \le a_2\right\},$$
$$R_i^{(\alpha)}(M^*) = \left\{(\varphi_1^T, \varphi_2^T) \in \Phi_i^{(\alpha)}:\ |\dot\varphi_1| \le a_1,\ |\dot\varphi_2| \le a_2\right\},$$

where the control M^* is given by (3.1).

According to (4.1), $R(M^*)$ is the domain on the configuration plane $(\varphi_1^T, \varphi_2^T)$ where the optimal regimes (3.1) without violating the angular velocity constraints (2.11).

As shown by the constructed optimal solutions for the piecewise-constant control laws (3.1), for the control torques $M_1(t)$ and $M_2(t)$ with two switching instants, there are two control regimes $\gamma = 0{,}1$ equivalent, in terms of time optimality, together with the one-switching control. Therefore, it suffices to consider only the regimes with $\gamma = 0$.

Due to the invariance of system (2.7)–(2.9) with respect to the change of variables $\varphi_i \to -\varphi_i$, $\varphi_1^T \to -\varphi_1^T$, $\varphi_2^T \to -\varphi_2^T$, and $M^* \to -M^*$, the domains $R_0^{(0,0)}$ and $R_0^{(1,0)}$, $R_0^{(0,1)}$ and $R_0^{(1,1)}$, $R_1^{(0)}$ and $R_1^{(1)}$, as well as $R_2^{(0)}$ and $R_2^{(1)}$, are pairwise symmetric about the origin. Hence, we will focus only on the domains $R_0^{(0,0)}$, $R_0^{(0,1)}$, $R_1^{(0)}$, and $R_2^{(0)}$.

The domains $R_i^{(0)}$, $i = 1, 2$, are constructed using the following procedure. Based on the analysis of the functions $\dot\varphi_i(\varphi_i)$, $i = 1, 2$, obtained under the optimal control regimes (3.2), (3.4), (3.6), (3.7) transferring system (2.7) from (2.8) to (2.9) for each degree of freedom, the angular velocity of the ith link achieves the maximum absolute value at the time instant t_0 if this link is rotated with the one-switching control, and at the time instants t_1 or t_2 if it is rotated with the two-switching control. Consequently, conditions (2.11) (or (4.1)) are not violated under the following relations:

$$\begin{aligned} \max_{t\in[0,T]} |\dot\varphi_1(t)| &= |\dot\varphi_1(t_j)| \le a_1, \quad j = 0, 1, 2, \quad 0 < t_1 \le t_0 \le t_2, \\ \max_{t\in[0,T]} |\dot\varphi_2(t)| &= |\dot\varphi_2(t_j)| \le a_2, \quad j = 0, 1, 2, \quad 0 < t_1 \le t_0 \le t_2. \end{aligned} \tag{4.2}$$

Substituting $t_0 = t_0(\varphi_1^T, \varphi_2^T)$, $t_1 = t_1(\varphi_1^T, \varphi_2^T)$, and $t_2 = t_2(\varphi_1^T, \varphi_2^T)$ from (3.3), (3.4), (3.6), and (3.7) into conditions (4.2) yields two inequalities in φ_1^T and φ_2^T, which define the domains

$$\begin{aligned} D_1^{(0)}(a_1) &= \left\{(\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)}:\ |\dot\varphi_1| \le a_1\right\}, \\ D_1^{(0)}(a_2) &= \left\{(\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)}:\ |\dot\varphi_2| \le a_2\right\}, \\ D_2^{(0)}(a_1) &= \left\{(\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}:\ |\dot\varphi_1| \le a_1\right\}, \\ D_2^{(0)}(a_2) &= \left\{(\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}:\ |\dot\varphi_2| \le a_2\right\}. \end{aligned} \tag{4.3}$$

Then the required domains are given by

$$R_1^{(0)} = D_1^{(0)}(a_1) \cap D_1^{(0)}(a_2), \qquad R_2^{(0)} = D_2^{(0)}(a_1) \cap D_2^{(0)}(a_2). \tag{4.4}$$

Let us proceed to the domains $R_0^{(0,0)}$, $R_0^{(0,1)}$, $R_1^{(0)}$, and $R_2^{(0)}$.

4.1. Construction of $R_0^{(0,0)}$ and $R_0^{(0,1)}$

In the domains $\Phi_0^{(0,0)}$ and $\Phi_0^{(0,1)}$, representing a pair of half-lines starting from the origin, both control torques M_1 and M_2 have one switching point (3.9) each. If the problem parameters satisfy the relation

$$a_1\left(I_1+1\right)\left(M_1^0-1\right)^{-1} < a_2 I_2, \tag{4.5}$$

then the angular velocity $\dot{\varphi}_1$ achieves its maximum absolute value a_1 earlier than $\dot{\varphi}_2$ achieves a_2. This conclusion follows from the simple integration of system (2.7), (2.8) up to the time instant $t = t_0$ (when $\dot{\varphi}_1$ hits the constraint).

In this case, $R_0^{(0,0)}$ and $R_0^{(0,1)}$ consist of two segments: one endpoint coincides with the origin $(0,0)$, and the other endpoints have the coordinates

$$A = (\varphi_1^A, \varphi_2^A), \qquad B = (\varphi_1^B, \varphi_2^B), \tag{4.6}$$

$$\begin{aligned}
\varphi_1^A &= a_1^2\left(I_1+1\right)\left(M_1^0-1\right)^{-1}, & \varphi_2^A &= a_1^2\left(I_1+1\right)^2 I_2^{-1}\left(M_1^0-1\right)^{-2},\\
\varphi_1^B &= a_1^2\left(I_1+1\right)\left(M_1^0+1\right)^{-1}, & \varphi_2^B &= -a_1^2\left(I_1+1\right)^2 I_2^{-1}\left(M_1^0+1\right)^{-2}.
\end{aligned}$$

When transferring system (2.7), (2.8) to points A and B at the switching instant $t = t_0$, the equality $\dot{\varphi}_1 = a_1$ and the inequality $\dot{\varphi}_2 < a_2$ hold simultaneously.

In the case of the reverse inequality (4.5),

$$a_1\left(I_1+1\right)\left(M_1^0-1\right)^{-1} > a_2 I_2, \tag{4.7}$$

on the contrary, the angular velocity $\dot{\varphi}_2$ achieves its maximum absolute value a_2 earlier. Then the corresponding domains $R_0^{(0,0)}$ and $R_0^{(0,1)}$ consist of two segments: one endpoint coincides with the origin $(0,0)$, and the other endpoints have the coordinates

$$C = (\varphi_1^C, \varphi_2^C), \qquad D = (\varphi_1^D, \varphi_2^D), \tag{4.8}$$

$$\begin{aligned}
\varphi_1^C &= a_2^2 I_2^2\left(M_1^0-1\right)\left(I_1+1\right)^{-1}, & \varphi_2^C &= a_2^2 I_2,\\
\varphi_1^D &= a_2^2 I_2^2\left(M_1^0+1\right)\left(I_1+1\right)^{-1}, & \varphi_2^D &= -a_2^2 I_2^2.
\end{aligned}$$

When transferring system (2.7), (2.8) to points C and D at the switching instant $t = t_0$, the equality $\dot{\varphi}_2 = a_2$ and the inequality $\dot{\varphi}_1 < a_1$ hold simultaneously.

4.2. Construction of $R_1^{(0)}$

Let $(\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)}$. Then, with single integration of system (2.7), (2.8) on the interval $[0, T]$ under the corresponding controls (3.3), (3.4) ($\gamma = 0$), we obtain the functions $\dot{\varphi}_i(\varphi_i)$, $i = 1, 2$, shown in Fig. 2.

According to the calculations,

$$\left|\dot{\varphi}_1(t)\right| \le \max_{t\in[0,T_1]} \left|\dot{\varphi}_1(t)\right| = \left|\dot{\varphi}_1(t_0)\right|, \tag{4.9}$$

$$\left|\dot{\varphi}_2(t)\right| \le \max_{t\in[0,T_1]} \left|\dot{\varphi}_2(t)\right| = \begin{cases} \left|\dot{\varphi}_2(t_1)\right|, & \varphi_2^T > 0,\\ \left|\dot{\varphi}_2(t_2)\right|, & \varphi_2^T < 0.\end{cases} \tag{4.10}$$

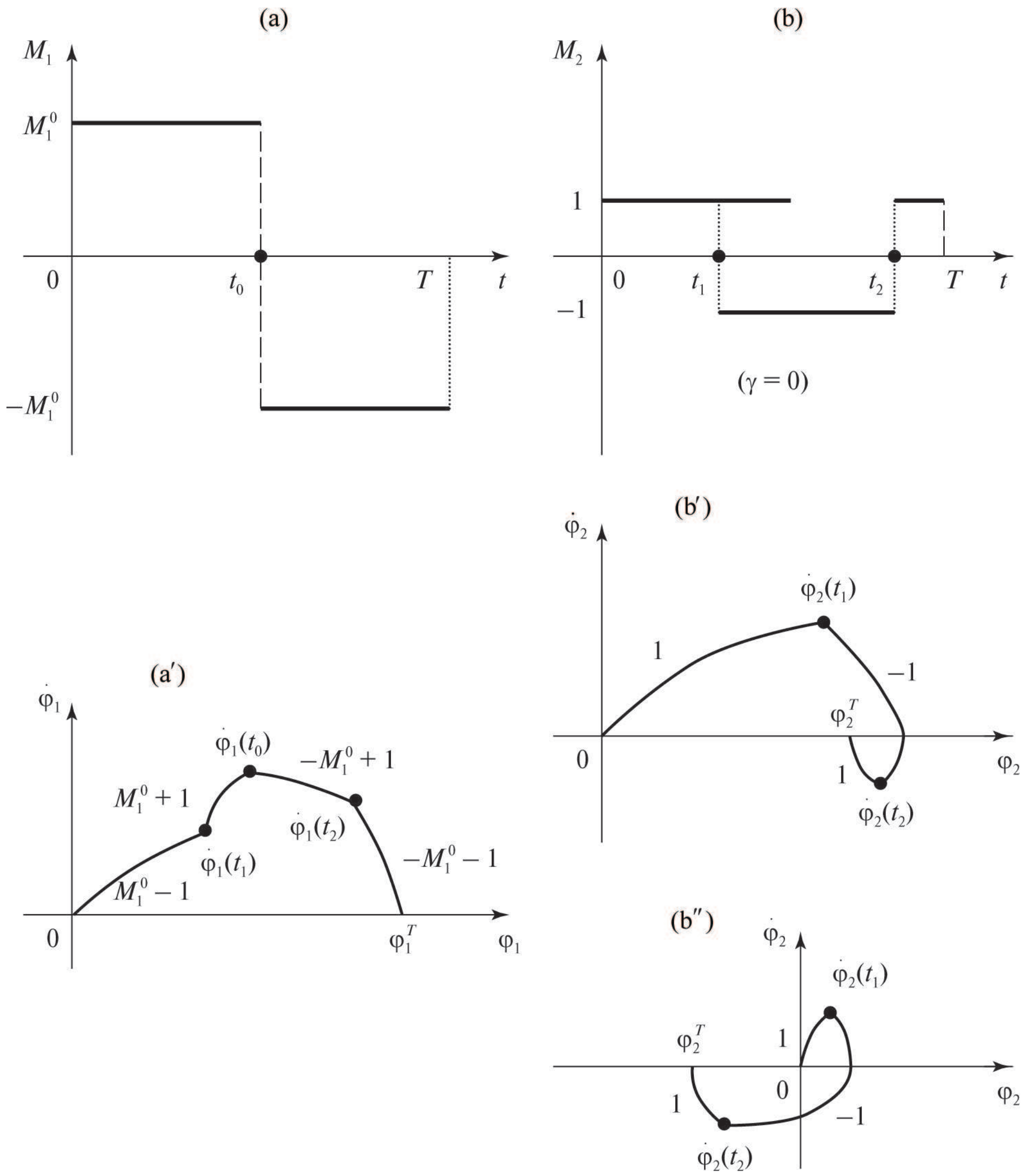

Fig. 2. If $(\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)}$, then (a) M_1 and (b) M_2 are relay functions with one and two switching points, respectively; (a′) $\dot{\varphi}_1(\varphi_1)$ and (b′), (b″) $\dot{\varphi}_2(\varphi_2)$ are the corresponding phase plots.

Considering the expressions for T, t_1, and t_2 in terms of t_0 (see (3.3) and (3.4)), $\dot{\varphi}_1(t_0)$, $\dot{\varphi}_2(t_1)$, and $\dot{\varphi}_2(t_2)$ in (4.9) and (4.10) can be represented as

$$\dot{\varphi}_1(t_0) = -I_2 \left(I_1 + 1\right)^{-1} \varphi_2^T t_0^{-1} + M_1^0 (I_1 + 1)^{-1} t_0 > 0, \tag{4.11}$$

$$\dot{\varphi}_2(t_1) = \left(\varphi_2^T t_0^{-1} + I_2^{-1} t_0\right)/2 > 0 \quad \text{for} \quad \varphi_2^T > 0, \tag{4.12}$$

$$\dot{\varphi}_2(t_2) = \left(\varphi_2^T t_0^{-1} + I_2^{-1} t_0\right)/2 < 0 \quad \text{for} \quad \varphi_2^T < 0. \tag{4.13}$$

After substituting t_0 from (3.3) into (4.1), the inequality $|\dot{\varphi}_1| \le a_1$ becomes

$$\varphi_2^T \ge M_1^0 I_2^{-1} a_1^{-2} \left(\varphi_1^T\right)^2 - \left(I_1 + 1\right) I_2^{-1} \varphi_1^T. \tag{4.14}$$

On the plane $(\varphi_1^T, \varphi_2^T)$, inequality (4.14) defines a domain with the following intersection with the domain $\Phi_1^{(0)}$ (3.2):

$$D_1^{(0)}(a_1) = \left\{ (\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)} \colon \begin{array}{l} -\dfrac{I_1+1}{I_2(M_1^0+1)}\varphi_1^T < \varphi_2^T < \dfrac{I_1+1}{I_2(M_1^0-1)}\varphi_1^T, \quad \varphi_1^T > 0 \\ \varphi_2^T \geq M_1^0 I_2^{-1} a_1^{-2} (\varphi_1^T)^2 - (I_1+1) I_2^{-1} \varphi_1^T, \quad \varphi_1^T > 0 \end{array} \right\}. \tag{4.15}$$

Due to (4.10), (4.12), and (4.13), the second inequality $|\dot\varphi_2| \leq a_2$ from (4.1) reduces to $t_0^2 - 2a_2 I_2 t_0 + I_2 \left|\varphi_2^T\right| \leq 0$, a quadratic inequality in t_0. Therefore, we have

$$t_0^- \leq t_0 \leq t_0^+, \qquad t_0^{+,-} = a_2 I_2 \pm \sqrt{a_2^2 I_2^2 - I_2 \left|\varphi_2^T\right|}. \tag{4.16}$$

Using the expression (3.3) for t_0, inequality (4.16) can be transformed to

$$\begin{gathered} \varphi_1^{T(-)} \leq \varphi_1^T \leq \varphi_1^{T(+)}, \\ \varphi_1^{T(+)} = M_1^0 \left(I_1+1\right)^{-1} \left(a_2 I_2 + \sqrt{a_2^2 I_2^2 - I_2 \left|\varphi_2^T\right|}\right)^2 - I_2 \left(I_1+1\right)^{-1} \varphi_2^T, \\ \varphi_1^{T(-)} = M_1^0 \left(I_1+1\right)^{-1} \left(a_2 I_2 - \sqrt{a_2^2 I_2^2 - I_2 \left|\varphi_2^T\right|}\right)^2 - I_2 \left(I_1+1\right)^{-1} \varphi_2^T. \end{gathered} \tag{4.17}$$

Within $\Phi_1^{(0)}$, the domain $D_1^{(0)}(a_2)$ is defined as follows:

$$D_1^{(0)}(a_2) = \left\{ (\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)} \colon \varphi_1^{T(-)} \leq \varphi_1^T \leq \varphi_1^{T(+)} \right\}, \tag{4.18}$$

where $\varphi_1^{T(+)}$ and $\varphi_1^{T(-)}$ are given by (4.17).

According to (4.4), we have

$$R_1^{(0)} = D_1^{(0)}(a_1) \cap D_1^{(0)}(a_2). \tag{4.19}$$

4.3. Construction of $R_2^{(0)}$

Let $(\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}$. Then, with single integration of system (2.7), (2.8) on the interval $[0, T]$ under the corresponding controls (3.6), (3.7) ($\gamma = 0$), we obtain the functions $\dot\varphi_i(\varphi_i)$, $i = 1, 2$, shown in Fig. 3.

Similar to the previous case, calculations yield the following results:

$$|\dot\varphi_1| \leq \max_{0 \leq t \leq T_2} |\dot\varphi_1(t)| = \begin{cases} |\dot\varphi_1(t_1)|, & \varphi_1^T > 0, \\ |\dot\varphi_1(t_2)|, & \varphi_1^T < 0, \end{cases} \tag{4.20}$$

$$|\dot\varphi_2| \leq \max_{0 \leq t \leq T_2} |\dot\varphi_2(t)| = |\dot\varphi_2(t_0)|, \tag{4.21}$$

where, in view of the expressions (3.6), (3.7) for T, t_1, and t_2 in terms of t_0,

$$\dot\varphi_1(t_1) = \left(M_1^0 - 1\right)(I_1+1)^{-1} \left\{ \left[(I_1+1)\varphi_1^T + I_2 \varphi_2^T \right] / \left(M_1^0 t_0 \right) + t_0 \right\} / 2 > 0, \tag{4.22}$$

$$\dot\varphi_1(t_2) = \left(M_1^0 + 1\right)(I_1+1)^{-1} \left\{ \left[(I_1+1)\varphi_1^T + I_2 \varphi_2^T \right] / \left(M_1^0 t_0 \right) - t_0 \right\} / 2 < 0, \tag{4.23}$$

$$\dot\varphi_2(t_0) = I_2^{-1} t_0 > 0. \tag{4.24}$$

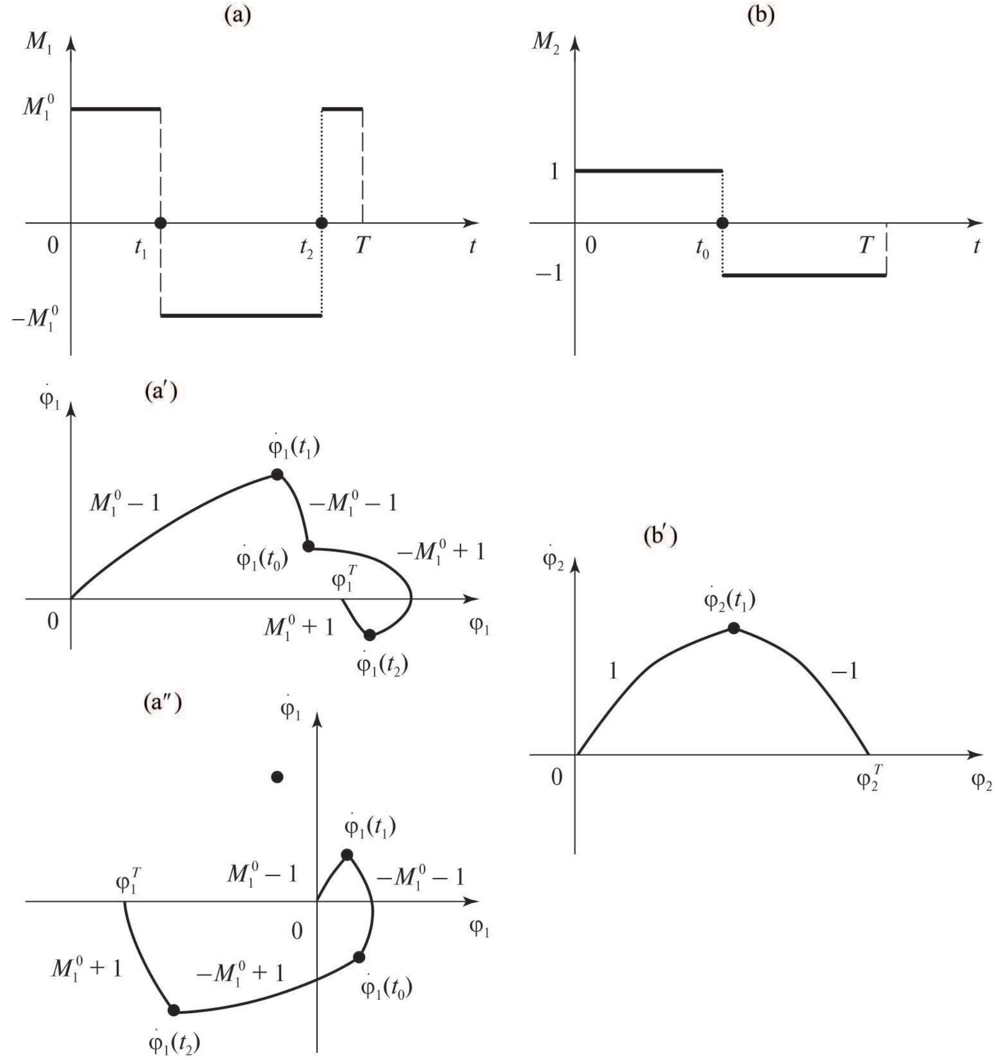

Fig. 3. If $(\varphi_1^T,\varphi_2^T)\in\Phi_2^{(0)}$, then (a) M_1 and (b) M_2 are relay functions with two and one switching points, respectively; (a′), (a″) $\dot\varphi_1(\varphi_1)$ and (b′) $\dot\varphi_2(\varphi_2)$ are the corresponding phase plots.

Based on the expression (3.6) for t_0 and (4.20)–(4.24), the inequalities $|\dot\varphi_1|\le a_1$ and $|\dot\varphi_2|\le a_2$ from (4.1) can be written as

$$\varphi_1^T\le -I_2\left(I_1+1\right)^{-1}\left(M_1^0+1\right)\varphi_2^T+2a_1M_1^0\left(M_1^0-1\right)^{-1}\sqrt{I_2\varphi_2^T},\qquad \varphi_1^T>0, \tag{4.25}$$

$$\varphi_1^T\ge I_2\left(I_1+1\right)^{-1}\left(M_1^0-1\right)\varphi_2^T-2a_1M_1^0\left(M_1^0+1\right)^{-1}\sqrt{I_2\varphi_2^T},\qquad \varphi_1^T<0, \tag{4.26}$$

$$\varphi_2^T\le I_2a_2^2. \tag{4.27}$$

According to (4.13) and (4.25)–(4.27), we have

$$D_2^{(0)}(a_1)=\left\{(\varphi_1^T,\varphi_2^T)\in\Phi_2^{(0)}:\ \begin{array}{ll}\varphi_1^T\le -\dfrac{I_2(M_1^0+1)}{I_1+1}\varphi_2^T+\dfrac{2a_1M_1^0}{M_1^0-1}\sqrt{I_2\varphi_2^T}, & \varphi_1^T>0\\[2ex] \varphi_1^T\ge \dfrac{I_2(M_1^0-1)}{I_1+1}\varphi_2^T-\dfrac{2a_1M_1^0}{M_1^0+1}\sqrt{I_2}\varphi_2^T, & \varphi_1^T<0\end{array}\right\}, \tag{4.28}$$

$$D_2^{(0)}(a_2)=\left\{(\varphi_1^T,\varphi_2^T)\in\Phi_2^{(0)}:\ \varphi_2^T\le I_2^{-1}a_2^2\right\}. \tag{4.29}$$

Hence, from (4.4) it follows that

$$R_2^{(0)} = D_2^{(0)}(a_1) \cap D_2^{(0)}(a_2). \tag{4.30}$$

To construct the domains $R_1^{(0)}$, $R_2^{(0)}$, $R_0^{(0,0)}$, and $R_0^{(0,1)}$, the manipulator was assumed to have the numerical dimensional parameters

$$\begin{gathered} L = 1\,\mathrm{m}, \quad m_2 = 10\,\mathrm{kg}, \quad I_1 = I_2 = (10/3)\,\mathrm{kg}\cdot\mathrm{m}^2, \\ M_1^0 = 2\,\mathrm{N}\cdot\mathrm{m}, \quad M_2^0 = 1\,\mathrm{N}\cdot\mathrm{m}. \end{gathered} \tag{4.31}$$

The links of such a manipulator are identical homogeneous rods. After passing to the dimensionless parameters (2.6), from (4.31) we obtain

$$L = 1, \quad m_2 = 1, \quad I_1 = I_2 = 1/3, \quad M_1^0 = 2, \quad M_2^0 = 1. \tag{4.32}$$

Let the dimensionless values of the angular velocities be

$$\begin{aligned} &(A)\;\; a_1 = 1,\; a_2 = 5.4, \\ &(B)\;\; a_1 = 1,\; a_2 = 0.75. \end{aligned} \tag{4.33}$$

In the case (4.33) (A), condition (4.5) holds; and the case (4.33) (B) corresponds to condition (4.17). According to the graphical plots, $D_{1,2}^{(0)}(a_1) \subset D_{1,2}^{(0)}(a_2)$ for the values (4.33) (A), and $D_{1,2}^{(0)}(a_2) \subset D_{1,2}^{(0)}(a_1)$ for the values (4.33) (B).

Consequently, from (4.19) and (4.30) it follows that

$$\begin{aligned} R_{1,2}^{(0)} &= D_{1,2}^{(0)}(a_1) \quad \text{in the case } (A), \\ R_{1,2}^{(0)} &= D_{1,2}^{(0)}(a_2) \quad \text{in the case } (B). \end{aligned} \tag{4.34}$$

In Fig. 4, the boundary of the domain (4.19) defined by the parabola (4.14) (marked with number (1)) intersects the lines $\Phi_0^{(0,0)}$ and $\Phi_0^{(0,1)}$ at points A and B (4.6), respectively. At each point of the boundary of (4.19), the conditions $\dot{\varphi} = a_1$ and $|\dot{\varphi}_2| < a_2$ hold. The curves given by equalities (4.25) and (4.26) (marked with numbers (2) and (3)) join on the axis φ_2^T and also intersect the lines $\Phi_0^{(0,0)}$ and $\Phi_0^{(0,1)}$ at points A (4.6) and B' (the centrally symmetric point to B (4.6)). Due to the aforementioned central symmetry, the boundaries of the domains $R_1^{(1)}$ and $R_2^{(1)}$ are marked by numbers $(1)'$, $(2)'$, and $(3)'$.

Remark. Depending on the value of a_2 in the case (4.33) (A), the junction point of curves (2) and (3) on the axis φ_2^T may lie below (Fig. 4) or above the line $\varphi_2^T = I_2 a_2^2$. In the latter situation, the boundary of the domain $R_2^{(0)}$ will contain a segment of this line.

According to Fig. 4, the domain R (4.1) narrows near the origin $(0,0)$, which can be explained as follows. The first link reaches point B (closer to the origin) with a greater acceleration, since the torque $M_1^0 + 1$ speeds it up faster, whereas at point A (farther from the origin) its acceleration is smaller due to the weaker action of the torque $M_1^0 - 1$. In both cases, $\dot{\varphi}_1$ achieves its maximum value earlier than $\dot{\varphi}_2$; moreover, with a greater acceleration this time instant occurs even earlier, further restricting the possible final states by shifting them closer to $(0,0)$. Thus, the greater acceleration the first link has, the closer to the origin the final point will be.

In Fig. 5, corresponding to the case (4.33) (B), numbers (1) and $(1)'$ mark the centrally symmetric boundaries of the domains $R_1^{(0)} = D_1^{(0)}(a_2)$ and $R_1^{(1)} = D_1^{(1)}(a_2)$, while numbers (2) and $(2)'$

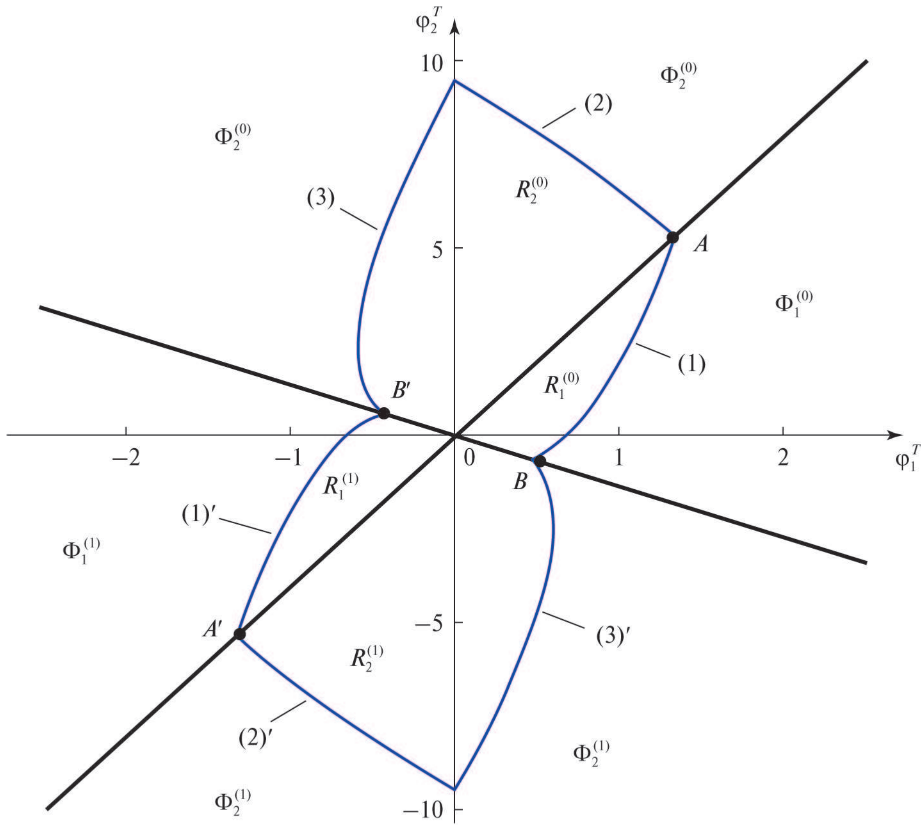

Fig. 4. The domains of final configurations reachable under the optimal regimes (3.1) with condition (4.5).

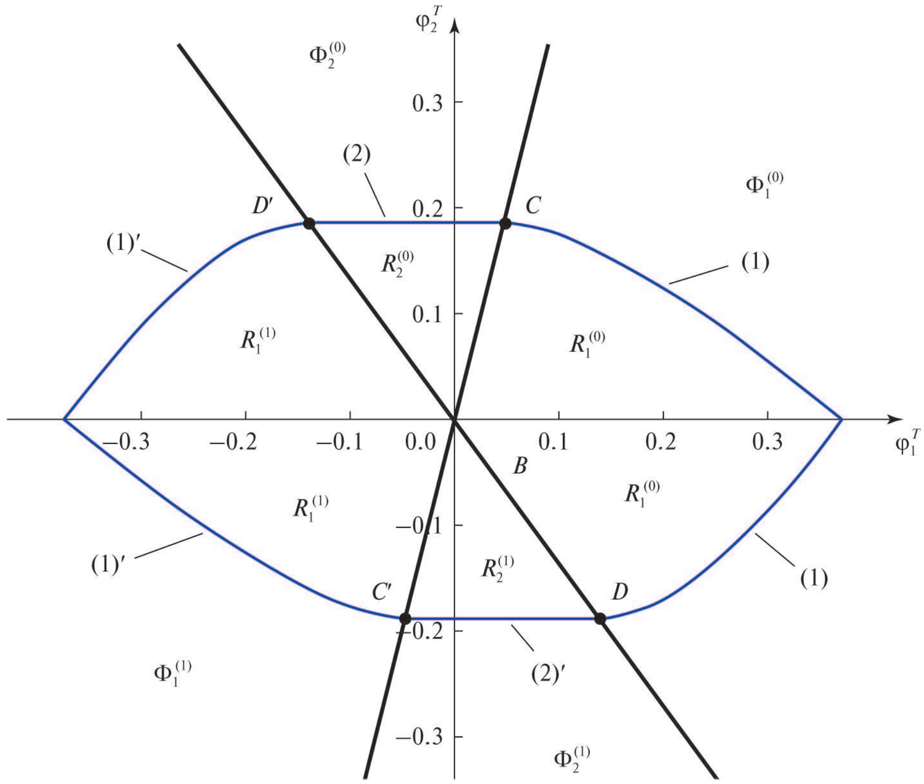

Fig. 5. The domains of final configurations reachable under the optimal regimes (3.1) with condition (4.7).

mark the boundaries of the domains $R_2^{(0)} = D_2^{(0)}(a_2)$ and $R_2^{(1)} = D_2^{(1)}(a_2)$, representing segments of the straight lines $\varphi_2^T = I_2 a_2^2$ and $\varphi_2^T = -I_2 a_2^2$. At each point of the boundary of (4.19), the conditions $\dot{\varphi} = a_1$ and $|\dot{\varphi}_2| < a_2$ hold. The line $\varphi_2^T = I_2 a_2^2$ intersects the lines $\Phi_0^{(0,0)}$ and $\Phi_0^{(1,0)}$ at points C (4.8) and D', the symmetric point to D (4.8), and the line $\varphi_2^T = -I_2 a_2^2$ intersects the lines $\Phi_0^{(0,1)}$ and $\Phi_0^{(1,1)}$ at points D (4.8) and C', the centrally symmetric point to C (4.8).

5. CONCLUSIONS

This paper has considered the model of a two-link manipulator with a statically balanced second link and angular velocity constraints. For this model, an algorithm has been developed to construct reachable domains on the configuration plane under time-optimal controls with a minimum number of switchings. Such controls ensure the manipulator's motion from an initial rest state to a given final rest state without violating angular velocity constraints; moreover, their limiting values can be achieved only at the switching instants, depending on the final configuration of the manipulator.

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Control of the Triaxial Orientation of a Rigid Body under a Delay in the Feedback Law

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Abstract—A rigid body under the action of a linear dissipative torque and an essentially nonlinear restoring one is considered. It is assumed that there is a constant delay in the restoring torque. Using a special construction of the Lyapunov–Krasovsky functional, it is proved that the given orientation of the body is asymptotically stable for any value of delay. In addition, the case is investigated where the body is subject to PID-like controller with linear differential part and essentially nonlinear integral and proportional ones. The admissible values of the controller parameters are determined, for which it solves the problem of triaxial stabilization. The results of numerical modeling are presented, confirming the established theoretical conclusions.

Keywords: rigid body, triaxial orientation, delay, Lyapunov–Krasovskii functional, PID controller

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1. INTRODUCTION

Control of the rotational motion of a rigid body is an actual problem of nonlinear mechanics. It has wide applications in the orientation of satellites, aircraft, underwater vehicles, and robotic systems [1–3]. One important special case of this problem is triaxial stabilization of a rigid body [1, 2]. A large number of papers have been devoted to this case (see, e.g., [1–6] and the bibliography cited therein). Various approaches to constructing stabilizing controls have been proposed. It should be noted that rigid body orientation control is a nonlinear problem, since the configuration space for it is not a linear space [7].

When designing controlled systems, it is often necessary to take into account delay that occurs in the feedback loop [8, 9]. Furthermore, delay can be intentionally introduced into the system to ensure stability or sufficient smoothness of transient processes [9–13]. The presence of delay significantly complicates stability analysis. The main approaches to solving this problem are the Razumikhin method and the Lyapunov–Krasovskii functional method [8, 9].

Some approaches to the synthesis of controls that ensure thriaxial stabilization of a rigid body taking into account a constant feedback delay were proposed in [7, 14–16]. Linear control torques were constructed, and the conditions for asymptotic stability of a given body orientation were formulated in terms of the solvability of systems of linear matrix inequalities, which result in rather stringent constraints on the system parameters and the magnitude of the delay. In [17, 18], linear controls of PID-like type were used to solve the thriaxial stabilization problem.

In this paper, we first consider a rigid body subject to a linear dissipative torque and an essentially nonlinear restoring torque. Nonlinear controls are more robust with respect to delays and nonstationary disturbances than linear ones (see [19–21]). In some cases, they allow for faster convergence of the norm of the closed-loop state vector and a reduction in the steady-state error compared to linear algorithms [22].

It is assumed that the restoring torque contains a constant delay. Using the Lyapunov–Krasovskii functional method, conditions are found that guarantee the asymptotic stability of a given body orientation for any delay value. Next, we consider the case where the body is subject to a PID controller with its differential part being linear and its integral and proportional parts being significantly nonlinear. Admissible values of the controller parameters are determined for which it solves the problem of triaxial body stabilization.

2. PROBLEM STATEMENT

Consider a rigid body rotating around a fixed point O coinciding with its mass center. Let $\omega(t)$ denote a vector of angular velocity and $Oxyz$ be principal central axes of inertia of the body.

Assume that two right triplets of mutually orthogonal unit vectors r_1, r_2, r_3 and s_1, s_2, s_3 are given, where the vectors r_1, r_2, r_3 are fixed in the frame $Oxyz$ and the vectors s_1, s_2, s_3 are fixed in the inertial space.

The angular motion of the body under the impact of a control torque M_u is modeled by the Euler equations

$$\Theta\dot{\omega}(t) + \omega(t) \times (\Theta\omega(t)) = M_u, \tag{1}$$

whereas, for the vectors s_1, s_2, s_3, we have the Poisson kinematic equations

$$\dot{s}_i(t) = -\omega(t) \times s_i(t), \qquad i = 1, 2, 3. \tag{2}$$

Here $\Theta = \operatorname{diag}\{A, B, C\}$ is the inertia tensor of the body in the axes $Oxyz$ (see [1, 2]).

The problem of triaxial stabilization of the body consists in the design of a torque M_u under which the system (1), (2) admits asymptotically stable equilibrium position

$$\omega = 0, \qquad s_i = r_i, \quad i = 1, 2, 3. \tag{3}$$

In [2, 4], it is shown that the required control can be chosen in the form of a sum of linear dissipative and restoring torques: $M_u = M_d + M_r$. Here

$$M_d = -D\omega(t), \tag{4}$$

$M_r = -L(t)$, D is a constant positive definite matrix,

$$L(t) = \alpha_1 s_1(t) \times r_1 + \alpha_2 s_2(t) \times r_2, \tag{5}$$

α_1, α_2 are positive coefficients. In [19], it was suggested to use essentially nonlinear restoring torque

$$M_r = -f^{\nu}(t)L(t), \tag{6}$$

instead of linear one, where the vector $L(t)$ is still determined by the formula (5), $\nu = \text{const} > 0$,

$$f(t) = \frac{1}{2}\left(\alpha_1\|s_1(t) - r_1\|^2 + \alpha_2\|s_2(t) - r_2\|^2\right),$$

and $\|\cdot\|$ is the Euclidean norm of a vector. The robustness of such a control with respect to nonstationary perturbations with zero mean values was proved.

In the present work, we will study conditions of the triaxial stabilization in the case where the body is under the influence of a linear torque of dissipative forces (4) and essentially nonlinear restoring torque, and there is a constant delay $\tau > 0$ in the restoring torque. Thus, the dynamical Euler equations can be rewritten in the form

$$\Theta\dot{\omega}(t) + \omega(t) \times (\Theta\omega(t)) = -f^{\nu}(t-\tau)L(t-\tau) - D\omega(t). \tag{7}$$

In addition, consider the case where PID controller of the form

$$M_u = -\beta f^{\nu}(t)L(t) - D\omega(t) - \int\limits_0^t f^{\nu}(\xi)G(t-\xi)L(\xi)d\xi \tag{8}$$

acts on the body, where β is a constant coefficient, $G(\zeta)$ is a continuous for $\zeta \in [0, +\infty)$ matrix. We will define admissible values of the controller parameters for which it provides the existence and the asymptotic stability of the equilibrium position (3) for the closed-loop system.

3. STABILIZATION WITH THE AID OF A DELAYED FEEDBACK LAW

Let the system (2), (7) be given. We will assume that initial functions $\phi(\xi)$ for it belong to the space $C([-\tau, 0], \mathbb{R}^{12})$ of continuous functions with the uniform norm $\|\phi\|_{\tau} = \max_{\xi\in[-\tau,0]} \|\phi(\xi)\|$.

Theorem 1. *For any constant positive delay τ, the system* (2), (7) *admits asymptotically stable equilibrium position* (3).

Proof. It is obvious that (3) is an equilibrium position of the closed-loop system. To prove the asymptotic stability of the equilibrium position, we choose a Lyapunov–Krasovskii functional as follows:

$$\begin{aligned} V(\omega(t), s_{1t}, s_{2t}) = f(t) + \frac{\gamma}{2}\omega^{\top}(t)\Theta\omega(t) + L^{\top}(t)D^{-1}\Theta\omega(t) \\ + \int\limits_{t-\tau}^{t} (\lambda(\xi+\tau-t)+\mu)(\|s_1(\xi)-r_1\|^{2\nu+2} + \|s_2(\xi)-r_2\|^{2\nu+2})d\xi \\ -L^{\top}(t)D^{-1}\int\limits_{t-\tau}^{t} f^{\nu}(\xi)L(\xi)d\xi, \end{aligned} \tag{9}$$

where γ, λ, μ are positive parameters, s_{jt} are restrictions of the corresponding components of the solution of the system: $s_{jt} : \xi \mapsto s_j(t+\xi)$, $\xi \in [-\tau, 0]$, $j = 1, 2$.

We obtain

$$\begin{aligned} f(t) + c_1\gamma\|\omega(t)\|^2 - c_2(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\|\omega(t)\| \\ +\mu\int\limits_{t-\tau}^{t} (\|s_1(\xi)-r_1\|^{2\nu+2} + \|s_2(\xi)-r_2\|^{2\nu+2})d\xi \\ -c_3(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\int\limits_{t-\tau}^{t}\left(\|s_1(\xi)-r_1\|^{2\nu+1} + \|s_2(\xi)-r_2\|^{2\nu+1}\right)d\xi \\ \le V(\omega_t, s_{1t}, s_{2t}) \le f(t) + c_2(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\|\omega(t)\| \\ + c_4\gamma\|\omega(t)\|^2 + (\lambda\tau+\mu)\int\limits_{t-\tau}^{t} (\|s_1(\xi)-r_1\|^{2\nu+2} + \|s_2(\xi)-r_2\|^{2\nu+2})d\xi \\ + c_3(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\int\limits_{t-\tau}^{t}\left(\|s_1(\xi)-r_1\|^{2\nu+1} + \|s_2(\xi)-r_2\|^{2\nu+1}\right)d\xi, \end{aligned}$$

where $c_l > 0$, $l = 1, 2, 3, 4$.

We differentiate the functional (9) with respect to the system (2), (7) and obtain

$$\dot V = -\gamma\omega^\top(t)D\omega(t) - f^\nu(t)L^\top(t)D^{-1}L(t) - L^\top(t)D^{-1}\Big(\omega(t)\times(\Theta\omega(t))\Big) + \gamma f^\nu(t-\tau)\omega^\top(t)L^\top(t-\tau)$$

$$+\Big(\alpha_1(\omega(t)\times s_1(t))\times r_1 + \alpha_2(\omega(t)\times s_2(t))\times r_2\Big)^\top D^{-1}\left(\int\limits_{t-\tau}^{t} f^\nu(\xi)L(\xi)d\xi - \Theta\omega(t)\right)$$

$$-\lambda\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi + (\lambda\tau+\mu)\big(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2}\big)$$

$$-\mu(\|s_1(t-\tau)-r_1\|^{2\nu+2}+\|s_2(t-\tau)-r_2\|^{2\nu+2}) \le (c_5-\gamma c_6)\,\|\omega(t)\|^2 - c_7 f^\nu(t)\|L(t)\|^2$$

$$+\gamma c_8\|\omega(t)\|(\|s_1(t-\tau)-r_1\|^{2\nu+1}+\|s_2(t-\tau)-r_2\|^{2\nu+1}) + c_9\|\omega(t)\|^2(\|s_1(t)-r_1\|+\|s_2(t)-r_2\|)$$

$$+c_{10}\|\omega(t)\|\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+1}+\|s_2(\xi)-r_2\|^{2\nu+1})d\xi - \lambda\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi$$

$$+(\lambda\tau+\mu)\big(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2}\big) - \mu(\|s_1(t-\tau)-r_1\|^{2\nu+2}+\|s_2(t-\tau)-r_2\|^{2\nu+2}),$$

where $c_5,\dots,c_{10}$ are positive coefficients.

We choose and fix a number $\epsilon\in(0,1)$. It is known [23], that there exists $\delta>0$ such that

$$\|L(t)\|^2 \ge \epsilon\left(\alpha_1^2\|s_1(t)-r_1\|^2+\alpha_2^2\|s_2(t)-r_2\|^2\right) \tag{10}$$

for $\|s_1(t)-r_1\|+\|s_2(t)-r_2\|<\delta$.

Using the relation (10) and Jensen's inequality (see [8, p. 87]), it is easy to show that if values of the parameters λ,μ,δ are sufficiently small and γ is sufficiently large, then the estimates

$$\frac{1}{2}\left(f(t)+\mu\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi + c_1\gamma\|\omega(t)\|^2\right)$$

$$\le V(\omega_t,s_{1t},s_{2t}) \le 2\left(f(t)+c_4\gamma\|\omega(t)\|^2 + (\lambda\tau+\mu)\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right), \tag{11}$$

$$\dot V \le -\frac{1}{2}\left(\gamma c_6\|\omega(t)\|^2 + \epsilon\, c_7 f^\nu(t)\left(\alpha_1^2\|s_1(t)-r_1\|^2+\alpha_2^2\|s_2(t)-r_2\|^2\right) + \lambda\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right) \tag{12}$$

hold for $\|s_{1t}-r_1\|_\tau+\|s_{2t}-r_2\|_\tau<\delta$. Hence (see [8, p. 53, 54]), the equilibrium position (3) is asymptotically stable. The proof is completed.

Remark 1. Unlike known conditions of the triaxial stabilization which were obtained with the aid of linear controls with delay [7, 14–16], Theorem 1 guarantees that, in the case of essentially nonlinear restoring torque, the asymptotic stability of the required equilibrium position occurs for any $\tau > 0$. However, it should be noticed that the delay increasing results in the decreasing the value of δ that determines the region in which estimates (11), (12) are valid. Thus, for a large delay, a control law may not ensure the asymptotic stability if the required position of the axes of the body-fixed coordinate system is sufficiently "far" from the current one. In such situations, to stabilize the program orientation, one should increase values of the parameters α_1, α_2 in the control torque.

4. STABILIZATION WITH THE AID OF A NONLINEAR PID CONTROLLER

Let the control torque be constructed by the formula (8). Consider the system composed of the kinematic equations (2) and dynamical Euler equations of the following form:

$$\Theta\dot{\omega}(t) + \omega(t) \times (\Theta\omega(t)) = -\beta f^{\nu}(t)L(t) - D\omega(t) - \int\limits_0^t f^{\nu}(\xi)G(t-\xi)L(\xi)d\xi. \tag{13}$$

Every solution of this system is defined by the initial conditions: $\omega(\xi + t_0) = w(\xi)$, $s_i(\xi + t_0) = \psi_i(\xi)$ for $\xi \in [-t_0, 0]$, $i = 1, 2, 3$. Here $t_0 \geq 0$, whereas the functions $w(\xi)$, $\psi_i(\xi)$ belong to the space $C([-t_0, 0], \mathbb{R}^3)$ of continuous functions $\varphi(\xi)$ with the norm $\|\varphi\|_{[-t_0,0]} = \max_{\xi\in[-t_0,0]} \|\varphi(\xi)\|$.

Assumption 1. The matrix $G(\zeta)$ possesses the following properties:

1) $\int\limits_0^{+\infty} \|G(\xi)\|d\xi < +\infty$,

2) $\int\limits_{\zeta}^{+\infty} \|G(\xi)\|d\xi \leq \sigma\|G(\zeta)\|$ for $\zeta \geq 0$, where $\sigma = \text{const} > 0$.

Remark 2. Assumption 1 is satisfied in the case when $G(\zeta) = \exp(-a\zeta)\tilde{G}$, where a is a positive constant, $\tilde{G}$ is a constant matrix. It should be noted that PID controllers with such kernels are widely applied in control theory (see, e.g., [8, 24]).

Let $\Xi = \beta I + \int\limits_0^{+\infty} G(\zeta)d\zeta$, where I is the identity matrix.

Assumption 2. The matrix $\Xi D + D\Xi^{\top}$ is positive definite.

Theorem 2. *If Assumptions 1 and 2 are satisfied, then the system* (2), (13) *admits asymptotically stable equilibrium position* (3).

Proof. It is obvious that (3) is an equilibrium position of the closed-loop system. To prove its asymptotic stability, we construct a Lyapunov–Krasovskii functional in the form

$$\begin{aligned} V(t, \omega(t), s_{1t}, s_{2t}) &= f(t) + \frac{\gamma}{2}\omega^{\top}(t)\Theta\omega(t) + L^{\top}(t)D^{-1}\Theta\omega(t) \\ &+ \mu \int\limits_0^t \int\limits_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi \\ &- L^{\top}(t)D^{-1} \int\limits_0^t \int\limits_{t-\xi}^{+\infty} G(\zeta)d\zeta f^{\nu}(\xi)L(\xi)d\xi, \end{aligned} \tag{14}$$

where γ, μ are positive parameters, s_{jt} are restrictions of the corresponding components of a solution of the system, $s_{jt} : \xi \mapsto s_j(t + \xi)$, $\xi \in [-t, 0]$, $j = 1, 2$.

The estimates

$$f(t)+c_1\gamma\|\omega(t)\|^2-c_2(\alpha_1\|s_1(t)-r_1\|+\alpha_2\|s_2(t)-r_2\|)\|\omega(t)\|$$

$$+\mu\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi-c_3(\alpha_1\|s_1(t)-r_1\|$$

$$+\alpha_2\|s_2(t)-r_2\|)\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+1}+\|s_2(\xi)-r_2\|^{2\nu+1})d\xi$$

$$\le V(t,\omega_t,s_{1t},s_{2t})\le f(t)+c_2(\alpha_1\|s_1(t)-r_1\|+\alpha_2\|s_2(t)-r_2\|)\|\omega(t)\|$$

$$+c_4\gamma\|\omega(t)\|^2+\mu\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi$$

$$+c_3(\alpha_1\|s_1(t)-r_1\|+\alpha_2\|s_2(t)-r_2\|)\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+1}+\|s_2(\xi)-r_2\|^{2\nu+1})d\xi,$$

$$\dot V\le(c_5-\gamma c_6)\|\omega(t)\|^2-f^\nu(t)L^\top(t)D^{-1}\Xi L(t)$$

$$-\mu\int\limits_0^t\|G(t-\xi)\|(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi+$$

$$+\mu\int\limits_0^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2})$$

$$+\gamma\beta c_7\|\omega(t)\|(\|s_1(t)-r_1\|^{2\nu+1}+\|s_2(t)-r_2\|^{2\nu+1})$$

$$+c_8\|\omega(t)\|^2(\alpha_1\|s_1(t)-r_1\|+\alpha_2\|s_2(t)-r_2\|)$$

$$+\gamma c_9\|\omega(t)\|\int\limits_0^t\|G(t-\xi)\|(\|s_1(\xi)-r_1\|^{2\nu+1}+\|s_2(\xi)-r_2\|^{2\nu+1})d\xi$$

$$+c_{10}\|\omega(t)\|\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+1}+\|s_2(\xi)-r_2\|^{2\nu+1})d\xi,$$

are valid for the functional (14) and its derivative with respect to the system (2), (13), where $c_1,\dots,c_{10}$ are positive coefficients.

As well as in the proof of Theorem 1, we choose a number $\epsilon\in(0,1)$ and find $\delta>0$ such that (10) holds for $\|s_1(t)-r_1\|+\|s_2(t)-r_2\|<\delta$. Taking into account Assumption 2, we obtain

$$f^\nu(t)L^\top(t)D^{-1}\Xi L(t)\ge\epsilon\, c_{11}(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2})$$

for $\|s_1(t)-r_1\|+\|s_2(t)-r_2\|<\delta$, where $c_{11}=\text{const}>0$.

In addition, using Assumption 1 and the Hölder inequality, we obtain

$$\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi$$

$$\le\sigma\int\limits_0^t\|G(t-\xi)\|(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi,$$

$$\int\limits_0^t\|G(t-\xi)\|\|s_j(\xi)-r_j\|^{2\nu+1}d\xi\le c_{12}\left(\int\limits_0^t\|G(t-\xi)\|\|s_j(\xi)-r_j\|^{2\nu+2}d\xi\right)^{\frac{2\nu+1}{2\nu+2}},\quad j=1,2,$$

where $c_{12}=\text{const}>0$.

With the aid of the derived estimates, it is easy to show that if values of δ and μ are sufficiently small and γ is sufficiently large, then the inequalities

$$\frac{1}{2}\left(f(t)+\mu\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi+c_1\gamma\|\omega(t)\|^2\right)$$

$$\le V(t,\omega_t,s_{1t},s_{2t})\le 2\left(f(t)+c_4\gamma\|\omega(t)\|^2\right.$$

$$\left.+\mu\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right),$$

$$\dot V\le-\frac{1}{2}\left(\gamma c_6\|\omega(t)\|^2+\epsilon\, c_{11}(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2})\right.$$

$$\left.+\mu\int\limits_0^t\|G(t-\xi)\|(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right)$$

hold for $\|s_{1t}-r_1\|_{[-t,0]}+\|s_{2t}-r_2\|_{[-t,0]}<\delta$. Hence (see [9, p. 211, 212]), the equilibrium position (3) is asymptotically stable. The proof is completed.

Remark 3. Theorem 2 states that the considered controller can provide the asymptotic stability of the program orientation even in the case where its proportional part is not restoring, i.e., where $\beta\le 0$. The destabilizing effect of this part is compensated via an appropriate choice of the integral term.

Indeed, if $\beta\le 0$, then one can choose the matrix $G(\zeta)$ in the form $G(\zeta)=ag(\zeta)I$, where a is a positive parameter, $g(\zeta)$ is a scalar continuous for $\zeta\in[0,+\infty)$ function possessing the following properties:

a) $\int\limits_0^{+\infty}|g(\xi)|d\xi<+\infty$,

b) $\int\limits_\zeta^{+\infty}|g(\xi)|d\xi\le\sigma|g(\zeta)|$ for $\zeta\ge 0$, where $\sigma=\text{const}>0$,

c) $\int\limits_0^{+\infty}g(\xi)d\xi>0$.

In this case, Assumption 1 is satisfied, whereas Assumption 2 comes down to the fulfillment of the inequality $\beta+a\int\limits_0^{+\infty}g(\zeta)d\zeta>0$, which is valid for sufficiently large value of the parameter a.

EXAMPLE

Consider a rigid body with principal central moments of inertia $A=20$, $B=25$, $C=35$, and choose the vectors $r_1=(1/\sqrt{2},1/\sqrt{2},0)^\top$, $r_2=(-1/\sqrt{2},1/\sqrt{2},0)^\top$, $r_3=(0,0,1)^\top$ as a program orientation. From here on, all physical quantities have dimensions in the SI system.

First, perform numerical simulation in the case where the angular motion of the body is described by the system (2), (7), and after that compare its results with those obtained for the system (2), (13). Let $\alpha_1=\alpha_2=1$, $\nu=0,3$, $D=2I$, $\tau=1$, $\beta=1$, $G(\zeta)=\exp(-2\zeta)I$.

In both cases, the following initial conditions were chosen for the numerical simulation: $t_0=0$, $\omega(t)=(0.3,-0.4,0.4)^\top$, $s_1(t)=(0,0,1)^\top$, $s_2(t)=(2/\sqrt{5},1/\sqrt{5},0)^\top$, $s_3(t)=(-1/\sqrt{5},2/\sqrt{5},0)^\top$ for

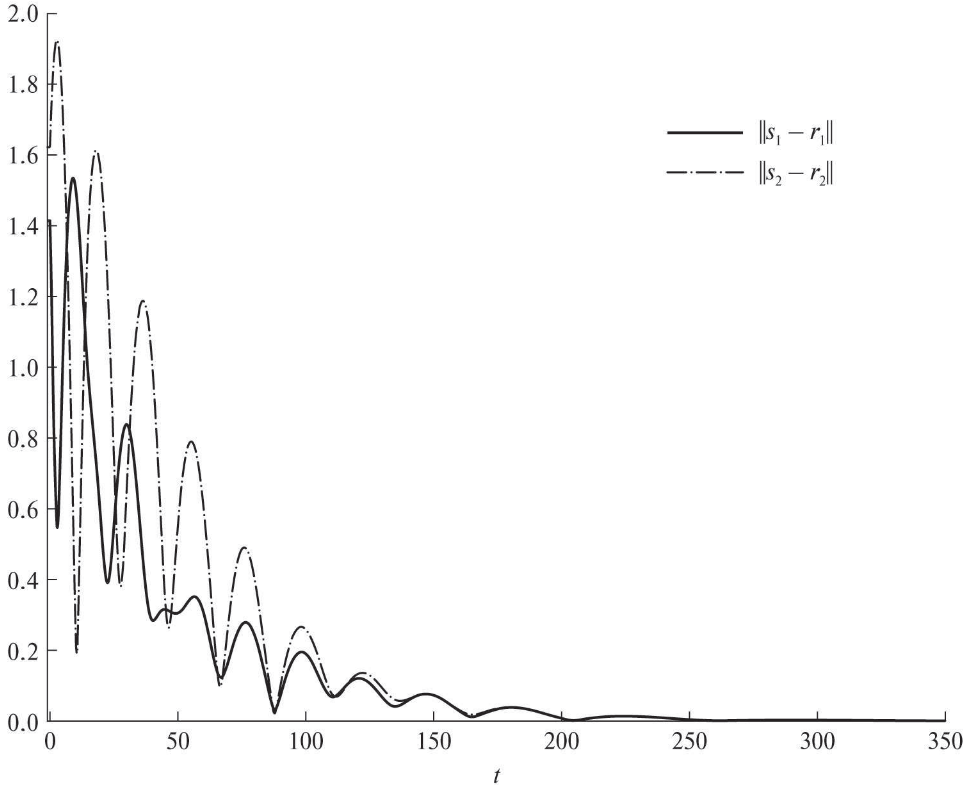

Fig. 1. Results of the numerical simulation under using the delay feedback law.

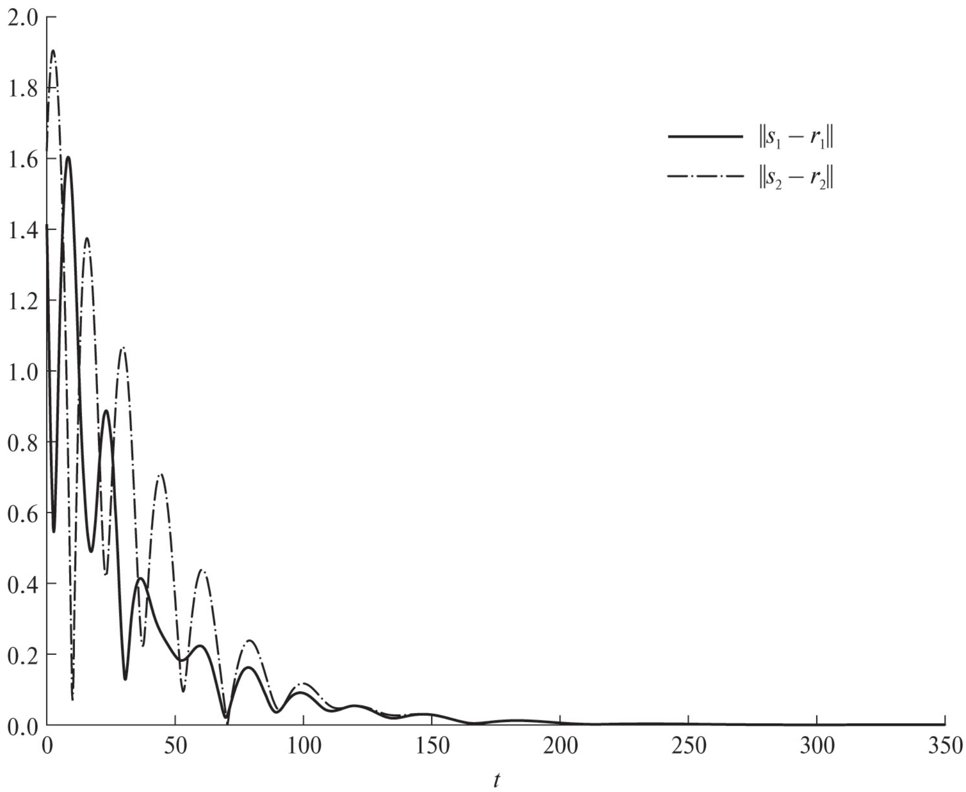

Fig. 2. Results of the numerical simulation under using the PID controller.

$t \in [-1, 0]$. In Figs. 1 and 2 the dependence of the values $\|s_1 - r_1\|$, $\|s_2 - r_2\|$ on time is shown. The simulation results demonstrate the asymptotic stability of the program orientation. However, under the using of PID controller, transient processes become smoother and their duration is reduced.

5. CONCLUSION

Two approaches to the design of control torques ensuring the triaxial stabilization of a rigid body are considered. The first one is based on the use of linear torque of dissipative forces and essentially nonlinear restoring torque containing a constant delay. It is proved that, unlike previously constructed linear control torques, the control proposed in this paper is robust with respect to delay. The second approach is based on the use of a PID controller with linear differential component and essentially nonlinear integral and proportional ones. Admissible values of the controller parameters for which it solves the triaxial stabilization problem are found. The established results imply that the asymptotic stability of program orientation can be ensured even when the proportional component of the controller is not restorative. The destabilizing effect of this component is compensated for by appropriately choosing the integral term.

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Theoretical Basis and Methods of Decision Analysis under Uncertainty with Qualitative Estimates of Probabilities and Preferences

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Abstract—The problems of decision-making are considered when preferences are estimated on an ordinal scale, and the possibilities of realizing the values of an uncertain factor are described by a qualitative probability (complete or only partial). Definitions of preference and indifference relations on a set of strategies are introduced. Simple decision rules are proposed that allow comparing strategies by preference. Examples of their application are given.

Keywords: decision-making under uncertainty, attitude to risk, quantitative probability, ordinal utility, preference relations

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1. INTRODUCTION

Practical decision-making problems often require the consideration of uncertain factors whose values are not known in advance. The most common approaches to such problems involve the use of a utility function, i.e., a function whose mathematical expectation models the preferences of the decision maker (DM) [1–3], or risk measures [4–6]. However, constructing a utility function is a non-trivial problem, and even assuming its existence is a strong hypothesis that may lead to paradoxes conclusions inconsistent with common sense or with the DMs expected outcomes [7, 8]. In addition, using the utility function involves using quantitative probability, i.e., the distribution of probabilities over multiple values of an uncertain factor must be set completely [9] or at least partially [10–12]. Risk measures, in turn, face the challenge of selecting an appropriate metric, as no universal measure exists and each has limitations that prevent a comprehensive risk assessment [13]. Furthermore, their calculation also requires knowledge of quantitative probabilities.

However, in complex decision-making problems of a socio-economic, political, or other nature, objective probabilities usually do not exist, necessitating reliance on expert assessments. Psychological research shows that quantitative evaluations are quite challenging for humans, and therefore, they tend to be insufficiently reliable and may contain significant errors [14].

Therefore, the development of methods for analyzing decisions under uncertainty that do not rely on quantitative probabilities is of theoretical and practical interest. On the other hand, in complex problems, the possible consequences of options are also easier and more reliably assessed qualitatively rather than quantitatively, i.e., on an ordinal scale. A method for analyzing such problems was first described in [15]. The purpose of this article is to present such methods in detail, with justification, and to illustrate their “operation”.

2. MATHEMATICAL MODEL OF PROBLEM SITUATION

The following mathematical model of the situation of decision-making under uncertainty is used:

$$\langle S, \Lambda, V, f, \succsim, R\rangle .$$

Here:

S is the set of strategies (plans, alternatives, options, actions, decisions).

$\Lambda = \{\lambda^1, \ldots, \lambda^n\}$ is a finite set of values of the *uncertain factor* ($n \geqslant 2$). Its elements are called *states of nature*, or *elementary events*, its subsets are called *events*.

V is the set of *numerical outcome estimates*, defined on an ordinal scale [16]. It is assumed that preferences increase on V: the greater the numerical estimate v, the more preferable it is.

f is the *scoring criterion*—a mapping $S \times \Lambda \to V$, i.e., $v = f(s, \lambda)$ is the numerical estimate of the outcome of using strategy s, if value λ of the uncertain factor is realized. Each strategy s is characterized by a vector of outcomes $x = \left(f\left(s, \lambda^1\right), \ldots, f\left(s, \lambda^n\right)\right)$. The set of all vector estimates x (both attainable and hypothetical) is $X = V^n$. Given the assumption about the nature of estimates from V, we consider the scoring criterion to be ordinal.

$\succsim$ is a qualitative probability, complete or only partial (the necessary information on qualitative probability is given in Section 3).

R is a non-strict preference relation of the DM on X: the statement xRy means that vector x is at least as preferable as y. It generates strict preference relation P and indifference relation I as follows: xPy if xRy and not yRx; xIy if both xRy and yRx. The statement xPy means that x is preferred to y, and xIy means that x and y are indifferent (equally preferred). It is assumed that the relation R is a quasi-order (it is reflexive: xRx for any $x \in X$, and transitive: xRy and yRz imply xRx for any $x, y, z \in X$), but only partial, i.e., not every x and y from X are comparable with respect to R: neither xRy nor yRx may hold. In other words, the relations I and P are the symmetric and asymmetric parts of the relation R. For the quasi-order R, the relation P turns out to be a strict partial order (it is irreflexive: yPy is false for any $y \in X$, and is transitive), and the relation I is an equivalence (it is symmetric: xRy implies yRx for any $x, y \in X$, it is reflexive and transitive). The relations R, P, I generate similar relations R_*, P_*, I_* on the set of strategies S as follows:

$$s' R_* s'' \Leftrightarrow x' R x'', \qquad s' P_* s'' \Leftrightarrow x' P x'', \qquad s' I_* s'' \Leftrightarrow x' I x'',$$

here $x' = \left(f\left(s', \lambda^1\right), \ldots, f\left(s', \lambda^n\right)\right)$ and $x'' = \left(f\left(s'', \lambda^1\right), \ldots, f\left(s'', \lambda^n\right)\right)$. The relations R_*, P_*, I_* are directly used to solve the analyzed decision-making problem, taking into account the type of its formulation. For example, if the optimal strategy should be selected, then a set of non-dominated strategies with respect to P_* is constructed, from which the best one must be selected (of course, if it is externally stable).

The relation R is not known in advance: it should be constructed based on given preferences on the set of estimates V and the qualitative probability $\succsim$. Methods for constructing this relation are discussed further in Sections 4 and 5.

3. QUALITATIVE PROBABILITY

Let us present the necessary information about qualitative probability. Let A, B, C denote subsets of the finite set of elementary events, or states of nature $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$; these subsets will be called "events". The number of all events in 2^Λ (including Λ itself and the empty set $\varnothing$) is equal to 2^n. Let $N = \{1, \ldots, n\}$.

Definition 1 [17]. A binary relation $\succsim$ on a set 2^Λ is called complete qualitative, or complete comparative probability, if it has the following properties, taken as axioms:

C1. Comparability: for any A and B, at least one of the relations $A \succsim B$ or $B \succsim A$ holds.

C2. Transitivity: $A \succsim B$ and $B \succsim C \Rightarrow A \succsim C$.

C3. Nontriviality: $\Lambda \succ \varnothing$ and $A \succsim \varnothing$ for any A.

C4. Additivity: if $A \cap C = B \cap C = \varnothing$, then $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$.

From properties C1 and C2 it follows that the complete qualitative probability is reflexive and therefore a connected quasi-order. Therefore, its asymmetric part $\succ$ is a (partial) order, and its symmetric part $\sim$ is an equivalence.

Let a quantitative probability Pr be defined on the elementary events as follows: $\Pr(\lambda^1) = p_1, \ldots, \Pr(\lambda^n) = p_n$, so that the probability of event A is equal to $\Pr A = \sum_{j:\lambda^j \in A} p_j$. The quantitative probability *numerically represents* the qualitative probability, or *is consistent* with the qualitative probability, if the following condition is satisfied: for any A, B

$$A \succsim B \Leftrightarrow \Pr A \geqslant \Pr B. \tag{1}$$

According to (1), $A \succ B \Leftrightarrow \Pr A > \Pr B$, $A \sim B \Leftrightarrow \Pr A = \Pr B$ for any A, B.

If a quantitative probability Pr is given, then the relation $\succsim$ generated by it according to (1) satisfies all properties C1–C4, i.e., it is a qualitative probability.

It turned out that any complete qualitative probability can be represented by some quantitative probability only for $n \leqslant 4$: in [18] a counterexample was constructed for $n = 5$.

In the following, to shorten the notation, we will also denote the elementary event λ^i by its number i, and write events A in multiplicative form. For example, the event $A = \{\lambda^1, \lambda^2, \lambda^4\}$ will be written as $\{1, 2, 4\}$ or 124, and instead of $\lambda^2 \in A$ we will write $2 \in A$.

Example 1 [18]. Let $n = 5$. Consider the relation $\succ$ defined by the chain:

$$\begin{aligned} &12\,345 \succ 2345 \succ 1345 \succ 1245 \succ 345 \succ 245 \succ 1235 \succ 235 \succ 145 \\ &\succ 1234 \succ 45 \succ 135 \succ 125 \succ 234 \succ 35 \succ 134 \succ 25 \succ 124 \succ 15 \\ &\succ 34 \succ 24 \succ 123 \succ 5 \succ 23 \succ 14 \succ 4 \succ 13 \succ 12 \succ 3 \succ 2 \succ 1 \succ \varnothing. \end{aligned} \tag{2}$$

It is straightforward to verify that all properties C1–C4 are satisfied, so the relation $\succsim$, defined by (2), is a complete qualitative probability. Suppose there are quantitative probabilities p_1, p_2, p_3, p_4, p_5 that represent this qualitative probability. From the above chain, we distinguish four relations: $4 \succ 13$, $23 \succ 14$, $15 \succ 34$, $134 \succ 25$. They correspond to four strict inequalities:

$$p_4 > p_1 + p_3, \quad p_2 + p_3 > p_1 + p_4, \quad p_1 + p_5 > p_3 + p_4, \quad p_1 + p_3 + p_4 > p_2 + p_5.$$

However, the system of these four inequalities is inconsistent. Indeed, adding the terms on the two sides of these inequalities, we obtain:

$$2p_1 + p_2 + 2p_3 + 2p_4 + p_5 > 2p_1 + p_2 + 2p_3 + 2p_4 + p_5, \text{ i.e., } 0 > 0.$$

In light of the above, it is clear that methods based on the assumption of the existence of quantitative probability (including the methods of utility theory) are, in principle, not applicable to qualitative probability, which cannot be represented quantitatively!

Definition 2 [19]. A binary relation $\succsim$ on a set 2^Λ is called a partial qualitative (or partial comparative) probability if it satisfies axioms C2–C4 and the axiom

C5. Reflexivity: $A \succsim A$ for any A.

A more detailed discussion of the properties of complete and partial qualitative probabilities (these properties are unusual and far from obvious) can be found in [20]. Here we note the following two properties:

C6. If $A_i \succsim B_i, i = 1, \dots, m, A_i \cap A_j = B_i \cap B_j = \varnothing, i \neq j$, then $A_1 \cup \dots \cup A_m \succsim B_1 \cup \dots \cup B_m$.

C7. $A \succsim B \Leftrightarrow \overline{B} \succsim \overline{A}$; therefore $A \succ B \Leftrightarrow \overline{B} \succ \overline{A}$ and $A \sim B \Leftrightarrow \overline{B} \sim \overline{A}$ (the authors have not encountered this property in the literature, and therefore its proof is given in the Appendix).

4. DEFINITION OF PREFERENCE AND INDIFFERENCE RELATIONS

Let us introduce definitions of preference and indifference relations on the set of vector valuations $X = V^n$, taking into account a given qualitative probability $\succsim$ (complete or only partial). We will introduce definitions of the relations $R^{\succsim}$, $P^{\succsim}$, $I^{\succsim}$ on the set of vector estimates $X = V^n$ taking into account a given qualitative probability $\succsim$ axiomatically, i.e., we will specify a system of axioms that have a clear practical meaning (from the point of view of preferences) and therefore these relations must satisfy. Since it makes no sense to take into account states of nature that are clearly not likely to occur when comparing strategies by preference, we will further use the axiom of nontriviality in a stronger formulation for qualitative probability: $A \succ \varnothing$ for any $A \neq \varnothing$.

Axiom 1. The relation $R^{\succsim}$ is reflexive.

Axiom 2. The relation $R^{\succsim}$ is transitive.

If these axioms are satisfied for $R^{\succsim}$, then the relation $P^{\succsim}$ is a strict partial order, and the relation $I^{\succsim}$ is an equivalence.

Axiom 3. If one component of a vector estimate x is greater than the corresponding component of a vector estimate y, and all other corresponding components are equal, then x is preferable to y: if $x_i > y_i$ and $x_j = y_j$, $j \in N = \{1, \dots, n\}, j \neq i$, then $xP^{\succsim}y$.

The Pareto relation $P^{\varnothing}$ is given by: $xP^{\varnothing}y \Leftrightarrow x_i \geqslant y_i, i \in N, x \neq y$.

According to Axioms 2 and 3, the relation $P^{\succsim}$ satisfies the Pareto axiom: if $x_i \geqslant y_i, i \in N, x \neq y$, then $xP^{\succsim}y$, i.e., $xP^{\varnothing}y \Rightarrow xP^{\succsim}y$.

Let A, B be two events. Let $x, y \in X$ be two vector estimates such that $x_i = y_j, i \in A, j \in B$, i.e., all their components with numbers from A and B are equal to a. Let $b \in V$ such that $b \neq a$, and $x' = (x \parallel x_i = b, i \in A) \in X$ (respectively, $y' = (y \parallel y_j = b, j \in B) \in X$) be the vector estimate obtained from x by replacing all its components with numbers from A with b (respectively, obtained from y by replacing all its components with numbers from B with b). We will say that vector estimates (x and y) are changed in the same way with respect to events A and B if their equal components corresponding to these events are replaced by the same value (i.e., x and y are replaced by x' and y', respectively). Furthermore, if $b > a$, then we will speak of an improvement of x and y (since in this case $x'P^{\varnothing}x$ and $y'P^{\varnothing}y$), and if $b < a$, then, on the contrary, we will speak of their deterioration (since $xP^{\varnothing}x'$ and $yP^{\varnothing}y'$). Note that when $A = \varnothing$ or $B = \varnothing$ the vector estimates x or, respectively, y do not change.

For the non-strict preference relation $R^{\succsim}$ and the relations $P^{\succsim}$ and $I^{\succsim}$ generated by it on the set of vector estimates X according to the meaning of these relations, the following properties are postulated.

Axiom 4. Let two vector estimates be improved in the same way with respect to two equally probable events; if these vector estimates are equal in preference, then they will remain equal in preference; if the first vector estimate is preferable to the second, then it will remain preferable to the second: if $A \sim B \succ \varnothing$ and $a < b$, then $xI^{\succsim}y \Rightarrow x'I^{\succsim}y'$, and $xP^{\succsim}y \Rightarrow x'P^{\succsim}y'$.

Axiom 5. If two vector estimates, the first of which is no less preferable than the second, are improved in the same way with respect to two events, the first of which is more probable than the second, then the first of the resulting vector estimates will be preferable to the second: if $A \succ B$ and $a < b$, then $xR^{\succsim}y \Rightarrow x'P^{\succsim}y'$.

Given that the relation $R^{\succsim}$ is reflexive, transitive, and satisfies Axiom 3, Axioms 4 and 5 define a recursive (inductive) definition of this relation if we start with relations $xI^{\succsim}x$ that are true for any $x \in X$, or with $xP^{\succsim}y$, if $xP^{\varnothing}y$ is true.

Example 2. Let $n = 5$, $V = \{1, 2, \ldots, 10\}$ and the complete qualitative probability be given by (2). Let us start with $x = (1, 1, 1, 1, 1)$. Since $2345 \succ 1235$, according to Axiom 5 we have $(1, 2, 2, 2, 2)\, P^{\succsim}\, (2, 2, 2, 1, 2)$. Further, since $245 \succ 123$, we have $(1, 3, 2, 3, 3)\, P^{\succsim}\, (3, 3, 3, 1, 2)$. Taking into account $4 \succ 13$ we obtain $(1, 3, 2, 4, 3)\, P^{\succsim}\, (4, 3, 4, 1, 2)$. Finally, $4 \succ \varnothing$, and therefore $(1, 3, 2, 5, 3)\, P^{\succsim}\, (4, 3, 4, 1, 2)$. It is convenient to represent these calculations as a double chain (arrows are drawn from more preferred vector estimates to less preferred ones, straight lines connect equal or identically preferred vector estimates):

$$\begin{array}{ccccccccc}
(1,1,1,1,1) & \leftarrow & (1,2,2,2,2) & \leftarrow & (1,3,2,3,3) & \leftarrow & (1,3,2,4,3) & \leftarrow & (1,3,2,5,3) \\
 & & \downarrow_{2345 \succ 1235} & & \downarrow_{245 \succ 123} & & \downarrow_{4 \succ 13} & & \downarrow_{4 \succ \varnothing} \\
(1,1,1,1,1) & \leftarrow & (2,2,2,1,2) & \leftarrow & (3,3,3,1,2) & \leftarrow & (4,3,4,1,2) & — & (4,3,4,1,2)\,.
\end{array}$$

Such double chains can be called *explanatory*: they clearly show why one vector estimate is preferable to another (or whether they are indifferent). The chain above is *ascending*, since the vector estimates in it improve. From Axioms 1–5, the following statements can be derived, also exploiting the deterioration of vector estimates.

Theorem 1. *Let two vector estimates be changed in the same way with respect to two equally probable events; if these vector estimates are equal in preference, then they will remain equal in preference; if the first vector estimate is preferable to the second, then it will remain preferable to the second: if $A \sim B \succ \varnothing$, then for any a and b, $xI^{\succsim}y \Rightarrow x'I^{\succsim}y'$, and $xP^{\succsim}y \Rightarrow x'P^{\succsim}y'$.*

The proofs of the theorems are given in the Appendix.

Theorem 2. *Let two vector estimates, the first of which is no less preferable than the second, be modified in the same way with respect to two events. If the first vector estimate is improved with respect to the more probable event or deteriorated with respect to the less probable event, then the first of the resulting vector estimates will be preferable to the second: if $(A \succ B$ and $a < b)$ or $(A \prec B$ and $a > b)$, then $xR^{\succsim}y \Rightarrow x'P^{\succsim}y'$.*

Theorems 1 and 2 are generalizations of Axioms 4 and 5 and can also be used to construct double chains. In writing the chains, we specifically use the statement $A \prec B$ instead of $B \succ A$ to further indicate that Theorem 2 is used with deteriorated vector estimates.

Example 3. Let $n = 5$, $V = \{1, 2, \ldots, 10\}$ and the complete qualitative probability be given by (2). Let us start with $x = (5, 5, 5, 5, 5)$. Since $1235 \prec 12\,345$, according to Theorem 2 we have $(4, 4, 4, 5, 4)\, P^{\succsim}\, (4, 4, 4, 4, 4)$. Further, since $1235 \prec 245$, we have $(3, 3, 3, 5, 3)\, P^{\succsim}\, (4, 3, 4, 3, 3)$. Taking into account $13 \prec 45$, we obtain $(2, 3, 2, 5, 3)\, P^{\succsim}\, (4, 3, 4, 2, 2)$. Finally, $1 \prec 4$, and therefore $(1, 3, 2, 5, 3)\, P^{\succsim}\, (4, 3, 4, 1, 2)$. We have a chain (this chain is *decreasing*):

$$\begin{array}{ccccccccc}
(5,5,5,5,5) & \rightarrow & (4,4,4,5,4) & \rightarrow & (3,3,3,5,3) & \rightarrow & (2,3,2,5,3) & \rightarrow & (1,3,2,5,3) \\
 & & \downarrow_{1235 \prec 12\,345} & & \downarrow_{1235 \prec 245} & & \downarrow_{13 \prec 45} & & \downarrow_{1 \prec 4} \\
(5,5,5,5,5) & \rightarrow & (4,4,4,4,4) & \rightarrow & (4,3,4,3,3) & \rightarrow & (4,3,4,2,2) & \rightarrow & (4,3,4,1,2)\,.
\end{array}$$

5. CONSTRUCTION OF PREFERENCE AND INDIFFERENCE RELATIONS

The decision rules presented below are based on the development of ideas from the theory of qualitative importance of criteria in multicriteria decision-making problems [21]. To formulate them, we assign to each pair of vector estimates x and y a non-decreasingly ordered set consisting of their components:

$$W(x,y) = \{x_1\} \cup \dots \cup \{x_n\} \cup \{y_1\} \cup \dots \cup \{y_n\} = \{w_1, \dots, w_q\},$$

where $q \leqslant 2n$ and depend on x and y, $w_1 < \dots < w_q$. We will assign $2(q-1)$ sets of component numbers to vector estimates:

$$M_k(x) = \{i \in N \mid x_i \geqslant w_k\}, \quad M_k(y) = \{i \in N \mid y_i \geqslant w_k\}, \quad k = 2, 3, \dots, q.$$
$$L_k(x) = \{i \in N \mid x_i \leqslant w_k\}, \quad L_k(y) = \{i \in N \mid y_i \leqslant w_k\}, \quad k = q-1, q-2, \dots, 1.$$

Theorem 3. *The relation $xR^{\succsim}y$ is true if and only if all of the following relations are true*

$$M_k(x) \succsim M_k(y), \quad k = 2, 3, \dots, q. \tag{3}$$

Furthermore, if in (3) *all $\succsim$ are $\sim$, then $xI^{\succsim}y$. Otherwise* (*when* (3) *contains $\succ$*) *$xP^{\succsim}y$ is true.*

Example 4. Under the conditions of Example 2 compare the vector estimates $x = (1, 3, 2, 5, 3)$ and $y = (4, 3, 4, 1, 2)$. Here $W(x,y) = \{1, 2, 3, 4, 5\}, q = 5$. According to (3), we have:

$$w_2 = 2, M_2(x) = \{2, 3, 4, 5\} \succ \{1, 2, 3, 5\} = M_2(y);$$
$$w_3 = 3, M_3(x) = \{2, 4, 5\} \succ \{1, 2, 3\} = M_3(y);$$
$$w_4 = 4, M_4(x) = \{4\} \succ \{1, 3\} = M_4(y); \qquad w_5 = 5, M_5(x) = \{4\} \succ \varnothing = M_5(y),$$

therefore $xP^{\succsim}y$ is true. It is easy to see that in the process of successive use of (3) the double chain from Example 2 is generated.

Due to the reflexivity and transitivity of qualitative probability $\succsim$ the relation $R^{\succsim}$ defined according to (3) is indeed a quasi-order, but in the general case its only partial. However, in the case of $q = 2$ and complete qualitative probability, any two vector estimates, according to (3), are comparable in preference.

Example 5. Let $V = \{1, 2\}$ and complete qualitative probability is given by (2). The complete order of the set X of all thirty-two ($= 2^5$) vector estimates, which are written in multiplicative form for brevity, is as follows:

$$22222\,P\,12222\,P\,21222\,P\,22122\,P\,11222\,P\,12122\,P\,22212\,P\,12212\,P$$
$$P\,21122\,P\,22221\,P\,11122\,P\,21212\,P\,22112\,P\,12221\,P\,11212\,P\,21221\,P$$
$$P\,12112\,P\,22121\,P\,21112\,P\,11221\,P\,12121\,P\,22211\,P\,11112\,P\,12211\,P$$
$$P\,21121\,P\,11121\,P\,21211\,P\,22111\,P\,11211\,P\,12111\,P\,21111\,P\,11111.$$

One can also formulate a decision rule equivalent to (3) and in some sense dual to it.

Theorem 4. *The relation $xR^{\succsim}y$ is true if and only if all of the following relations are true*

$$L_k(x) \precsim L_k(y), \quad k = q-1, q-2, \dots, 1. \tag{4}$$

Furthermore, if in (4) *all $\precsim$ are $\sim$, then $xI^{\succsim}y$. Otherwise (when* (4) *contains $\prec$) $xP^{\succsim}y$ is true.*

Considering the sets $M_k(x)$ and $L_k(x)$ as subsets of the universal set N, we can write: $\overline{M}_{k+1}(x) = L_k(x)$ and $\overline{M}_{k+1}(y) = L_k(y)$, $k = 1, \dots, q-1$, and then in light of property C7 it is immediately clear that conditions (3) and (4) are equivalent.

Example 6. Under the conditions of Example 3 compare the vector estimates $x = (1, 3, 2, 5, 3)$ and $y = (4, 3, 4, 1, 2)$. Here $W(x, y) = \{1, 2, 3, 4, 5\}, q = 5$. According to (4), we have:

$$w_4 = 4, L_4(x) = \{1, 2, 3, 5\} \prec \{1, 2, 3, 4, 5\} = L_4(y);$$
$$w_3 = 3, L_3(x) = \{1, 2, 3, 5\} \prec \{2, 4, 5\} = L_3(y);$$
$$w_2 = 2, L_2(x) = \{1, 3\} \prec \{4, 5\} = L_2(y); \qquad w_1 = 1, L_1(x) = \{1\} \prec \{4\} = L_1(y),$$

so $xP^{\succsim}y$ is true. In the process of successive use of (4) the double chain from Example 3 is generated.

Remark. The relation of non-strict preference on a set of strategies for decision-making problems with qualitative estimates of probabilities and preferences was introduced earlier in [22] similar to the relation of non-strict preference for groups of criteria comparable in importance, but it was weaker than the one considered here, and no analytical decision rule was established for it.

6. CONCLUSION

This article examines a new, previously unexplored formulation of a decision analysis problem, in which preferences are assessed on an ordinal scale, and the probability of realizing the values of an uncertain factor is expressed using qualitative probability. It is not assumed that qualitative probability can be represented quantitatively. An approach to analyzing such problems is developed and effective decision rules are proposed that enable pairwise comparison of strategies by preference and the elimination of unacceptable (dominated) ones.

It is known that the use of qualitative estimates alone typically does not completely resolve the problem of identifying a single optimal strategy, but rather only narrows the set of candidates. This facilitates further analysis (in particular, it requires obtaining fewer quantitative preference estimates, and simpler ones, such as interval estimates). Therefore, obtaining and using only qualitative estimates typically occurs in the early stages of iterative procedures for solving choice problems [20, 23].

The results obtained can be considered a first step toward developing a decision analysis methodology focused on cases in which preferences and event probabilities are assessed qualitatively. Further research could focus on taking into account people's attitudes toward risk, incorporating more general types of preference relations into the results set (e.g., partial relations, which are common to multicriteria problems), and developing more sophisticated scales for representing preference relations (e.g., the first ordered metric scale).

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APPENDIX

A.1. PROOF OF PROPERTY C7

Consider case 1, when $A \cap B = \varnothing$. This implies $A \subseteq \overline{B}$ and $B \subseteq \overline{A}$. Denote $C = \overline{A \cup B} = \overline{A} \cap \overline{B}$. It is clear that $A \cap C = B \cap C = \varnothing$. Therefore, $A \cup C = \left(A \cup \overline{A}\right) \cap \left(A \cup \overline{B}\right) = A \cup \overline{B}$. Then from $A \subseteq \overline{B}$ it follows that $A \cup C = \overline{B}$. Similarly, $B \cup C = \overline{A}$. Since $A \cap C = B \cap C = \varnothing$, according to property C4 we obtain $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C \Leftrightarrow \overline{B} \succsim \overline{A}$.

Now consider the general case 2, when $A \cap B \neq \varnothing$. Denote $D = A \cap B$. According to C4, we have $A \succsim B \Leftrightarrow A \setminus D \succsim B \setminus D$. Because $A \setminus D \cap B \setminus D = \varnothing$, as proved in Case 1, we have $A \setminus D \succsim B \setminus D \Leftrightarrow \overline{B \setminus D} \succsim \overline{A \setminus D}$. Restate $\overline{A \setminus D} = \overline{A} \cup D$ and $\overline{B \setminus D} = \overline{B} \cup D$. Since $\overline{A} \cap D = \overline{B} \cap D = \varnothing$, it follows from property C4 that $\overline{B} \cup D \succsim \overline{A} \cup D \Leftrightarrow \overline{B} \succsim \overline{A}$. As a result, we obtain the required $A \succsim B \Leftrightarrow \overline{B} \succsim \overline{A}$. Property C7 is proved.

The proofs of Theorems 1 and 2 are much simpler if we use Theorem 3. Therefore, we first give the proof of Theorem 3 that does not use Theorems 1 and 2.

A.2. PROOF OF THEOREM 3

Introduce the operator for parallel improvement of vector estimates:

$$\Uparrow [A, B, a, b]\,, \quad \text{where} \quad A, B \in 2^{\Lambda}, \quad a, b \in \mathbb{R}, \quad a < b.$$

The domain of an operator is the set of pairs of vector estimates (x, y) such that $x_i = y_j = a$, $i \in A$, $j \in B$. The result of applying this operator to a pair (x, y) is the pair of vector estimates (x', y') such that $x' = (x \parallel x_i = b, i \in A)$, $y' = (y \parallel y_j = b, j \in B)$. Therefore, this operator preserves the statements of Axioms 4 and 5 of the relation $R^{\succsim}$. For any vector estimates x and y the relation $xR^{\succsim}y$ is true if and only if there exists a vector estimate z and a superposition of improvement operators

$$(x, y) = \Uparrow [A_L, B_L, a_L, b_L] \ldots \Uparrow [A_1, B_1, a_1, b_1]\,(z, z), \tag{A.1}$$

where $A_l \succsim B_l$, $l = 1, \ldots, L$. It is clear that the components of the vector estimate z must satisfy the conditions $z_i \leqslant x_i$ and $z_i \leqslant y_i, i \in N$. For convenience, we will further assume that $z_i \geqslant w_1 = \min_{j \in N}\{x_j, y_j\}$ is satisfied for all $i \in N$. Otherwise, the values of the corresponding components can be increased using the operator $\Uparrow [\{i\}, \{i\}, z_i, w_1]$.

The sequence of application of operators (A.1) corresponds to a double chain, starting with the vector estimate z. Therefore, from now on, an expression of the form (A.1) will also be called a double chain.

Comparison of vector estimates x and y using decision rule (3) can be represented as a superposition of improvement operators of the following form:

$$(x, y) = \Uparrow [M_q(x), M_q(y), w_{q-1}, w_q] \ldots \Uparrow [M_2(x), M_2(y), w_1, w_2]\,(e, e), \tag{A.2}$$

where $e = (w_1, \ldots, w_1)$. The existence of the double chain (A.2) proves the sufficient condition of Theorem 3.

Let us proceed to the proof of necessity. Let a double chain of the form (A.1) be given. Denote $U = W(x, y) \cup \{a_1\} \cup \cdots \cup \{a_L\} \cup \{b_1\} \cup \cdots \cup \{b_L\} = \{u_1, \ldots, u_r\}$, where $r \leqslant 2n + 2L$, $u_1 < \cdots < u_r$. Because, as assumed, $z_i \geqslant w_1, i \in N$, we should have $u_1 = w_1$. Besides, we have $u_r = w_q$, since the quantities $a_l, b_l, l = 1, \ldots, L$, cannot exceed the finite values of the components of the vector estimates x and y. Thus, the boundaries of the non-uniform grids U and $W(x, y)$ coincide, but in the general case U can be finer than $W(x, y)$, i.e., $W(x, y) \subseteq U$.

Construct a double chain of the form (A.2) using (A.1), following the steps:

1) Move from the initial vector estimate z to e.

2) Divide all operators into operators with a step along the adjacent nodes of the non-uniform grid U.

3) Sort the operators in ascending order of the step number k on the grid U.

4) Combine operators with matching k into a common operator.

5) Combine the operators within a step along the grid $W(x, y)$ into a common operator.

Step 1. The pair (z,z) can easily be obtained from (e,e) by successively applying the improvement operators $\Uparrow[\{i\},\{i\},w_1,z_i]$ for all $i \in N$ such that $z_i > w_1$. Recall that in (A.1) the assumption was made that $z_i \geqslant w_1, i \in N$. We substitute the resulting expression into (A.1) instead of (z,z).

Step 2. The increment interval in an arbitrary improvement operator can be divided according to the following rule:

$$\Uparrow[A,B,a,b] = \Uparrow[A,B,c,b] \Uparrow[A,B,a,c], \text{where } a < c < b. \tag{A.3}$$

Using property (A.3), each operator $\Uparrow[A,B,u_k,u_t], k < t$, in the existing chain can be expressed through operators with steps along the grid U:

$$\Uparrow[A,B,u_k,u_t] = \Uparrow[A,B,u_{t-1},u_t] \ldots \Uparrow[A,B,u_{k+1},u_{k+2}] \Uparrow[A,B,u_k,u_{k+1}].$$

Step 3. Consider any two adjacent operators in the resulting chain:

$$\ldots \Uparrow[C,D,u_{t-1},u_t] \Uparrow[A,B,u_{k-1},u_k] \ldots$$

If $k \leqslant t$, we leave the operators in the same sequence. The case $k > t$ is possible only when $(A \cup B) \cap (C \cup D) = \varnothing$, otherwise the result of the operator $\Uparrow[A,B,u_{k-1},u_k]$ will be outside the domain of the operator $\Uparrow[C,D,u_{t-1},u_t]$. Because these two operators affect the non-intersecting sets of components of the vector estimates, we can permute these two operators in the overall sequence, without affecting the overall result.

Using the considered permutation of adjacent operators, we sort the entire sequence of operators in ascending order of k.

Step 4. For each $k = 2, \ldots, r$, consider all operators that increase the components of the vector estimates from u_{k-1} to u_k. Due to the sorting performed in the previous step, all these operators form a subsequence:

$$\Uparrow[A_{km},B_{km},u_{k-1},u_k] \ldots \Uparrow[A_{k2},B_{k2},u_{k-1},u_k] \Uparrow[A_{k1},B_{k1},u_{k-1},u_k].$$

Consider the interval $w_{t-1} < u_k \leqslant w_t$ on a coarser grid $W(x,y)$. For each $k = 2, \ldots, r$, it exists, since $w_1 < u_2$ and $u_r = w_q$. It turns out that during the transformation (A.1) from the vector estimate e to the vector estimate x, the components with numbers from the sets $A_{k1}, \ldots, A_{km}$ increased from u_{k-1} to u_k, and we have $x_i \geqslant u_k$. At the same time the sets $A_{k1}, \ldots, A_{km}$ are pairwise disjoint, since the same component of the vector estimate cannot be increased from u_{k-1} to u_k more than once. Since the final values of the components of the vector estimate x belong to $W(x,y)$, we have $x_i \geqslant w_t$. And this by definition means that $N_t(x) = A_{k1} \cup A_{k2} \cup \cdots \cup A_{km}$. Similarly, $N_t(y) = B_{k1} \cup B_{k2} \cup \cdots \cup B_{km}$ and the sets $B_{k1}, \ldots, B_{km}$ are pairwise disjoint.

Then, from $A_{ki} \succsim B_{ki}$, $i = 1, \ldots, m$, according to property C6 of qualitative probability (see Section 3), it follows that $N_t(x) \succsim N_t(y)$. And the considered subsequence of operators can be folded into one: $\Uparrow[N_t(x),N_t(y),u_{k-1},u_k]$.

Step 5. The previous reasoning is the same for all steps $[u_{k-1},u_k]$ of the grid U that fall inside the step $[w_{t-1},w_t]$ of the grid $W(x,y)$, namely, $[u_{k_{t-1}},u_{k_{t-1}+1}]$, $[u_{k_{t-1}+1},u_{k_{t-1}+2}], \ldots, [u_{k_t-1},u_{k_t}]$, where $u_{k_{t-1}} = w_{t-1}$, $u_{k_t} = w_t$. Moreover, the entire interval $[w_{t-1},w_t]$ must be covered by these steps of the grid U, since in the vector estimates x and y the components with numbers from $N_t(x)$ and $N_t(y)$, respectively, eventually increased from w_{t-1} to w_t. Therefore, the operators $\Uparrow[N_t(x),N_t(y),u_{k-1},u_k]$ for all $k = k_{t-1}+1, \ldots, k_t$ can be folded into one operator according to property (A.3):

$$\Uparrow[N_t(x),N_t(y),w_{t-1},w_t] = \Uparrow[N_t(x),N_t(y),u_{k_t-1},u_{k_t}] \ldots \Uparrow[N_t(x),N_t(y),u_{k_{t-1}},u_{k_{t-1}+1}].$$

The required chain (A.2) is obtained. The necessary condition of Theorem 3 is proved.

Lemma 1. *If $A = B \succ \varnothing$, then for any a and b, $xI^{\succsim}y \Rightarrow x'I^{\succsim}y'$, and $xP^{\succsim}y \Rightarrow x'P^{\succsim}y'$.*

A.3. PROOF OF LEMMA 3

The condition of Lemma 1 for $a < b$ follows from Axiom 4. Suppose that $xR^{\succsim}y$ holds for $A = B$ and $a > b$. Consider the union of sets $W = W(x, y) \cup W(x', y')$. It differs from $W(x, y)$ by adding the number b if it was not in $W(x, y)$. And it differs from $W(x', y')$ by adding the number a if it was not in $W(x', y')$. Let $W = \{w_1, \ldots, w_q\}, w_1 < \cdots < w_q$. Adding elements to $W(x, y)$ does not affect the fulfillment of the conditions of the decision rule (3), therefore, according to Theorem 3 from $xR^{\succsim}y$ it follows that

$$M_k(x) \succsim M_k(y), \quad k = 2, 3, \ldots, q. \tag{A.4}$$

In the case of $xI^{\succsim}y$ in (A.4), $M_k(x) \sim M_k(y)$ is true for all k, and in the case of $xP^{\succsim}y$ in (A.4), $M_k(x) \succ M_k(y)$ is true for at least one k.

Denote $w_{k_a} = a$, $w_{k_b} = b$. Note that for $k \leqslant k_b$ and $k > k_a$ the set equalities $M_k(x') = M_k(x)$ and $M_k(y') = M_k(y)$ hold. And for $k_b < k \leqslant k_a$ we have $M_k(x') \cup A = M_k(x)$, $M_k(x') \cap A = \varnothing$ and $M_k(y') \cup A = M_k(y)$, $M_k(y') \cap A = \varnothing$. Then, according to the additivity axiom, from $M_k(x) \sim M_k(y)$ it follows that $M_k(x') \sim M_k(y')$, and from $M_k(x) \succ M_k(y)$ it follows that $M_k(x') \succ M_k(y')$. Therefore, according to Theorem 3 taking into account (A.4) in the case $xI^{\succsim}y$ we obtain $x'I^{\succsim}y'$, and in the case $xP^{\succsim}y$ we obtain $x'P^{\succsim}y'$. Lemma 1 is proved.

A.4. PROOF OF THEOREM 1

For $a < b$ the statement of Theorem 1 follows from Axiom 4. Suppose that $xR^{\succsim}y$ holds for $a > b$. Let us deteriorate the vector estimates x and y with respect to the same event $C = A \cup B$: $x'' = (x \parallel x_i = b, i \in C)$, $y'' = (y \parallel y_i = b, i \in C)$. Then, according to Lemma 1 $x''R^{\succsim}y''$ holds. Note that the vector estimates x' and y' can be obtained by improving in the same way (from b to a) the vector estimates x'' and y'' with respect to the events $B \setminus A$ and $A \setminus B$, respectively. But according to the additivity axiom, $A \sim B$ implies $B \setminus A \sim A \setminus B$. Then according to Axiom 4 we obtain $x'R^{\succsim}y'$. Theorem 1 is proved.

A.5. PROOF OF THEOREM 2

For $A \succ B$ and $a < b$ the statement of Theorem 2 follows from Axiom 5. Suppose that $xR^{\succsim}y$ holds for $A \prec B$ and $a > b$. Let us deteriorate the vector estimates x and y with respect to the same event $C = A \cup B$: $x'' = (x \parallel x_i = b, i \in C)$, $y'' = (y \parallel y_i = b, i \in C)$. Then, according to Lemma 1, $x''R^{\succsim}y''$ holds. Note that the vector estimates x' and y' can be obtained by improving in the same way (from b to a) the vector estimates x'' and y'' with respect to the events $B \setminus A$ and $A \setminus B$, respectively. But according to the additivity axiom, $A \prec B$ implies $B \setminus A \succ A \setminus B$. Then according to Axiom 5 we obtain $x'P^{\succsim}y'$. Theorem 2 is proved.

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